ATOMIC CHARACTERIZATION OF THE HARDY SPACE $H^1_L(\mathbb{R})$
OF ONE-DIMENSIONAL SCHRÖDINGER OPERATORS
WITH NONNEGATIVE POTENTIALS

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Abstract. Given a Schrödinger operator $L = \frac{d^2}{dx^2} - V(x)$ on $\mathbb{R}$ with nonnegative potential $V$, we present an atomic characterization of the associated Hardy space $H^1_L(\mathbb{R})$.

1. Introduction

Consider a Schrödinger operator $L = \frac{d^2}{dx^2} - V(x)$ on $\mathbb{R}$ with locally integrable, nonnegative potential $V \in L^1_{loc}(\mathbb{R})$, $V \geq 0$, $V \not\equiv 0$. It is well known that $L$ generates a semigroup of operators $\{K_t : t \geq 0\}$ in $L^p(\mathbb{R})$, $1 \leq p < \infty$, with nonnegative integral kernels $\{k_t : t \geq 0\}$, see, e.g., [2]. We define the Hardy space $H^1_L(\mathbb{R})$ associated with the operator $L$ as

$$H^1_L(\mathbb{R}) = \left\{ f \in L^1(\mathbb{R}) : \|f\|_{H^1_L} = \sup_{t \geq 0} \|K_t f(\cdot)\|_{L^1(\mathbb{R})} < \infty \right\}.$$

The aim of this note is to describe an atomic decomposition of $H^1_L(\mathbb{R})$. We start by introducing some definitions and notation.

Let $\{I_j : j \in \mathbb{N}\}$ be a cover of $\mathbb{R}$ by dyadic intervals (i.e., intervals of the form $[m2^k, (m+1)2^k]$, $k, m \in \mathbb{Z}$) with disjoint interiors. For any two closed dyadic intervals $I, J \subset \mathbb{R}$, we say that they are neighbors of each other if their intersection $I \cap J$ consists of a single point. Assume that the dyadic cover $\mathcal{I} = \{I_j : j \in \mathbb{N}\}$ has the following additional property: for each $I_j$ there exist its neighbors $I_j', I_j''$, and $I_j'^*, I_j''^*$ have lengths comparable to the length of $I_j$ with uniform comparison constants $0 < C_1, C_2 < \infty$.

Throughout this paper we shall denote the length of an interval $I$ by $|I|$. Moreover, given an interval $I \subset \mathbb{R}$, we denote by $I^* \subset \mathbb{R}$ the interval with the same center as $I$, but with length equal to $1 + \alpha$ times the length of $I$. The parameter $\alpha > 0$ will depend on constants in Lemma 2.1 and, thus, shall be chosen later. At this point we shall only mention that $\alpha$ needs to be sufficiently small, in particular, so that $\{I_j^* : j \in \mathbb{N}\}$ forms a locally finite cover of $\mathbb{R}$.

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We say that $a$ is an $H^1_L(\mathbb{R})$-atom if either $a$ is a classical atom supported in $I^*_j$ for some $j \in \mathbb{N}$ (that is, $\int a = 0$ and $|a| \leq |I^*_j|^{-1}$, see, e.g., [3, Definition 1.2.3]) or if $a = \|I_j\| I_j$ for some $j \in \mathbb{N}$. Next we define the Hardy space $H^1_L(\mathbb{R})$ associated with $I$ to be

\begin{equation}
H^1_L(\mathbb{R}) = \left\{ f \in L^1(\mathbb{R}) : f(x) = \sum_{j \in \mathbb{N}} \lambda_j a_j(x), \sum_{j \in \mathbb{N}} |\lambda_j| < \infty \right\}
\end{equation}

for any family $\{a_j : j \in \mathbb{N}\}$ of $H^1_L(\mathbb{R})$-atoms. We let

$$\|f\|_{H^1_L(\mathbb{R})} = \inf\left\{ \sum_{j \in \mathbb{N}} |\lambda_j| < \infty \right\},$$

where the infimum is taken over all representations of $f$ in (1.1).

Our goal is to prove the following complete atomic characterization of Hardy spaces $H^1_L(\mathbb{R})$.

**Theorem 1.1.** Let $L = \frac{d^2}{dx^2} - V(x)$ on $\mathbb{R}$, $V \in L^1_{\text{loc}}(\mathbb{R})$, $V \geq 0$, $V \not\equiv 0$. Then, there exists a family of dyadic intervals $I = \{I_j : j \in \mathbb{N}\}$ such that

$$H^1_L(\mathbb{R}) = H^1_T(\mathbb{R}).$$

2. **Auxiliary lemmas and Proof of Theorem 1.1**

The proof of Theorem 1.1 is not self-contained and it uses a slightly modified argument from [1]. For the statements and proofs of auxiliary lemmas in this section we need the following definition: given an interval $I$, we denote by $I^#$ an interval with the same center as $I$ but with twice the length. We begin with the following lemma.

**Lemma 2.1.** Let $I = \{I_j : j \in \mathbb{N}\}$ be a family of maximal dyadic intervals defined by the stopping time condition

$$|I| \int_{I^#} V \leq 1.$$

Then, $I^{**} = \{I_j^{**} : j \in \mathbb{N}\}$ forms a locally finite cover of $\mathbb{R}$ and, for each $I_j \in I$, its neighbors are well defined and have lengths comparable to $|I_j|$.

Lemma 2.1 seems to be known. Since we do not know the exact reference, we provide here a complete proof.

**Proof.** Denote by $I^d$ the smallest dyadic interval containing $I$ and bigger than $I$. For $x \in \mathbb{R}$ denote by $I_x$ the maximal dyadic interval $I$ such that

$$x \in I \text{ and } |I| \int_{I^#} V \leq 1.$$

Since $\lim_{I \to 0} |I| \int_{I^#} V = 0$ and $\lim_{I \to \infty} |I| \int_{I^#} V = \infty$, the interval $I_x$ is well defined. Moreover, $\mathbb{R} = \bigcup_{x \in \mathbb{R}} I_x$. Using standard properties of dyadic intervals we can choose a countable subfamily $I$ of $\{I_x : x \in \mathbb{R}\}$ such that $\mathbb{R} = \bigcup_{j=1}^{\infty} I_j$ and the intervals $\{I_j : j \in \mathbb{N}\}$ have disjoint interiors.

Fix $I \in I$. Assume that for each $\varepsilon > 0$ there exists some dyadic interval $J \in I$ such that the following conditions hold:

$$\text{dist} \{I, J\} \leq |I| \text{ and } |J| \leq \varepsilon |I|.$$
Choosing $\varepsilon$ sufficiently small, one can verify that $(J^d)### \subset I#####$. By the definition of $J$ we obtain

$$1 \geq |I| \int_{I#####} V \geq |I| \int_{(J^d)#####} V = \frac{|I|}{|J^d|} |J^d| \int_{(J^d)#####} V \geq \frac{|I|}{|J^d|} \geq \frac{1}{2\varepsilon}.$$ 

Since $\varepsilon$ is arbitrary, the above estimate yields a contradiction. Thus, there exist universal constants $0 < C_1, C_2 < \infty$, such that for each $I \in \mathcal{I}$ its neighbors are well defined and their lengths are comparable to $|I|$, with universal comparison constants $C_1$ and $C_2$.

We close the proof of Lemma 2.1 by choosing $\alpha$ sufficiently small so that $\mathcal{I}^{*} = \{I_j^{*} : j \in \mathbb{N}\}$ is a locally finite cover of $\mathbb{R}$. □

In Lemmas 2.2 and 2.3 below we indicate the necessary modifications of [1] required to prove Theorem 1.1. Let $\mathcal{I}$ be as described in Lemma 2.1. We start by recalling condition $(D)$ from [1]:

$$\exists C, \varepsilon > 0, \text{ such that } \forall y \in I^{*}, \forall j \geq 0, \quad \int k_{2j|I|^{2}}(x, y)dx \leq Cj^{-1-\varepsilon}.$$ 

In order to verify that condition $(D)$ is satisfied, we shall use the well-known method of superharmonic weights.

**Lemma 2.2.** Let $\mathcal{I}$ be the cover of $\mathbb{R}$ defined in Lemma 2.1 and let $I \in \mathcal{I}$. There exists a constant $C > 0$ such that

$$\forall y \in I#####, \forall j \geq 0, \quad \int k_{2j|I|^{2}}(x, y)dx \leq C2^{-j}. $$ 

**Proof.** By the definition of $I \in \mathcal{I}$, we may choose an interval $J$ with the properties $I##### \subset J \subset (I^d)#####$

and

$$|J| \int_{J} V = 1.$$ 

Define the function

$$\phi_I(x) = \int_{J} V(y) \left( C|J| + \frac{1}{2}|x - y| \right) dy,$$

where $C > 0$ is some positive constant. With appropriate choice of the constant $C$, function $\phi_I$ satisfies $\phi_I(x) \geq 1 + \frac{|x|}{4|J|}$ and $\phi_I(y) \leq C'$ for some other constant $C' > 0$. We shall show first that $L(\phi_I)(x) \leq 0$. Indeed, we have

$$\frac{d^2}{dx^2}|x - y| = 2\delta_0(x - y),$$ 

where $\delta_x$ denotes the Dirac delta at $x \in \mathbb{R}$ (point mass). Consequently,

$$L(\phi_I)(x) = \frac{d^2}{dx^2} \phi_I(x) - V(x)\phi_I(x) = V(x)(1_{J}(x) - \phi_I(x)) \leq 0.$$ 

This, in turn, yields

$$\frac{d}{dt} \int K_t(\delta_y)(x)\phi_I(x)dx = \int K_t(\delta_y)(x)L(\phi_I)(x)dx \leq 0.$$ 

By the above,

$$\int K_t(\delta_y)(x)\phi_I(x)dx \leq \phi_I(y).$$

(2.1)
Without loss of generality, we may assume that $|I| = 1$. Then, (2.1) implies in particular that
\begin{equation}
\forall \ t \geq 0, \quad \int_{0}^{t} k_t(0, y) \, dy \leq C(1 + t)^{-\frac{1}{2}}. \tag{2.2}
\end{equation}

Indeed, in order to see (2.2), we shall consider the function
\[ \rho(t) = \int k_t(0, y) \, dy. \]
Observe that $\|k_t\|_{L^\infty} \leq C t^{-\frac{1}{2}}$ implies that we have
\[ k_{2t}(0, z) = \int k_t(0, y) k_t(y, z) \, dy \leq C t^{-\frac{1}{2}} \int k_t(0, y) \, dy = C \rho(t) t^{-\frac{1}{2}}. \]
Hence, using (2.1), we get for $R > 0$
\[ \rho(2t) \leq \int_{|x| \leq R} k_{2t}(0, y) \, dy + R^{-1} \int_{|x| \geq R} k_{2t}(0, y)(1 + |y|) \, dy \leq 2RC \rho(t) t^{-\frac{1}{2}} + CR^{-1}. \]
Taking a minimum with respect to $R$, we obtain
\[ \rho(2t) \leq C t^{-\frac{1}{2}} (\rho(t))^\frac{1}{2}. \]
Iterating the above inequality yields the desired estimate (2.2). Lemma 2.2 follows.

Let $\{\phi_j : j \in \mathbb{N}\}$ be a smooth resolution of identity associated with $\{I_j : j \in \mathbb{N}\}$; that is, let it be a family of smooth functions with the properties
\[ \text{supp } \phi_j \subset I_j^*, \quad 0 \leq \phi_j \leq 1, \]
\[ \forall x \in \mathbb{R}, \quad \left| \frac{d^n}{dx^n} \phi_j(x) \right| \leq C_n |I_j|^{-n}, \]
\[ \forall x \in \mathbb{R}, \quad \sum_{j \in \mathbb{N}} \phi_j(x) = 1. \]
Here, we choose the parameter $\alpha$ depending on the constants in Lemma 2.1. Moreover, we choose $\alpha$ to be sufficiently small, for the purpose of the next lemma, which is an analog of Lemma 3.11 of [1].

The kernel of the heat semigroup generated by $\Delta$ is denoted by $p_t$:
\[ p_t(x) = (4\pi t)^{-1/2} \exp(-x^2/4t), \]
and $P_t$ denotes the associated convolution operator. For the purpose of comparison, it may be noted here that $P_t$ arises from the special case of a Schrödinger operator with zero potential.

Lemma 2.3. Let $I = \{I_j : j \in \mathbb{N}\}$ be the cover of $\mathbb{R}$ defined in Lemma 2.1 and let $\{\phi_j : j \in \mathbb{N}\}$ be a smooth resolution of identity associated with $I$ and satisfying the above properties. The following estimate holds:
\[ \forall j \in \mathbb{N}, \quad \left\| \sup_{0 < t \leq |I_j|^2} (K_t - P_t)(\phi_j f) \right\|_{L^1(\mathbb{R})} \leq C \|\phi_j f\|_{L^1(\mathbb{R})}. \]
Proof. We write \( V(x) = V'(x) + V''(x) \), where \( V'(x) = 1_{I_{j}^*}(x)V(x) \) and \( V''(x) = V(x) - V'(x) \). Using the perturbation formula, i.e., \( P_t = K_t + \int_{0}^{t} P_{t-s}VK_s ds \) (see, e.g., [1]) we obtain

\[
(K_t - P_t)(\phi_j f)(x) = \int_{0}^{t} P_{t-s}V'K_s(\phi_j f)(x) ds + \int_{0}^{t} P_{t-s}V''K_s(\phi_j f)(x) ds.
\]

In order to estimate the first summand in (2.3), we start by noting that

\[
\int_{0}^{t} p_s(x) ds \leq t^{\frac{1}{2}} \exp(-x^2/4t).
\]

By the Feynman-Kac formula, which states that \( k_t(x,y) \leq p_t(x-y) \), we obtain that there exists a constant \( \gamma > 0 \), depending on the constants in Lemma 2.1, such that for \( 0 < t \leq |I_j|^2 \),

\[
\int_{0}^{t} P_{t-s}V'K_s(\delta_y)(x) ds \leq \left( \sup_{\frac{1}{2} \leq s \leq t} p_s \right) * \left( V' \int_{0}^{t} P_s(\delta_y) ds \right)(x) \leq C t^{1/2} p_t * V'(x)
\]

\[
\leq C |I_j| p_{|I_j|^2} * V'(x)
\]

\[
\leq C |I_j|^{-1} \exp(-\gamma x^2 / 4|I_j|^2)|I_j| \int_{\mathbb{R}} V'(x) dx
\]

\[
\leq C |I_j|^{-1} \exp(-\gamma x^2 / 4|I_j|^2).
\]

In the above calculation \( C \) denotes an arbitrary positive constant. Similar to the estimation in (2.4) we may show that

\[
\int_{\frac{1}{2}}^{t} P_{t-s}V'K_s(\delta_y)(x) ds \leq t^{-1/2} \int_{\frac{1}{2}}^{t} P_{t-s}(V')(x) ds \leq C |I_j|^{-1} \exp(-\gamma x^2 / 4|I_j|^2).
\]

Thus, we have shown that \( \int_{0}^{t} P_{t-s}V'K_s(\delta_y)(x) ds \leq C |I_j|^{-1} \exp(-\gamma x^2 / 4|I_j|^2) \), independently of \( y \) and \( t \leq |I_j|^2 \), which, in turn, implies that

\[
\left\| \sup_{0 < t \leq |I_j|^2} \int_{0}^{t} P_{t-s}V'K_s(\phi_j f)(x) ds \right\|_{L^1(\mathbb{R})} \leq C \| \phi_j f \|_{L^1(\mathbb{R})}.
\]

The second summand in (2.3) has been estimated in [1, Lemma 3.11]. For the sake of readability, we recall that argument here. We observe that for \( y \in (I_{j}^*)^c \), \( x \in I_{j}^* \), and for \( 0 < s \leq t \leq |I_j|^2 \), we have

\[
p_{t-s}(x-y) \leq C |I_j|^{-1} \exp(-c|x-y|^2 / |I_j|^2).
\]

The above inequality, together with [1, Lemma 3.11], is used to estimate that

\[
\left\| \sup_{0 < t \leq |I_j|^2} \int_{0}^{t} P_{t-s}V''K_s(\phi f)(x) ds \right\|_{L^1(I_{j}^*)} \leq C \left\| \frac{1}{|I_j|} \phi_j f \right\|_{L^1(\mathbb{R})}.
\]

This, in turn, implies that

\[
\left\| \sup_{0 < t \leq |I_j|^2} \int_{0}^{t} P_{t-s}V''K_s(\phi f)(x) ds \right\|_{L^1(I_{j}^*)} \leq C \left\| \phi_j f \right\|_{L^1(\mathbb{R})}.
\]

\[\Box\]
Remark. We would like to note here that the estimate of \( V'' \) in [1] is obtained using an additional assumption, the so-called condition \((K)\), which states that there exist constants \( C, \epsilon > 0 \) such that for all \( x \in \mathbb{R}^d \), for any dyadic cube \( Q \in \mathcal{Q} \), and for \( t \leq \text{diam}(Q)^2 \),

\[
(2.5) \quad \int_0^{2t} (1_Q \ast V) * p_s(x) ds \leq C \left( \frac{t}{\text{diam}(Q)} \right)^\epsilon.
\]

In the proof of Lemma 2.3 above, we see that (2.5) always holds for \( V \in L^1_{\text{loc}}(\mathbb{R}) \), \( \epsilon = 1/2 \). This may be seen, e.g., as a consequence of (2.4).

Let us now recall the main result of [1] (Theorem 2.2) in the form adapted to the presentation in this paper.

**Theorem 2.4.** Let \( \mathcal{I} = \{I_j : j \in \mathbb{N}\} \) be the cover of \( \mathbb{R} \) derived in Lemma 2.1 and let \( \{\phi_j : j \in \mathbb{N}\} \) be the associated resolution of identity. Assume that the following conditions are satisfied:

1. \( \forall y \in I_{\ast\ast\ast\ast}, \forall j \in \mathbb{N}, \int k_{2^j |I|^2}(x,y) dx \leq C 2^{-j/2}; \)
2. \( \forall j \in \mathbb{N}, \|\sup_{0 < t \leq |I_j|^2} (K_t - P_t)(\phi_j f)\|_{L^1(\mathbb{R})} \leq C \|\phi_j f\|_{L^1(\mathbb{R})}. \)

Then, \( H^1_1(\mathbb{R}) = H^1_{1/2}(\mathbb{R}) \).

**Proof.** Follows by direct inspection of the proof of the main theorem in [1]. \( \square \)

**Proof of Theorem 1.1.** Combining Theorem 2.4 with Lemmas 2.1–2.3, we obtain the thesis of Theorem 1.1. \( \square \)

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