NOTE ON A REMARKABLE SUPERPOSITION FOR A NONLINEAR EQUATION

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Abstract. We give a simple proof of—and extend—a superposition principle for the equation \( \text{div}(|\nabla u|^{p-2}\nabla u) \leq 0 \), discovered by Crandall and Zhang. An integral representation comes as a byproduct. It follows that a class of Riesz potentials is \( p \)-superharmonic.

1. Introduction

The Newtonian potentials
\[
V(x) = c_n \int \frac{\rho(y)\,dy}{|x - y|^{n-2}}, \quad \rho \geq 0,
\]
are important examples of superharmonic functions in the \( n \)-dimensional Euclidean space, \( n \geq 3 \). They are obtained through a superposition of fundamental solutions
\[
\frac{A_j}{|x - y_j|^{n-2}}, \quad A_j \geq 0,
\]
of the Laplace equation. The equation \( \Delta V(x) = -\rho(x) \) holds.

For the \( p \)-Laplace equation
\[
-\text{div}(|\nabla u|^{p-2}\nabla u) = 0
\]
it was recently discovered by M. Crandall and J. Zhang that a similar superposition of fundamental solutions is possible. Indeed, they proved in [CZ] that sums like
\[
\sum A_j |x - a_j|^{\frac{p-n}{p-1}} \quad (2 < p < n)
\]
are \( p \)-superharmonic functions, where \( A_j \geq 0 \). They also included exponents other than the natural \( (p-n)/(p-1) \) and allowed \( p \) to vary between 1 and \( \infty \). The Riesz potentials
\[
\int \frac{\rho(y)\,dy}{|x - y|^{(n-p)/(p-1)}}
\]
appear as the limit of such sums.
The purpose of our note is to give an alternative proof of the following theorem for the Riesz potentials

\[ V_\alpha(x) = \int |x - y|^\alpha \rho(y)dy. \]

**Theorem.** Let \( \rho \in C_0(\mathbb{R}^n), n \geq 2, \) be a nonnegative function. We have three cases depending on \( p \):

(i) \( 2 < p < n \). The function \( V_\alpha \) is \( p \)-superharmonic, if

\[ \frac{p-n}{p-1} \leq \alpha < 0. \]

(ii) \( p > n \). The function \( V_\alpha \) is \( p \)-subharmonic, if

\[ \alpha \geq \frac{p-n}{p-1}. \]

If \( p = \infty \), we may take \( \alpha \geq 1 \).

(iii) \( p = n \). The function \( V_0(x) = \int \log(|x - y|)\rho(y)dy \)

is \( n \)-subharmonic.

Before proceeding, we make a comment about the case \( 1 < p < 2 \), which exhibits a puzzling behaviour. While the fundamental solution

\[ |x - a|^{\frac{p-n}{p-1}} \]

is \( p \)-superharmonic in the whole \( \mathbb{R}^n \), the sum

\[ |x - a|^{\frac{p-n}{p-1}} + |x - b|^{\frac{p-n}{p-1}} \]

is not, assuming of course that \( a \neq b \). The sum is \( p \)-subharmonic when \( x \neq a \) and \( x \neq b \), but it is not \( p \)-subharmonic in the whole \( \mathbb{R}^n \). A \( p \)-subharmonic function cannot take the value \(+\infty\) in its domain of definition because of the comparison principle. This was about \( p < 2 \).

We recall from [L] that \( p \)-superharmonic functions are defined as lower semicontinuous functions \( v : \mathbb{R}^n \rightarrow (0, \infty] \) that obey the comparison principle with respect to the \( p \)-harmonic functions. A more direct characterization is available for smooth functions. When \( p \geq 2 \) the function \( v \in C^2(\mathbb{R}^n) \) is \( p \)-superharmonic if and only if

\[ -\text{div}(|\nabla v(x)|^{p-2} \nabla v(x)) \geq 0 \]

at each point \( x \). From the identity

\[ \text{div}(|\nabla v|^{p-2} \nabla v) = |\nabla v|^{p-4} \left\{ |\nabla v|^2 \Delta v + (p-2)\Delta_\infty v \right\}, \]

where

\[ \Delta_\infty v = \sum_{i,j=1}^n \frac{\partial}{\partial x_i} \frac{\partial}{\partial x_j} \frac{\partial^2 v}{\partial x_i \partial x_j} \]

is the \( \infty \)-Laplacian operator, we can read off that the pointwise inequality

\[ |\nabla v|^2 \Delta v + (p-2)\Delta_\infty v \leq 0 \]

is an equivalent characterization of \( p \)-superharmonic functions \( v \) in \( C^2(\mathbb{R}^n) \). (Incidentally, this is also valid in the case \( 1 < p < 2 \). See [JLM].)

Thus we have a practical definition for functions of class \( C^2 \). The polar set \( \Xi = \{ x : v(x) = +\infty \} \) can be exempted if \( v \) is lower semicontinuous in \( \mathbb{R}^n \) and
\( v \in C^2(\mathbb{R}^n \setminus \Xi) \). An example is the fundamental solution \( |x - a|^{(p - n)/(p - 1)}, 1 < p < n \), where the point \( x = a \) is exempted.

Finally, we mention that in Section 3 we need the concept of \textit{weak \( p \)-supersolution}. We say that \( v \in W^{1,p}_{\text{loc}}(\mathbb{R}^n) \) is a weak \( p \)-supersolution if

\[
\int_{\mathbb{R}^n} \left\langle |\nabla v|^{p - 2}\nabla v, \nabla \varphi \right\rangle \, dx \geq 0
\]

holds for all nonnegative \( \varphi \in C_0^\infty(\mathbb{R}^n) \). In [L] it was established that \textit{locally bounded} \( p \)-superharmonic functions are weak \( p \)-supersolutions. On the other hand, lower semicontinuous weak \( p \)-supersolutions are \( p \)-superharmonic functions.

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\textbf{2. Proof of the Theorem}

We assume \( n \geq 2 \). The following calculations are formal, but are easy to justify because \( \alpha > 2 - n \). Notice that we have

\[
\alpha \geq \frac{p - n}{p - 1} > 2 - n \quad \text{when} \quad p > 2.
\]

Differentiating

\[
V(x) = \int |x - y|^\alpha \rho(y) \, dy
\]

under the integral sign we obtain

\[
\frac{\partial V}{\partial x_i} = \alpha \int |x - y|^{\alpha - 2} (x_i - y_i) \rho(y) \, dy
\]

and

\[
\frac{\partial^2 V}{\partial x_i \partial x_j} = \alpha (\alpha - 2) \int |x - y|^{\alpha - 4} (x_i - y_i) (x_j - y_j) \rho(y) \, dy
\]

\[
+ \alpha \delta_{ij} \int |x - y|^{\alpha - 2} \rho(y) \, dy.
\]

Aiming at \( \Delta_\infty V \), we write the product of the integrals in

\[
\frac{\partial V}{\partial x_i} \frac{\partial V}{\partial x_j} \frac{\partial^2 V}{\partial x_i \partial x_j}
\]

as a triple integral in disjoint variables.\(^*\) This yields the formula

\[
\Delta_\infty V(x) = \alpha^3 \int |x - c|^{\alpha - 2} \rho(c) \, dc \left| \int |x - a|^{\alpha - 2} (x - a) \rho(a) \, da \right|^2
\]

\[
+ \alpha^3 (\alpha - 2) \int |x - c|^{\alpha - 2} \rho(c) \left| \frac{x - c}{|x - c|} \right| \int |x - a|^{\alpha - 2} (x - a) \rho(a) \, da \right|^2 \, dc
\]

in vector notation. Keeping \( \alpha \) within the prescribed range we have only harmless singularities.

\(^*\) The principle is clear from the example

\[
\left( \int e^x \, dx \right)^2 \int e^{2x} \, dx = \iiint e^{a+b+2c} \, da \, db \, dc.
\]
By the Cauchy-Schwarz inequality we have
\[ \left| \int \frac{x - c}{|x - c|} \left| x - a \right|^{\alpha - 2} (x - a) \rho(a) \, da \right| \leq \left| \int |x - a|^{\alpha - 2} (x - a) \rho(a) \, da \right|. \]
In easily understandable notation we can therefore write the above formula as
\[ \Delta_\infty V(x) = \alpha^3 A(x) + \alpha^3 (\alpha - 2) B(x) \]
where
\[ 0 \leq B(x) \leq A(x). \]
From this we can read off that \( \Delta_\infty V(x) \geq 0 \) when \( \alpha \geq 1 \). This settles the case \( p = \infty \). There is a more succinct representation. Lagrange’s identity
\[ |X \wedge Y|^2 = \frac{1}{2} \sum (X_i Y_j - X_j Y_i)^2 = |X|^2 |Y|^2 - \langle X, Y \rangle^2 \]
for vectors enables us to write
\[ C(x) = A(x) - B(x) \]
\[ = \int |x - c|^{\alpha - 2} \rho(c) \left| \frac{x - c}{|x - c|} \wedge \int |x - a|^{\alpha - 2} (x - a) \rho(a) \, da \right|^2 \, dc. \]
Notice that \( C(x) \geq 0 \). Thus we have arrived at the representation formula
\[ \Delta_\infty V(x) = \alpha^3 C(x) + \alpha^3 (\alpha - 1) B(x), \]
which is particularly appealing for \( \alpha = 1 \) and
\[ V(x) = \int |x - y| \rho(y) \, dy. \]
Continuing the calculations, we find that
\[ \Delta V(x) = \alpha(\alpha - 2 + n) \int |x - c|^{\alpha - 2} \rho(c) \, dc, \]
and hence, after some simple manipulations
\[ |\nabla V|^2 \Delta V = \alpha^3 (\alpha - 2 + n) A. \]
It follows that
\[ |\nabla V|^2 \Delta V + (p - 2) \Delta_\infty V \\
= \alpha^3 (n + \alpha + p - 4) A(x) + \alpha^3 (\alpha - 2)(p - 2) B(x) \\
= \alpha^3 [(2 - \alpha)(p - 2) C(x) + (n - p + \alpha(p - 1)) B(x)]. \]
In this formula we have command over the sign of
\[ |\nabla V|^2 \Delta V + (p - 2) \Delta_\infty V, \]
_at least in the cases needed for the theorem_. We may add that the logarithmic integral in the borderline case \( p = n \) requires a separate, but similar, calculation leading to
\[ |\nabla V_0|^2 \Delta V_0 + (n - 2) \Delta_\infty V_0 \\
= 2(n - 2) C(x) \]
where \( \alpha = 0 \) in the expression for \( C(x) \). This concludes our proof of the Theorem.
It is remarkable that the factor \( n - p + \alpha(p - 1) \) in front of \( B(x) \) reveals the natural exponent \( \alpha = (p - n)/(p - 1) \); the term vanishes for this \( \alpha \). Thus

\[
|\nabla V|^2 \Delta V + (p - 2) \Delta_{\infty} V = \alpha^3(2 - \alpha)(p - 2)C(x)
\]

when \( \alpha = (p - n)/(p - 1) \), and \( p > 2 \).

To this one may add that the method is rather flexible. For example, in the case of a variable exponent it works for

\[
\int |x - y|^\alpha(y) \rho(y) dy.
\]

One can also consider \( V(x) + \langle a, x \rangle \) with an extra linear term.

3. Riesz potentials

So far, we have assumed that the nonnegative density \( \rho \) in the Riesz potential

\[
V(x) = \int |x - y|^\alpha \rho(y) dy
\]

is smooth. The restriction can easily be relieved because of the following theorem: the pointwise limit of an increasing sequence of \( p \)-superharmonic functions is either a \( p \)-superharmonic function or identically \( +\infty \). Thus we immediately reach the case with lower semicontinuous \( \rho \)'s. We point out that the discrete case

\[
\sum A_j |x - a_j|^\alpha
\]

follows if one regards the integrals as sums in disguise and takes into account a special reasoning concerning the poles \( a_j \).

We can do more than that. Indeed, we can allow rather general measures.

**Proposition.** Let \( \mu \) be a Radon measure on \( \mathbb{R}^n \) satisfying the growth condition

\[
\int_{|y| \geq 1} |y|^\alpha d\mu(y) < \infty.
\]

The theorem holds for the Riesz potentials

\[
V(x) = \int |x - y|^\alpha d\mu(y).
\]

In other words, we have replaced \( \rho(y) dy \) with \( d\mu(y) \). The growth condition guarantees that \( V(x) < \infty \) almost everywhere. In fact \( V(x) \equiv \infty \) if \( \int |y|^\alpha d\mu(y) = +\infty \). See [P, Theorem 3.4, p. 78] about this.

**Proof.** Because of the increasing limit

\[
V(x) = \lim_{R \to \infty} \int_{|y| < R} |x - y|^\alpha d\mu(y)
\]

we may, in the proof, assume that the measure \( \mu \) has compact support. To simplify the exposition, we confine ourselves to the case

\[
\alpha = \frac{p - n}{p - 1}, \quad 2 < p < n.
\]
The passage from integrals of the type $\int |x - y|^\alpha \rho(y)dy$ to the more general kind with the Radon measure is accomplished through a regularization, for example

$$\rho_t(y) = \frac{1}{(4\pi t)^{n/2}} \int e^{-\frac{|x-y|^2}{4t}} d\mu(\xi),$$

where the heat kernel is present. We have

$$\int \rho_t(y)dy = \int d\mu(\xi) = \mu(\mathbb{R}^n) = M.$$

Let us denote

$$V_k(x) = \int |x - y|^\alpha \rho_{t_k}(y)dy \quad (k = 1, 2, 3, \cdots)$$

where $t_k \to 0^+$ as $k \to \infty$. According to the theorem each $V_k$ is $p$-superharmonic. It is not difficult to see that $V_k \to V$ a.e., at least for a subsequence. The Proposition follows from the general theorem about a.e. convergence in [KM, Theorem 1.17], which assures that the limit function is $p$-superharmonic.

In the present situation a more direct proof is possible. It is based on a compactness argument in $W^{1,p}_{loc}$. 

**Alternative proof.** A direct calculation yields

$$\int_{B_R} |\nabla V_k|^{p-1}dx \leq 2 \left( M \frac{n-p}{n-1} \right)^{p-1} \omega_{n-1} R, \quad k = 1, 2, 3, \cdots.$$ 

To obtain such a bound, free of $k$, we proceed as follows:

\[
|\nabla V_k(x)| \leq |\alpha| \int |x - y|^{\alpha-1} \rho_k(y)dy,
\]

\[
|\nabla V_k(x)|^{p-1} \leq |\alpha|^{p-1} \int |x - y|^{(\alpha-1)(p-1)} \rho_k(y)dy \left( \int \rho_k(y)dy \right)^{p-2}
\]

\[
= |\alpha|^{p-1} M^{p-2} \int |x - y|^{1-n} \rho_k(y)dy,
\]

\[
\int_{B_R} |\nabla V_k(x)|^{p-1}dx \leq |\alpha|^{p-1} M^{p-2} \int \rho_k(y) \int_{B_R} |x - y|^{1-n} dx dy.
\]

The inner integral can be estimated as

$$\int_{B_R} |x - y|^{1-n} dx \leq 2R \omega_{n-1}$$

since $y \in B_R$. This yields the desired bound.

According to the celebrated Banach-Saks theorem there exists a sequence of indices $k_1 < k_2 < \cdots$ such that for the arithmetic means

$$W_j = \frac{V_{k_1} + V_{k_2} + \cdots + V_{k_j}}{j}$$

we have that $\nabla W_j \to \nabla V$ strongly in $L^{p-1}(B_R)$. Now we take advantage of the linear structure by concluding that each $W_j$ is a $p$-superharmonic function, because it can be written as a Riesz potential with the density $(\rho_{k_1} + \cdots + \rho_{k_j})/j$. Hence

$$\int \langle |\nabla W_j|^{p-2} \nabla W_j, \nabla \varphi \rangle dx \geq 0.$$
for each nonnegative test function \( \varphi \in C_0^\infty(\mathbb{R}^n) \). Given \( \varphi \), we take a ball \( B_R \) containing its support. The strong convergence of the sequence \( \{ \nabla W_j \} \) in \( L^{p-1}(B_R) \) and the elementary vector inequality

\[
|b|^{p-2}b - |a|^{p-2}a \leq (p - 1)|b - a|(|b| + |a|)^{p-2}, \quad p \geq 2,
\]

enable us to pass to the limit under the integral sign so that also

\[
\int \langle |\nabla V|^{p-2}\nabla V, \nabla \varphi \rangle dx \geq 0.
\]

We could conclude that \( V \) is a weak supersolution if we knew that \( V \) belongs to \( W^{1,p}_{loc}(\mathbb{R}) \). Unfortunately, this is not always the case. For example, the fundamental solution is not in \( W^{1,p}_{loc}(\mathbb{R}^n) \). A simple correction is required. Also the cut-off functions

\[
W_j^L = \min \{ W_j(x), L \}
\]

are weak \( p \)-supersolutions. The ordinary Caccioppoli estimate

\[
\int \zeta^p|\nabla W_j^L|^pdx \leq p^pL^p \int |\nabla \zeta|^pdx
\]

is available; cf. [L, Corollary 2.5]. By weak lower semicontinuity it also holds for

\[
V^L = \min \{ V(x), L \}.
\]

Therefore, \( \nabla V^L \in L^p_{loc}(\mathbb{R}^n) \), so that \( V^L \) is in the right Sobolev space.

As before, we can conclude that

\[
\int \langle |\nabla V^L|^{p-2}\nabla V^L, \nabla \varphi \rangle dx \geq 0,
\]

but this time it follows that \( V^L \) is a weak \( p \)-supersolution. Then the increasing limit \( V = \lim_{L \to \infty} V^L \) is \( p \)-superharmonic.

Strictly speaking, the conclusion is that the function

\[
\tilde{V}(x) = \text{ess lim inf}_{y \to x} V(y)
\]

is \( p \)-superharmonic, because it is the increasing limit of the \( p \)-superharmonic functions \( \text{ess lim inf}_{y \to x} V^L(y) \) as \( L \to \infty \). See [KM, Proposition 1.7]. The Lemma below concludes our proof.

\[\square\]

**Lemma** (Brelot). At each point \( x_0 \) we have

\[
V(x_0) = \tilde{V}(x_0) = \text{ess lim inf}_{x \to x_0} V(x),
\]

when \( 2 - n < \alpha < 0 \).

For the sake of completeness we present the proof.

**Proof.** The function \( |x - y|^{\alpha} \) is superharmonic and therefore we have

\[
\int_{B(x_0,r)} |x - y|^\alpha dx \leq |x_0 - y|^\alpha
\]

*This proof does not work if one cuts the original functions \( V_j \) instead.*
for the volume average over the ball $B(x_0, r)$. It follows that

$$\int_{B(x_0, r)} V(x) dx = \int_{B(x_0, r)} \int |x - y|^{\alpha} d\mu(y) dx$$

$$= \int \left( \int_{B(x_0, r)} |x - y|^{\alpha} dx \right) d\mu(y)$$

$$\leq \int |x_0 - y|^{\alpha} d\mu(y) = V(x_0).$$

This is merely a restatement of the fact that $V$ is a superharmonic function (in the ordinary sense).

It follows from Fatou’s lemma that $V$ is lower semicontinuous. Hence

$$V(x_0) \leq \liminf_{x \to x_0} V(x) \leq \text{ess lim inf}_{x \to x_0} V(x)$$

$$\leq \liminf_{r \to 0} \int_{B(x_0, r)} V(x) dx \leq V(x_0)$$

where the last inequality was proved above. Thus equality holds at each step. □

References


