

UNIVERSALITY OF RANK 6 PLÜCKER RELATIONS AND GRASSMANN CONE PRESERVING MAPS

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ABSTRACT. The Plücker relations define a projective embedding of the Grassmann variety $Gr(p, n)$. We give another finite set of quadratic equations which defines the same embedding, and whose elements all have rank 6. This is achieved by constructing a certain finite set of linear maps $\bigwedge^p k^n \rightarrow \bigwedge^2 k^4$, and pulling back the unique Plücker relation on $\bigwedge^2 k^4$. We also give a quadratic equation depending on $(p + 2)$ parameters having the same properties.

1. INTRODUCTION

1.1. Plücker relations. Throughout, let k be a field, and let e_1, \dots, e_n be a basis of the vector space k^n . Define the coordinates $\{\prod_{i_1, \dots, i_p}\}_{1 \leq i_1 < \dots < i_p \leq n}$ on $\bigwedge^p k^n$ by

$$(1) \quad \bigwedge^p k^n \ni \omega = \sum_{1 \leq i_1 < i_2 < \dots < i_p \leq n} \prod_{i_1, i_2, \dots, i_p} e_{i_1} \wedge e_{i_2} \wedge \dots \wedge e_{i_p},$$

and extend them to arbitrary indices in $\{1, \dots, n\}^p$ by making them antisymmetric.

An element $\omega \in \bigwedge^p k^n$ is called *decomposable* if it can be written in the form $\omega = v_1 \wedge v_2 \wedge \dots \wedge v_p$ for some $v_i \in k^n$; otherwise it is called *indecomposable*. The *Grassmann cone* $\Gamma^p k^n = \{\omega \in \bigwedge^p k^n \mid \omega = v_1 \wedge v_2 \wedge \dots \wedge v_p \text{ for some } v_i \in k^n\}$ is the set of decomposable elements in $\bigwedge^p k^n$. The *Plücker relations* [3, 7, 8, 9]

$$(2) \quad P_{A,B}(\omega) := \sum_{i=1}^{p+1} (-1)^{i-1} \prod_{a_1 a_2 \dots a_{p-1} b_i} \prod_{b_1 b_2 \dots b_{p+1} \setminus b_i} = 0,$$

where $A = \{a_1, \dots, a_{p-1}\}$, $B = \{b_1, \dots, b_{p+1}\} \subset \{1, \dots, n\}$, and where the $\setminus b_i$ at the end of the indices indicates the absence of b_i from the indices, hold if and only if $\omega \in \Gamma^p k^n$, making $\Gamma^p k^n$ a k^\times -invariant affine variety.¹ The quotient $Gr(p, n) := (\Gamma^p k^n \setminus \{0\})/k^\times \subset \mathbb{P}(\bigwedge^p k^n)$ is the Grassmann variety.

We do not need to consider all the choices of indices A and B : Since rearranging the elements of A or B only affects $P_{A,B}$ by total change in sign, it suffices to consider A and B whose elements are listed in increasing order. Moreover, if $A \subset B$,

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¹To be precise, while (2) cut out $\Gamma^p k^n$ set-theoretically for any k [9, Theorem 1], more general Plücker relations may be needed to define it scheme-theoretically if $\text{char}(k) > 0$ [1]. In this paper we shall concentrate on the set-theoretic aspect, so the simpler Plücker relations as in (2) suffice.

then $P_{A,B} = 0$, and if $A \setminus (A \cap B)$ is a one point set $\{a\}$, then exchanging a with any element of $B \setminus (A \cap B)$ only affects $P_{A,B}$ by total change in sign. So we take

$$\mathcal{P}(p, n) = \left\{ P_{A,B} \left| \begin{array}{l} A, B \subset \{1, \dots, n\}, A = \{a_1, \dots, a_{p-1}\}, B = \{b_1, \dots, b_{p+1}\} \\ \text{with } a_1 < \dots < a_{p-1} \text{ and } b_1 < \dots < b_{p+1}, A \not\subset B, \text{ and} \\ \text{if } A \setminus (A \cap B) = \{a\} \text{ then } a < b \text{ for any } b \in B \setminus (A \cap B) \end{array} \right. \right\}$$

as a set of generators of Plücker relations, and it suffices to define the Grassmann cone: $\Gamma^p k^n = \{\omega \in \wedge^p k^n \mid P(\omega) = 0 \text{ for all } P \in \mathcal{P}(p, n)\}$.

By definition we have $\mathcal{P}(p, n) = \emptyset$ if $\min\{p, n - p\} \leq 1$. The first nontrivial case $(p, n) = (2, 4)$ yields $\mathcal{P}(2, 4) = \{P_{\{1\}\{2,3,4\}}\}$, so $\omega = \sum_{1 \leq i < j \leq 4} \Pi_{ij} e_i \wedge e_j \in \wedge^2 k^4$ is decomposable if and only if

$$(3) \quad P_{\{1\}\{2,3,4\}}(\omega) = \Pi_{12}\Pi_{34} - \Pi_{13}\Pi_{24} + \Pi_{14}\Pi_{23} = 0.$$

The *rank* of a quadratic form is the rank of the symmetric matrix which defines it (unless $\text{char}(k) = 2$, in which case a different definition of rank is used [4]). The rank of $P_{A,B}$ as a quadratic form on $\wedge^p k^n$ is twice the number $|B \setminus (A \cap B)|$ of nonvanishing terms in (2). So the set $\mathcal{P}(p, n)$ consists of quadratic forms of every even rank from 6 up to $2 \min\{p, n - p\} + 2$, and the Plücker relations in $\mathcal{P}(p, n)$ all have rank 6 only when $\min\{p, n - p\} = 2$. The literature on algebraic geometry occasionally demonstrates an interest in the rank of the Plücker relations [13, 14], with particular attention paid to the simplest, rank 6 case.

1.2. Grassmann cone preserving maps. A linear map $G: \wedge^p k^n \rightarrow \wedge^{p'} k^{n'}$ is said to be a Grassmann cone preserving map (GCP map for short) if

$$G(\Gamma^p k^n) \subset \Gamma^{p'} k^{n'}.$$

The *induced map* $\wedge^p L: \wedge^p k^n \rightarrow \wedge^p k^{n'}$ of a linear map $L: k^n \rightarrow k^{n'}$ is a GCP map. Another GCP map is the *dual isomorphism* $\delta: \wedge^p k^n \rightarrow \wedge^{n-p} k^n$, defined by

$$e_{i_1} \wedge \dots \wedge e_{i_p} \mapsto e_{j_1} \wedge \dots \wedge e_{j_{n-p}},$$

where $i_1 < \dots < i_p, j_1 < \dots < j_{n-p}$ and $\{i_1, \dots, i_p\} \cup \{j_1, \dots, j_{n-p}\} = \{1, \dots, n\}$.

Every nontrivial GCP map (those which do *not* send the whole of $\wedge^p k^n$ to decomposables) can be written as a composition of maps of these two types [18].

1.3. Motivation. In the theory of classical integrable systems, it is well-known that the KP hierarchy of soliton equations, written in an appropriate (Hirota) form, is nothing but the Plücker relations for an infinite dimensional Grassmannian in the space of functions [2, 15]. Somewhat remarkably, in that setting a single 3-term (i.e., rank 6) quadratic functional equation with parameters suffices to encode the entire hierarchy [6, 12, 17], and the same 3-term equation, even without the parameters, can characterize Jacobians of curves among all the principally polarized abelian varieties [10]. However, the literature on algebraic geometry does not appear to have a similar result on the “universality” of the 3-term Plücker relation (3), i.e., that the 3-term relation is in a sense the only Plücker relation one needs.

In this paper, we will show such universality by pulling back (3) by various GCP maps to obtain a finite set of rank 6 equations on $\wedge^p k^n$ which suffices to cut out the cone of decomposables $\Gamma^p k^n \subset \wedge^p k^n$ set-theoretically. Inspired by the soliton theory observation, we will also construct a parameter-dependent rank 6 equation which determines the decomposability of ω .

2. A SET OF POLYNOMIALS WITH QUADRIC RANK 6

We will define another set of quadratic forms $\mathcal{P}'(p, n)$ to be used later to cut out the Grassmann cone $\Gamma^p k^n$ just as $\mathcal{P}(p, n)$ does. First we introduce a convenient notation: for a 2-vector $\vec{i}_j = (i_j^{(0)}, i_j^{(1)}) \in \{1, \dots, n\}^2$, let

$$\prod_{i_1 i_2 \dots i_{j-1} \vec{i}_j i_{j+1} \dots i_p} = \prod_{i_1 i_2 \dots i_{j-1} i_j^{(0)} i_{j+1} \dots i_p} + \prod_{i_1 i_2 \dots i_{j-1} i_j^{(1)} i_{j+1} \dots i_p},$$

and extend it inductively to the case where two or more indices are 2-vectors.

Definition 2.1. For $\mathcal{A} = \{\alpha_1, \alpha_2, \alpha_3, \alpha_4\} \subset \{1, \dots, n\}$, $\mathcal{B} = \{\vec{\beta}_1, \vec{\beta}_2, \dots, \vec{\beta}_m\}$ with $\vec{\beta}_i = (\beta_i^{(0)}, \beta_i^{(1)}) \in \{1, \dots, n\}^2$, and $\mathcal{C} = \{\gamma_1, \dots, \gamma_{p-m-2}\} \subset \{1, \dots, n\}$, let

$$(4) \quad P'_{\mathcal{A}, \mathcal{B}, \mathcal{C}} = \pi_{12}\pi_{34} - \pi_{13}\pi_{24} + \pi_{14}\pi_{23}, \quad \text{where} \quad \pi_{ij} = \prod_{\alpha_i \alpha_j \vec{\beta}_1 \dots \vec{\beta}_m \gamma_1 \dots \gamma_{p-2-m}}.$$

Note that $P'_{\mathcal{A}, \mathcal{B}, \mathcal{C}}$ is nothing but (3) with the six variables Π_{ij} replaced by π_{ij} . Thus, it has rank at most six as a quadratic form of the original coordinates $\Pi_{i_1 \dots i_p}$. Moreover, if $\alpha_i, \beta_j^{(\mu)}$ and γ_l are all distinct it has rank exactly six.

Lemma 2.2. Each $P'_{\mathcal{A}, \mathcal{B}, \mathcal{C}}$ in (4) can be written as a \mathbb{Z} -linear combination of elements of $\mathcal{P}(p, n)$:

$$P'_{\mathcal{A}, \mathcal{B}, \mathcal{C}} = \sum_{\vec{\mu}, \vec{\nu} \in \{0, 1\}^m} P_{\{\beta_1^{(\mu_1)} \dots \beta_m^{(\mu_m)} \gamma_1 \dots \gamma_{p-m-2} \alpha_1\} \{\alpha_2 \alpha_3 \alpha_4 \beta_1^{(\nu_1)} \dots \beta_m^{(\nu_m)} \gamma_1 \dots \gamma_{p-m-2}\}}.$$

Proof. For each fixed choice of $\vec{\mu}$ and $\vec{\nu}$ the first three terms in expansion (2) of P are exactly those that appear in (4). The next m terms, i.e., the terms involving $\beta_i^{(\nu_i)}$, are zero if $\mu_i = \nu_i$, and cancel if $\mu_i \neq \nu_i$ with the term where μ_i and ν_i are switched. The last $p - m - 2$ terms, the terms involving γ_i , are all zero. \square

The lemma implies that if some $P'_{\mathcal{A}, \mathcal{B}, \mathcal{C}}(\omega) \neq 0$, then ω is indecomposable. Conversely, in Section 4 we will see that some $P'_{\mathcal{A}, \mathcal{B}, \mathcal{C}}(\omega) \neq 0$ if ω is indecomposable. However, *not* every element of $\mathcal{P}(p, n)$ is a linear combination of rank 6 quadratic forms $P'_{\mathcal{A}, \mathcal{B}, \mathcal{C}}$. For instance, a tedious but straightforward computation shows that the Plücker relation $P_{\{1, 2, 3\}, \{4, 5, 6, 7, 8\}}$ cannot be a linear combination of $P'_{\mathcal{A}, \mathcal{B}, \mathcal{C}}$.

The quadratic forms $P'_{\mathcal{A}, \mathcal{B}, \mathcal{C}}$ are not independent. Clearly, rearranging the elements of each index set affects them by at most a change in sign. If the indices $\alpha_i, \beta_j^{(\varepsilon)}$ and γ_l are not all distinct, then $P'_{\mathcal{A}, \mathcal{B}, \mathcal{C}}$ is a linear combination of those with distinct indices. There are other, less obvious linear relations like

$$\sum_{i \in \mathbb{Z}/3\mathbb{Z}} P'_{\{\nu_i, \alpha_1, \alpha_2, \alpha_3\}, \mathcal{B} \cup \{(\nu_{i+1}, \nu_{i+2})\}, \mathcal{C}} - \sum_{0 \leq i, j \leq 2; i \neq j} P'_{\{\nu_i, \alpha_1, \alpha_2, \alpha_3\}, \mathcal{B}, \mathcal{C} \cup \{\nu_j\}} = 0.$$

However, instead of using these identities to find a basis for the linear span of the set of all $P'_{\mathcal{A}, \mathcal{B}, \mathcal{C}}$, let us introduce a subset just suited for our purpose of set-theoretic characterization of the Grassmann cone.

Definition 2.3. Let $\mathcal{P}'(p, n)$ be the set of all $P'_{\mathcal{A}, \mathcal{B}, \mathcal{C}}$ where the triples $\mathcal{A}, \mathcal{B}, \mathcal{C}$ satisfy

$$(5) \quad \left\{ \begin{array}{l} \text{the indices } \alpha_i, \beta_j^{(\mu)} \text{ and } \gamma_l \text{ are all distinct and such that} \\ \alpha_i < \alpha_{i+1}, \quad \beta_i^{(0)} < \beta_i^{(1)}, \quad \beta_i^{(\nu)} < \beta_{i+1}^{(\nu)}, \quad \beta_i^{(0)} < \alpha_1, \quad \beta_i^{(1)} < \alpha_3, \quad \gamma_i < \gamma_{i+1}. \end{array} \right.$$

It is a simple exercise in combinatorics to find the number of elements of $\mathcal{P}'(p, n)$:

$$|\mathcal{P}'(p, n)| = \sum_{m=0}^M \frac{n!}{(2m+4)!(p-m-2)!(n-p-m-2)!} (C_{m+2} - C_{m+1}),$$

where $M = \min\{p, n - p\} - 2$, and $C_r = \frac{1}{r+1} \binom{2r}{r}$ is the Catalan number [16]. Rewriting this, we see that in general $\mathcal{P}'(p, n)$ is a much smaller set than $\mathcal{P}(p, n)$:

$$|\mathcal{P}'(p, n)| = \sum_{m=0}^M \frac{3m+3}{(2m+4)(2m+3)} a_m \leq \frac{1}{4} a_0 + \sum_{m=1}^M a_m = |\mathcal{P}(p, n)|,$$

where the equality holds if and only if $M \leq 0$, and where

$$a_m := \frac{n!}{(m+1)!(m+3)!(p-m-2)!(n-p-m-2)!}$$

is the number of pairs of subsets $A, B \subset \{1, \dots, n\}$ such that $|A| = p-1$, $|B| = p+1$ and $|A \cap B| = p-m-2$, which is also the number of the rank $2m+6$ elements in $\mathcal{P}(p, n)$ if $m > 0$, and is four times this number if $m = 0$.

3. DECOMPOSABILITY AND GCP MAPS TO $\Gamma^2 k^4$

In this section we will construct a finite set of GCP maps from $\bigwedge^p k^n$ to $\bigwedge^2 k^4$, indexed by the same triples $\mathcal{A}, \mathcal{B}, \mathcal{C}$ as in Definition 2.3, such that if ω is indecomposable, then for *some* GCP map G in this set $G(\omega)$ is also indecomposable. First we define a linear map from k^n to k^{p+2} determined by these indices and prove a lemma addressing a question of vector geometry.

For any $S \subset \{1, \dots, n\}$ let $k^S = \bigoplus_{i \in S} k e_i \subset k^n$. For any $S, T \subset \{1, \dots, n\}$ and a map $f: S \rightarrow T$, define $f^*: k^T \rightarrow k^S$ and $f_*: k^S \rightarrow k^T$ by $\sum_{j \in T} a_j e_j \mapsto \sum_{i \in S} a_{f(i)} e_i$ and $\sum_{i \in S} a_i e_i \mapsto \sum_{i \in S} a_i e_{f(i)}$, respectively. Thus if i_S is the natural inclusion $S \subset \{1, \dots, n\}$, then $\pi_S := i_S^*: k^n \rightarrow k^S$ is the projection $\sum_{i=1}^n a_i e_i \mapsto \sum_{i \in S} a_i e_i$. Writing $S = \{\xi_1, \dots, \xi_{|S|}\}$ with $\xi_1 < \dots < \xi_{|S|}$, we also define $\tau_S: S \rightarrow \{1, \dots, |S|\}$ by $\tau_S(\xi_i) = i$, so that $\tau_{S*}: k^S \xrightarrow{\sim} k^{|S|}$, $\tau_{S*}(e_{\xi_i}) = e_i$, is an isomorphism.

Definition 3.1. Suppose \mathcal{A}, \mathcal{B} and \mathcal{C} are as in Definition 2.1, subject to condition (5). Let $B_\varepsilon = \{\beta_1^{(\varepsilon)}, \dots, \beta_m^{(\varepsilon)}\}$ ($\varepsilon = 0, 1$), and let $S' = \mathcal{A} \cup B_1 \cup \mathcal{C}$ and $S = S' \cup B_0$. Let $\varphi: B_0 \xrightarrow{\sim} B_1$ be the order-preserving map $\varphi(\beta_i^{(0)}) = \beta_i^{(1)}$, $i = 1, \dots, m$, and extend it to a map from S to S' , still denoted by φ , by $\varphi(i) = i$ outside B_0 . Noting that $|S'| = p+2$, we let $X := X_{\mathcal{A}, \mathcal{B}, \mathcal{C}}: k^n \rightarrow k^{p+2}$ be the composition of linear maps $\tau_{S'*} \circ \varphi_* \circ \pi_S: k^n \rightarrow k^S \rightarrow k^{S'} \rightarrow k^{p+2}$, i.e., writing $S' = \{\xi_1, \dots, \xi_{|S'|}\}$ with $\xi_1 < \dots < \xi_{|S'|}$ we have

$$X(e_i) = \begin{cases} e_j & \text{if } i = \xi_j, \text{ or if } \exists l [i = \beta_l^{(0)} \text{ and } \beta_l^{(1)} = \xi_j], \\ 0 & \text{otherwise.} \end{cases}$$

Lemma 3.2. Let V_0 and V_1 be p -dimensional subspaces of k^n . If $q := q(V_0, V_1) := p - \dim(V_0 \cap V_1) = \dim(V_0 + V_1) - p$ satisfies $q \geq 2$, there exist \mathcal{A}, \mathcal{B} and \mathcal{C} subject to (5) so that

- $\hat{V}_0 = X_{\mathcal{A}, \mathcal{B}, \mathcal{C}}(V_0)$ and $\hat{V}_1 = X_{\mathcal{A}, \mathcal{B}, \mathcal{C}}(V_1)$ are p -dimensional subspaces of k^{p+2} ,
- $\dim(\hat{V}_0 \cap \hat{V}_1) = p - 2$,
- $\{X_{\mathcal{A}, \mathcal{B}, \mathcal{C}}(e_{\alpha_i}) \mid i = 1, 2, 3, 4\}$ is linearly independent modulo $(\hat{V}_0 \cap \hat{V}_1)$.

Proof. Take a minimal $S \subset \{1, \dots, n\}$ such that $\pi_S|_{V_\varepsilon}: V_\varepsilon \rightarrow k^S$ ($\varepsilon = 0, 1$) are injective, and such that $q_S := q(\pi_S(V_0), \pi_S(V_1)) \geq 2$. Let $V'_\varepsilon := \pi_S(V_\varepsilon)$. We have

$$q_S \leq q$$

since $V'_0 \cap V'_1 \supset \pi_S(V_0 \cap V_1) \simeq V_0 \cap V_1$ implies $p - q_S \geq p - q$, and

$$(6) \quad p + q_S = |S|$$

since the minimality of S implies $V'_0 + V'_1 = \pi_S(V_0 + V_1) = k^S$.

If $q_S = 2$ so that $\dim k^S/(V'_0 \cap V'_1) = 4$, choose $\mathcal{A} \subset S$ such that $|\mathcal{A}| = 4$ and the composition of the maps $k^{\mathcal{A}} \hookrightarrow k^S \rightarrow k^S/(V'_0 \cap V'_1)$ is an isomorphism, and let $\mathcal{B} = \emptyset$ and $\mathcal{C} = S \setminus \mathcal{A}$. It is easy to see that $X_{\mathcal{A}, \mathcal{B}, \mathcal{C}}$ satisfies the desired properties.

If $q_S > 2$, then for any $i \in S$ the minimality of S implies that $\pi_{S \setminus \{i\}}|_{V_0}$ or $\pi_{S \setminus \{i\}}|_{V_1}$ cannot be injective (otherwise we would have $q_{S \setminus \{i\}} = q_S - 1 \geq 2$), thus

$$(7) \quad e_i \in V'_0 \quad \text{or} \quad e_i \in V'_1 \quad (\forall i \in S).$$

Take any $B_0, B_1 \subset S$ such that

$$(8) \quad k^{B_\varepsilon} \subset V'_\varepsilon \quad \text{and} \quad k^{B_\varepsilon} \cap V'_{1-\varepsilon} = \{0\} \quad (\varepsilon = 0, 1).$$

Note that (8) implies $B_0 \cap B_1 = \emptyset$ and $|B_\varepsilon| \leq q_S$. Let $\delta = 0$ if $|B_0| \leq |B_1|$; otherwise let $\delta = 1$. Let $S' = S \setminus B_\delta$. Take any inclusion $\varphi: B_\delta \hookrightarrow B_{1-\delta} \subset S'$, and extend it to a map from S to S' by letting $\varphi(i) = i$ for $i \in S'$. As a consequence of (8) we have $\ker \varphi_* \cap V'_\varepsilon = \{0\}$, so that $V''_\varepsilon := \varphi_*(V'_\varepsilon)$ are p -dimensional subspaces of $k^{S'}$. Since $V''_0 + V''_1 = \varphi_*(V'_0 + V'_1) = \varphi_*(k^S) = k^{S'}$ has dimension $|S| - |B_\delta| = p + q_S - |B_\delta|$ (see (6)), we have

$$(9) \quad q(V''_0, V''_1) = q_S - |B_\delta| = q_S - \min\{|B_0|, |B_1|\}.$$

If we further assume that (B_0, B_1) is a maximal pair satisfying (8), then $|B_0| = |B_1| = q_S$. To prove this, it suffices to show that B_δ is not maximal if $|B_\delta| < q_S$. By (7) we have

$$e_i \in V''_0 \quad \text{or} \quad e_i \in V''_1 \quad (\forall i \in S'),$$

but since $|\{i \in S' \mid e_i \in V''_\varepsilon\}| \leq \dim(V''_\varepsilon) = p$ ($\varepsilon = 0, 1$) and $|S'| = p + q_S - |B_\delta|$, if $|B_\delta| < q_S$, then there exists $i \in S'$ such that

$$(10) \quad e_i \notin V'_{1-\delta}.$$

Let $B'_\delta = B_\delta \cup \{i\}$. Since $i \in S' = S \setminus B_\delta$ we have $i \notin B_\delta$, so that $B'_\delta \supsetneq B_\delta$. By (10) we have $e_i \notin V'_{1-\delta}$; hence by (7) we have $e_i \in V'_\delta$. This and the first formula in (8) (with $\varepsilon = \delta$) yield

$$(11) \quad k^{B'_\delta} = k^{B_\delta} \oplus ke_i \subset V'_\delta.$$

Using (10), the first formula in (8) with $\varepsilon = 1 - \delta$ and the second formula in (8) with $\varepsilon = \delta$, we have

$$e_i \notin \varphi_*^{-1}(V'_{1-\delta}) = V'_{1-\delta} \oplus \bigoplus_{i \in B_\delta} k(e_i - e_{\varphi(i)}) = V'_{1-\delta} \oplus k^{B_\delta},$$

hence $k^{B'_\delta} \cap V'_{1-\delta} = \{0\}$, which together with (11) shows that B_δ is not maximal.

Take maximal B_ε , denote them by \overline{B}_ε , and write $\overline{B}_\varepsilon = \{b_1^\varepsilon, \dots, b_{q_S}^\varepsilon\}$ with $b_1^\varepsilon < \dots < b_{q_S}^\varepsilon$. We then let $B_\varepsilon = \{b_1^\varepsilon, \dots, b_{q_S-2}^\varepsilon\} \subset \overline{B}_\varepsilon$,

$$\mathcal{A} = \{b_{q_S-1}^0, b_{q_S}^0, b_{q_S-1}^1, b_{q_S}^1\} = (\overline{B}_0 \setminus B_0) \cup (\overline{B}_1 \setminus B_1),$$

$$\mathcal{B} = \{(b_i^0, b_i^1) \mid i = 1, \dots, q_S - 2\} \quad \text{and} \quad \mathcal{C} = S \setminus (\overline{B}_0 \cup \overline{B}_1) = S \setminus (\mathcal{A} \cup B_0 \cup B_1).$$

As subsets of \overline{B}_ε , B_ε satisfy (8). Taking order-preserving $\varphi: B_0 \xrightarrow{\sim} B_1$ in the above argument (note $\delta = 0$ since $|B_0| = |B_1|$) we obtain $\varphi: S \rightarrow S'$ and $X := X_{\mathcal{A},\mathcal{B},\mathcal{C}} = \tau_{S'^*} \circ \varphi_* \circ \pi_S$ as in Definition 3.1. The map X is regular on each V_ε (so that $\hat{V}_\varepsilon = \tau_{S'^*} V_\varepsilon''$ has dimension p), and $X(e_{\alpha_i}) = \tau_{S'^*}(e_{\alpha_i})$ remain linearly independent modulo $\hat{V}_0 \cap \hat{V}_1 = \tau_{S'^*}(V_0'' \cap V_1'')$. Finally, applying (9) to B_ε we have $q(V_0'', V_1'') = q_S - |B_\delta| = 2$, and $\dim(\hat{V}_0 \cap \hat{V}_1) = \dim(V_0'' \cap V_1'') = p - 2$ as desired. \square

Now, let us define a GCP map associated to the same indices:

Definition 3.3. The Grassmann cone preserving map $G := G_{\mathcal{A},\mathcal{B},\mathcal{C}}: \bigwedge^p k^n \rightarrow \bigwedge^2 k^4$ is defined as

$$G_{\mathcal{A},\mathcal{B},\mathcal{C}} = \bigwedge^2 Z \circ \delta \circ \bigwedge^p X_{\mathcal{A},\mathcal{B},\mathcal{C}},$$

where $X_{\mathcal{A},\mathcal{B},\mathcal{C}}$ is as in Definition 3.1, $\delta: \bigwedge^p k^{p+2} \rightarrow \bigwedge^2 k^{p+2}$ is the dual isomorphism, and $Z: k^{p+2} \rightarrow k^4$ is the linear map

$$Z(e_i) = \begin{cases} e_j & \text{if } e_i = X_{\mathcal{A},\mathcal{B},\mathcal{C}}(e_{\alpha_j}), \\ 0 & \text{otherwise.} \end{cases}$$

Theorem 3.4. *The element $\omega \in \bigwedge^p k^n$ is indecomposable if and only if there exists a choice of indices $\mathcal{A}, \mathcal{B}, \mathcal{C}$ so that $G_{\mathcal{A},\mathcal{B},\mathcal{C}}(\omega) \in \bigwedge^2 k^4$ is indecomposable.*

Proof. Since $G_{\mathcal{A},\mathcal{B},\mathcal{C}}$ is a GCP map, if ω is decomposable $G_{\mathcal{A},\mathcal{B},\mathcal{C}}(\omega)$ is also decomposable for any choice of \mathcal{A}, \mathcal{B} , and \mathcal{C} . So, let us assume that ω is indecomposable and show that for an appropriate choice of the indices, its image in $\bigwedge^2 k^4$ is also indecomposable.

Suppose first that ω can be written as a sum of *two* decomposable elements

$$(12) \quad \omega = v_1 \wedge \cdots \wedge v_p + w_1 \wedge \cdots \wedge w_p.$$

Then ω is indecomposable if and only if [18] the p -dimensional subspaces $V_0 = \langle v_1, \dots, v_p \rangle$ and $V_1 = \langle w_1, \dots, w_p \rangle$ of k^n satisfy $\dim(V_0 \cap V_1) \leq p - 2$. Applying Lemma 3.2 gives us a choice of $\mathcal{A}, \mathcal{B}, \mathcal{C}$ such that

$$(\bigwedge^p X_{\mathcal{A},\mathcal{B},\mathcal{C}})(\omega) = \omega_1 \wedge \omega_2 \in \bigwedge^p k^{p+2},$$

where ω_1 is an indecomposable element of the second exterior power of the 4-dimensional space $k^{A'} := X_{\mathcal{A},\mathcal{B},\mathcal{C}}(k^{\mathcal{A}})$, and ω_2 is a nonzero element of $\bigwedge^{p-2}(\hat{V}_0 \cap \hat{V}_1)$. Since $k^{A'} \cap (\hat{V}_0 \cap \hat{V}_1) = \{0\}$, letting $A'' := \{1, \dots, p+2\} \setminus A'$ we have $\omega_{2,0} := (\bigwedge^{p-2} \pi_{A''})\omega_2 \neq 0$, which implies $\bigwedge^2 \pi_{A'} \circ \delta(\pi_1 \wedge \pi_2) = \bigwedge^2 \pi_{A'} \circ \delta(\pi_1 \wedge \pi_{2,0}) = c\delta_{A'}(\pi_1)$, where $c \neq 0$ and $\delta_{A'}$ is the dual isomorphism on $\bigwedge^* k^{A'}$. Since $Z = \tau_{A'^*} \circ \pi_{A'}$, this implies $G_{\mathcal{A},\mathcal{B},\mathcal{C}}(\omega) = \bigwedge^2 Z \circ \delta(\omega_1 \wedge \omega_2) = (\bigwedge^2 \tau_{A'^*})(c\delta_{A'}(\omega_1)) \neq 0$.

This proves the claim when ω can be written as a sum of two decomposable elements. This is always true in $\bigwedge^2 k^4$, but not in general. We proceed by induction on the dimension n with the case $n = 4$ as our initialization.

Regarding k^{n-1} as the subspace $\langle e_1, \dots, e_{n-1} \rangle$ of $k^n = \langle e_1, \dots, e_n \rangle$, let $\omega_1 \in \bigwedge^{p-1} k^{n-1}$ and $\omega_2 \in \bigwedge^p k^{n-1}$ be such that

$$\omega = \omega_1 \wedge e_n + \omega_2.$$

Now, if ω_1 is indecomposable, we make use of the induction hypothesis on $\bigwedge^{p-1} k^{n-1}$ to get \mathcal{A}, \mathcal{B} , and \mathcal{C} so that $G_{\mathcal{A},\mathcal{B},\mathcal{C}}(\omega_1)$ is indecomposable. If we consider instead

$\mathcal{C}' = \mathcal{C} \cup \{n\}$ which has cardinality one greater than \mathcal{C} , then $G_{\mathcal{A},\mathcal{B},\mathcal{C}'}(\omega_1 \wedge e_n) = G_{\mathcal{A},\mathcal{B},\mathcal{C}}(\omega_1)$ and $G_{\mathcal{A},\mathcal{B},\mathcal{C}'}(\omega_2) = 0$ since $\bigwedge X(\omega_2) = 0$. Consequently, $\mathcal{A}, \mathcal{B}, \mathcal{C}'$ satisfies the claim.

On the other hand, if ω_2 is indecomposable we make use of the induction hypothesis to obtain \mathcal{A}, \mathcal{B} and \mathcal{C} . In this case, without any further modification we have that $G(\omega) = G(\omega_2)$ is indecomposable since the absence of the number n in the indices will result in $\bigwedge X$ which annihilates $\omega_1 \wedge e_n$.

The only other possibility is that both ω_1 and ω_2 are decomposable, which returns us to the case that was proved initially. □

4. DETERMINING DECOMPOSABILITY USING THE ELEMENTS OF $\mathcal{P}'(p, n)$

Applying Theorem 3.4, we now show that the rank 6 quadratic forms in $\mathcal{P}'(p, n)$ are capable of characterizing decomposables, like the Plücker relations in $\mathcal{P}(p, n)$.

Theorem 4.1. *For any $\omega \in \bigwedge^p k^n$, all the elements of $\mathcal{P}'(p, n)$ vanish at ω if and only if ω is decomposable.*

Proof. By Theorem 3.4, ω is decomposable if and only if $G_{\mathcal{A},\mathcal{B},\mathcal{C}}(\omega) \in \bigwedge^2 k^4$ is decomposable for all choices of $\mathcal{A}, \mathcal{B}, \mathcal{C}$ as in Definition 2.3 and $G_{\mathcal{A},\mathcal{B},\mathcal{C}}$ as in Definition 3.3. However, the decomposability of an element of $\bigwedge^2 k^4$ is determined by the single Plücker relation (3).

So it is sufficient to note that substituting the 6 coordinates of $G_{\mathcal{A},\mathcal{B},\mathcal{C}}(\omega)$ into (3) yields precisely the equation $P'_{\mathcal{A},\mathcal{B},\mathcal{C}} = 0$. To see this, note that writing

$$G_{\mathcal{A},\mathcal{B},\mathcal{C}}(\omega) = \sum_{1 \leq i_1 < i_2 \leq 4} \hat{\Pi}_{i_1 i_2} e_{i_1} \wedge e_{i_2}$$

we can compute $\hat{\Pi}_{i_1 i_2}$ directly. It is necessarily independent of $\prod_{j_1 j_2 j_3 \dots j_p}$ unless exactly two of $\alpha_1, \dots, \alpha_4$, one each of $\beta_j^{(0)}$ and $\beta_j^{(1)}$ for each $1 \leq j \leq m$ and all of $\gamma_1, \dots, \gamma_{p-m-2}$ are represented amongst j_1, \dots, j_p , since the wedge products of which these are the coefficients are in the kernel of G . Moreover, of the remaining elements the only restriction on those which arise in $\hat{\Pi}_{i_1 i_2}$ is that α_{i_1} and α_{i_2} are not present amongst the indices. In particular,

$$\hat{\Pi}_{i_1 i_2} = (\epsilon_{ij}) \prod_{\alpha_1 \alpha_2 \alpha_3 \alpha_4 \bar{\beta}_1 \dots \bar{\beta}_m \gamma_1 \dots \gamma_{p-2-m} \setminus \alpha_{i_1} \alpha_{i_2}},$$

where $\epsilon_{ij} \in \{1, -1\}$ and substituting these into (3) results in $\pm P'_{\mathcal{A},\mathcal{B},\mathcal{C}}$. □

5. PARAMETER DEPENDENT FORMULATION

In this section we will introduce a GCP map depending polynomially on the $p + 2$ free parameters x_1, \dots, x_{p+2} , and use it to determine the decomposability of $\omega \in \bigwedge^p k^n$. Let $R = k[x_1, \dots, x_{p+2}]$, and let Q be the field of fractions of R . We call $\eta \in \bigwedge^a R^b$ decomposable if $\eta \otimes_R Q \in \bigwedge^a Q^b$ is decomposable, i.e., if $\eta \otimes_R Q \in \Gamma^a Q^b$. If η is decomposable, then for any $\mathfrak{p} \in \text{Spec } R$ there exist $v_1, \dots, v_a \in (R_{\mathfrak{p}})^b$ such that $\eta = v_1 \wedge \dots \wedge v_a$. Here $R_{\mathfrak{p}}$ is the localization of R at \mathfrak{p} .

Definition 5.1. Let X be the $(p+2) \times n$ matrix whose (i, j) entry is x_i^{j-1} , regarded as a Q -linear map $X: Q^n \rightarrow Q^{p+2}$. Let $Z = (I \ O)$ be the $4 \times (p+2)$ matrix consisting of a 4×4 unit matrix and a $4 \times (p-2)$ zero matrix, also regarded as a Q -linear map $Z: Q^{p+2} \rightarrow Q^4$. The GCP map $\bigwedge^2 Z \circ \delta \circ \bigwedge^p X: \bigwedge^p Q^n \rightarrow \bigwedge^2 Q^4$ restricted to

$\bigwedge^p k^n \subset \bigwedge^p Q^n$ defines $G: \bigwedge^p k^n \rightarrow \bigwedge^2 R^4$. Let $H = G^* P_{\{1\}\{2,3,4\}}: \bigwedge^p k^n \rightarrow R$, i.e., for $\omega \in \bigwedge^p k^n$, writing

$$G(\omega) = \sum_{1 \leq i < j \leq 4} \hat{\Pi}_{ij} e_i \wedge e_j, \quad \hat{\Pi}_{ij} \in R,$$

we let

$$H(\omega) = \hat{\Pi}_{12} \hat{\Pi}_{34} - \hat{\Pi}_{13} \hat{\Pi}_{24} + \hat{\Pi}_{14} \hat{\Pi}_{23} \in R.$$

Lemma 5.2. *For $0 \leq q \leq n$, let V' be a q -dimensional subspace of k^n . Then the Q -linear map X restricted to $V' \otimes_k Q$ has the maximal rank $r := \min\{p + 2, q\}$, i.e., one of the $r \times r$ -minors of matrix X is a nonzero element of R .*

Proof. For any matrix M , denote by $M_{i_1 \dots i_a}^{j_1 \dots j_b}$ the $a \times b$ matrix obtained by taking the rows i_1, \dots, i_a and columns j_1, \dots, j_b of M . Let $U = (u_{ij})$ be an $n \times q$ matrix whose columns form a basis of V' . It suffices to show that $D := \det((XU)_{1 \dots r}^{1 \dots r})$ is not the zero polynomial. Using the Cauchy-Binet formula we have

$$D = \sum_{n \geq h_1 > \dots > h_r \geq 1} \det X_{1 \dots r}^{h_r \dots h_1} \det U_{h_r \dots h_1}^{1 \dots r} = \Delta' \sum_{n-r \geq \lambda_1 \geq \dots \geq \lambda_r \geq 0} c_\lambda s_\lambda(x_1, \dots, x_r),$$

where Δ' is $(-1)^{r(r-1)/2}$ times the Vandermonde determinant $\prod_{1 \leq i < j \leq r} (x_i - x_j)$, $s_\lambda = s_\lambda(x_1, \dots, x_r)$ is the Schur function for the partition $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_r)$, [11], and $c_\lambda = \det U_{h_r \dots h_1}^{1 \dots r}$ with $\lambda_i = h_i - (r - i + 1)$. Since the s_λ are linearly independent over k , and since some $c_\lambda \in k$ do not vanish, D is not the zero polynomial. \square

Theorem 5.3. *The following four conditions for $\omega \in \bigwedge^p k^n$ are equivalent:*

- (1) ω is decomposable;
- (2) $\bigwedge^p X(\omega) \in \bigwedge^p R^{p+2}$ is decomposable;
- (3) $G(\omega) \in \bigwedge^2 R^4$ is decomposable;
- (4) $H(\omega) = 0$.

Proof. Since H is the pull-back by G of the Plücker relation for $\Gamma^2 k^4$, conditions 3 and 4 are equivalent. Since G is a GCP map, 1 implies 3. So we have only to prove that 3 implies 2 and 2 implies 1.

We first prove that 3 implies 2. Suppose $\alpha := \bigwedge^p X(\omega)$ is indecomposable. Then $\beta := \delta(\alpha) \in \bigwedge^2 k^{p+2}$ is also indecomposable since $\delta(\beta) = \alpha$ and δ is a GCP map. Hence for some $P_{A,B} \in \mathcal{P}(2, p + 2)$ we have

$$(13) \quad P_{A,B}(\beta) = \Pi_{i_0 i_1} \Pi_{i_2 i_3} - \Pi_{i_0 i_2} \Pi_{i_1 i_3} + \Pi_{i_0 i_3} \Pi_{i_1 i_2} \neq 0,$$

where $A = \{i_0\}$ and $B = \{i_1, i_2, i_3\}$ with $1 \leq i_1 < \dots < i_4 \leq p + 2$, and $\beta = \sum_{1 \leq i < j \leq p+2} \Pi_{ij} e_i \wedge e_j$. Permutation of the parameters $(x_i) \mapsto (x_{\sigma(i)})$, where $\sigma \in \mathfrak{S}(p + 2)$, does not affect the indecomposability of α and β , and it yields the action $\Pi_{ij} \mapsto \varepsilon(\sigma, i, j) \Pi_{\sigma(i)\sigma(j)}$ on the coordinates Π_{ij} of β . Here $\varepsilon(\sigma, i, j) = \pm 1$ depends on i and j , but in such a way that its net effect on each term of (13) is the same for all three terms. Hence for all choices of (i_0, \dots, i_3) , $P_{A,B}(\beta)$ are the same polynomials upon renaming the variables x_i and a possible change in sign. In particular, we can take $(i_0, i_1, i_2, i_3) = (1, 2, 3, 4)$, i.e., $P_{\{1\}\{2,3,4\}}(\beta) = P_{\{1\}\{2,3,4\}}(\bigwedge^2 Z(\beta)) \neq 0$. Thus $\bigwedge^2 Z(\beta) = G(\omega)$ is indecomposable, proving 3 implies 2.

Now we prove that 2 implies 1. Suppose ω is indecomposable. We will prove the indecomposability of $\bigwedge^p X(\omega)$.

First consider the case where ω is the sum of *two* decomposable elements, $\omega = v_1 \wedge \dots \wedge v_p + w_1 \wedge \dots \wedge w_p$. Setting $V = \langle v_1, \dots, v_p \rangle_Q$ and $W = \langle w_1, \dots, w_p \rangle_Q$

we have, as seen in the proof of Theorem 3.4, that $\dim V = \dim W = p$ and $\dim V \cap W \leq p - 2$, so that $\dim V + W \geq p + 2$. Using Lemma 5.2 with $V' = V, W$ and $V + W$ respectively, we have $\dim XV = \dim XW = p$ and $\dim(XV + XW) = \dim(X(V + W)) = p + 2$, so that $\dim(XV \cap XW) = p - 2$. Hence $\wedge^p X(\omega) = Xv_1 \wedge \cdots \wedge Xv_p + Xw_1 \wedge \cdots \wedge Xw_p$ is indecomposable.

Next we study the general case by induction on n . If $n = 4$, then $\det X$ is a nonzero element of R , so that 2 implies 1 is obvious. Suppose the assertion holds for $n - 1$. Let $\omega = \omega_1 \wedge e_n + \omega_2$ with $\omega_1 \in \wedge^{p-1} k^{n-1}$ and $\omega_2 \in \wedge^p k^{n-1}$, where we regard $k^{n-1} \subset k^n$ as in the proof of Theorem 3.4. If both ω_1 and ω_2 are decomposable, it reduces to the case studied above.

Suppose ω_1 is indecomposable. Let X_0 be the $(p + 1) \times (n - 1)$ matrix obtained from X by removing the last row and column. Since $Xe_n = \sum_{i=1}^{p+2} x_i^{n-1} e_i$, we have

$$(14) \quad \begin{aligned} \wedge^p X(\omega) &= \wedge^{p-1} X(\omega_1) \wedge Xe_n + \wedge^p X(\omega_2) \\ &= (-1)^{p+n} x_{p+2}^{n-1} \wedge^{p-1} X_0(\omega_1) \wedge e_{p+2} + \text{lower degree terms in } x_{p+2}. \end{aligned}$$

By the induction hypothesis $\wedge^{p-1} X_0(\omega_1)$ is indecomposable, so the right-hand side of (14) is also indecomposable, as seen by expanding the Plücker relations $P_{A,B}(\wedge^p X(\omega))$ with $A \cap B \ni p + 2$ in powers of x_{p+2} and taking the coefficients of x_{p+2}^{2n-2} .

Suppose ω_1 is decomposable and ω_2 is not. Then $\wedge^{p-1} X(\omega_1)$ is decomposable, and by the induction hypothesis $\wedge^p X(\omega_2)$ is not. We prove by contradiction that $\wedge^p X(\omega)$ cannot be decomposable:

Let $\pi'_{i_1, \dots, i_{p-1}}$ (resp. ρ_{i_1, \dots, i_p}) be the coordinates of $\wedge^{p-1} X(\omega_1)$ (resp. $\wedge^p X(\omega_2)$). Thus $\wedge^p X(\omega_1 \wedge e_n) = \wedge^{p-1} X(\omega_1) \wedge Xe_n$ has the coordinates

$$(15) \quad \pi_{i_1, \dots, i_p} = \sum_{\nu=1}^p (-1)^{p-\nu} \pi'_{i_1, \dots, i_p \setminus i_\nu} x_{i_\nu}^{n-1}.$$

By assumption, $\{\pi_{i_1, \dots, i_p}\}$ satisfies all of the Plücker relations in $\mathcal{P}(p, p + 2)$, and $\{\rho_{i_1, \dots, i_p}\}$ does not satisfy some relation, say $P_{A,B}$, in $\mathcal{P}(p, p + 2)$. Here, using the symmetry argument as we used in the proof of “3 implies 2” above, we may assume $A = \{1, 5, 6, \dots, p + 2\}$ and $B = \{2, 3, 4, 5, 6, \dots, p + 2\}$. Denoting the sequence of indices $5, 6, \dots, p + 2$ by λ , and hence $\pi_{ij\lambda} = \pi_{ij56\dots p+2}$, etc., we thus have

$$(16) \quad \pi_{12\lambda}\pi_{34\lambda} - \pi_{13\lambda}\pi_{24\lambda} + \pi_{14\lambda}\pi_{23\lambda} = 0,$$

$$(17) \quad \rho_{12\lambda}\rho_{34\lambda} - \rho_{13\lambda}\rho_{24\lambda} + \rho_{14\lambda}\rho_{23\lambda} \neq 0.$$

If $\wedge^p X(\omega)$ is decomposable, then $\{\pi_{i_1, \dots, i_p} + \rho_{i_1, \dots, i_p}\}$, the coordinates of ω , must satisfy all the relations in $\mathcal{P}(p, p + 2)$ and so the above $P_{A,B}$ in particular; thus by using (16) we have

$$(18) \quad \begin{aligned} 0 &= (\pi_{12\lambda} + \rho_{12\lambda})(\pi_{34\lambda} + \rho_{34\lambda}) - (\pi_{13\lambda} + \rho_{13\lambda})(\pi_{24\lambda} + \rho_{24\lambda}) \\ &\quad + (\pi_{14\lambda} + \rho_{14\lambda})(\pi_{23\lambda} + \rho_{23\lambda}) \\ &= (\pi_{12\lambda}\rho_{34\lambda} - \pi_{13\lambda}\rho_{24\lambda} + \pi_{14\lambda}\rho_{23\lambda}) + (\rho_{12\lambda}\pi_{34\lambda} - \rho_{13\lambda}\pi_{24\lambda} + \rho_{14\lambda}\pi_{23\lambda}) \\ &\quad + (\rho_{12\lambda}\rho_{34\lambda} - \rho_{13\lambda}\rho_{24\lambda} + \rho_{14\lambda}\rho_{23\lambda}). \end{aligned}$$

By the definition of a linear map X , $\rho_{ij\lambda}$ are polynomials in $x_r, r = i, j, 5, \dots, p + 2$, with no x_r having the $(n - 1)$ st or higher power, and $\pi_{ij\lambda}$ are polynomials in the same set of x_r with x_r^{n-1} appearing only as the last factor in each term of (15).

Moreover, for each $r \in \{1, \dots, 4\}$, each term on the right-hand side of (18) is the product of a polynomial which depends on x_r and another which does not. Thus comparing the coefficients of x_r^{n-1} ($r = 2, 3, 4$) on both sides of (18), we have

$$\pi'_{1\lambda}\rho_{r'r''\lambda} - \pi'_{r'\lambda}\rho_{1r''\lambda} + \pi'_{r''\lambda}\rho_{1r'\lambda} = 0, \quad (r', r'') = (3, 4), (4, 2), (2, 3).$$

After using these to eliminate $\rho_{23\lambda}$, $\rho_{24\lambda}$ and $\rho_{34\lambda}$, the left-hand side of (17) becomes

$$\rho_{12\lambda} \frac{\pi'_{3\lambda}\rho_{14} - \pi'_{4\lambda}\rho_{13}}{\pi'_{1\lambda}} - \rho_{13\lambda} \frac{\pi'_{2\lambda}\rho_{14\lambda} - \pi'_{4\lambda}\rho_{12\lambda}}{\pi'_{1\lambda}} + \rho_{14\lambda} \frac{\pi'_{2\lambda}\rho_{13\lambda} - \pi'_{3\lambda}\rho_{12\lambda}}{\pi'_{1\lambda}} = 0,$$

which is a contradiction. Hence $\bigwedge^p X(\omega)$ is indecomposable, completing the proof of “2 implies 1”. \square

6. CONCLUDING REMARKS

We obtained a set $\mathcal{P}'(p, n)$ of rank 6 quadratic forms on $\bigwedge^p k^n$, which is in general much smaller than the set of standard Plücker relations, and yet is capable of determining the decomposable elements in $\bigwedge^p k^n$. Every element of $\mathcal{P}'(p, n)$ is obtained by pulling back the one nontrivial Plücker relation on $\bigwedge^2 k^4$. This means:

1. The elements of $\mathcal{P}'(p, n)$ define quadric hypersurfaces which are isomorphic to each other by a GCP map, and the Grassmannian is obtained as the intersection of those isomorphic quadrics. This may be of interest to geometers who have already noted and used the previously known fact that this was true when $p = 2$ [13, 14].

2. The Grassmannian $Gr(p, n)$ is the intersection of the pullbacks of $Gr(2, 4)$ under all GCP maps from $\bigwedge^p k^n$ to $\bigwedge^2 k^4$.

3. The 3-term Plücker relation is in a sense “universal”, providing an explanation for the special role played by the 3-term relation in applications like soliton theory [5, 6]. In this respect, upon suitable identification of variables, condition (4) in Theorem 5.3 can be viewed as a parameter-dependent Fay-Hirota type difference equation to characterize KP tau-functions [12, 17] in the case of rational solutions.

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