COMPLEX MONGE-AMPÈRE OF A MAXIMUM

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Abstract. We derive a formula for \((dd^c u)^n\) where \(u = \max_j u_j\) is a finite maximum. As an application, we compute the complex equilibrium measures of some generalized polyhedra.

Plurisubharmonic (psh) functions play a primary role in pluripotential theory. They are closely related to the operator \(dd^c = 2i\partial \bar{\partial}\) (with notation \(d = \partial + \bar{\partial}\) and \(d^c = i(\bar{\partial} - \partial)\)), which serves as a generalization of the Laplacian from \(\mathbb{C}\) to \(\mathbb{C}^{\dim}\) for \(\dim > 1\). If \(u\) is smooth of class \(C^2\), then for \(1 \leq n \leq \dim\), the coefficients of the exterior power \((dd^c u)^n\) are given by the \(n \times n\) subdeterminants of the matrix \((\partial^2 u/\partial z_i \partial \bar{z}_j)\). The top exterior power corresponds to \(n = \dim\), and in this case we have the determinant of the full matrix, which gives the complex Monge-Ampère operator. The extension of the (nonlinear) operator \((dd^c)^n\) to nonsmooth functions has been studied by several authors (see, for instance, [3]). Here it will suffice to define \((dd^c)^n\) on psh functions which are continuous.

For an open set \(E \subset \mathbb{C}^{\dim}\), the set of psh functions on \(E\) forms a cone which is closed under the operations of addition and of taking finite maxima. The relation between \((dd^c)^n\) and the additive structure is given by the formula

\[
(dd^c(u_1 + u_2))^n = \sum_{n_1 + n_2 = n} \frac{n!}{n_1!n_2!} (dd^c u_1)^{n_1} \wedge (dd^c u_2)^{n_2},
\]

which holds in both the classical and generalized sense. Here we make the connection between \((dd^c)^n\) and the operation of taking a finite maximum. A few examples of this are known already: if \(u = \max(0, \log(|z_1|^2 + \cdots + |z_k|^2))\), then \((dd^c u)^k\) is a multiple of surface measure on \(\{|z_1|^2 + \cdots + |z_k|^2 = 1\}\). And if \(u = \max(0, x_1, \ldots, x_k)\), then \((dd^c u)^k\) is a multiple of surface measure on \(\{x_1 = \cdots = x_k = 0\}\). The case \((dd^c \max(u_1, u_2, u_3))^2\), \(u_j\)'s pluriharmonic, is given in [4]. A different sort of formula for \((dd^c \max(u_1, u_2))^k\) is given in [2], which, as pointed out by the referee, follows easily from the formula of this paper.

More generally, let the functions \(u_1, \ldots, u_m\) be smooth, and let

\[u := \max(u_1, \ldots, u_m)\]
be their maximum. It follows that $dd^c u_j$ and $dd^c u$ are locally bounded below, and for the purpose of defining $(dd^c)^n$, $u_j$ and $u$ may be treated as psh functions. Taking the maximum stratifies $E$ as follows: for each $J \subset \{1, \ldots, m\}$ there is

$$E_J = E_J(u_1, \ldots, u_m) := \{z \in E : u(z) = u_j(z) \ \forall j \in J, \ u(z) > u_i(z) \ \forall i \notin J\}.$$  

The sets $E_J$ form a partition of $E$, and as $J$ increases, the subsequent $E_J$‘s lie inside the “boundaries” of the previous ones. Since $(dd^c u)^n$ is representable by integration, we may decompose it into a sum over the elements of the partition

$$(dd^c u)^n = \sum_J (dd^c u)^n|_{E_J}. \tag{1}$$

The first terms in (1) are easy to identify: $E_j$ is open and thus $(dd^c u)^n|_{E_j} = (dd^c u_j)^n|_{E_j}$. For the rest of the terms, we write $J = \{j_1, \ldots, j_\ell\}$, $j_1 < \cdots < j_\ell$, so $|J| = \ell$ is the number of elements, and we define the forms

$$(\delta_j^c)^n = \delta_j^c(u_1, \ldots, u_m) := d^c(u_{j_1} - u_{j_2}) \wedge \cdots \wedge d^c(u_{j_{\ell-1}} - u_{j_\ell}), \tag{2}$$

$$\sigma_j^n = \sigma_j^n(dd^c u_1, \ldots, dd^c u_m) := \sum_{\beta_1 + \cdots + \beta_\ell = n} (dd^c u_{j_1})^{\beta_1} \wedge \cdots \wedge (dd^c u_{j_\ell})^{\beta_\ell}. \tag{3}$$

If $E_J$ is smooth, we let $[E_J]$ denote the current of integration over $E_J$, where we orient $E_J$ so that the current $\delta_j^c \wedge [E_J]$ is positive (see Lemma 1). In this paper we identify the terms of the summation (1) as integrations on the strata $E_J$:

**Theorem 1.** Let $u_j \in C^3$, $1 \leq j \leq m$, be given, and set $u = \max(u_1, \ldots, u_m)$. If all of the sets $E_J$ are smooth, then

$$(dd^c u)^n = \sum_J \sigma_J^n|_{E_J} - |J| + 1 \wedge \delta_j^c \wedge [E_J], \tag{\star}$$

where the sum is taken over all $J \subset \{1, \ldots, m\}$ with $1 \leq |J| \leq n + 1$.

A function that arises frequently is $v = \log(|f_1|^2 + \cdots + |f_N|^2)$, where the $f_i$’s are holomorphic. Set $f = (f_1, \ldots, f_N) : \mathbb{C}^\dim \to \mathbb{C}^N$ and let $\pi : \mathbb{C}^N - 0 \to \mathbb{P}^{N-1}$ be the projection. The powers $(dd^c v)^n$ may be determined on $\{v > -\infty\}$ using the fact that $dd^c \log(|z|^2 + \cdots + |z|^2) = \pi^* \omega_{FS}$, where $\omega_{FS}$ denotes the Fubini-Study Kähler form on $\mathbb{P}^{N-1}$. That is, $(dd^c v)^n$ is the pullback of $\omega_{FS}^n$ under $(\pi \circ f)^\circ$. In particular, $(dd^c v)^n = 0$ if $n \geq N$.

Such functions and their maxima arise naturally with generalized polyhedra. For $1 \leq \alpha \leq A$, let $p_{\alpha,1}, \ldots, p_{\alpha,N_\alpha}$ be polynomials. Let $\deg_\alpha = \max_{1 \leq i \leq N_\alpha}(\deg(p_{\alpha,i}))$ be the maximum of the degrees. Define

$$u_\alpha = \frac{1}{2 \deg_\alpha} \log \sum_{i=1}^{N_\alpha} |p_{\alpha,i}|^2$$

and

$$K = \{z \in \mathbb{C}^\dim : \max_{1 \leq \alpha \leq A} (|p_{\alpha,1}|^2 + \cdots + |p_{\alpha,N_\alpha}|^2) \leq 1\}.$$  

A useful fact is $(dd^c(\max(u_{\alpha,1}, \ldots, u_{\alpha,i})))^N = 0$ on the set where the maximum is finite, whenever $N \geq N_{\alpha_1} + \cdots + N_{\alpha_\ell}$. This may be seen because (\star) involves sums of terms of the form $(dd^c u_{\alpha_1})^{\beta_1} \wedge \cdots \wedge (dd^c u_{\alpha_k})^{\beta_k}$ with $\beta_1 + \cdots + \beta_k = N + 1 - k$, which means that for some $i$, $\beta_i \geq N_{\alpha_i}$ and thus $(dd^c u_{\alpha_i})^{\beta_i} = 0$. 

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**Theorem 2.** Let us set $u := \max(0, u_1, \ldots, u_A)$. Suppose that $u \geq \log^+ |z| - C$ and that for every $J$ such that $E_J - K \neq \emptyset$ we have $\sum_{j \in J} N_j \leq \dim$. Then $u$ is the psh Green function for $K$. If the sets $E_J$ are smooth at points of $K$, then the equilibrium measure $\mu_K := (\frac{1}{2\pi} dd^c u)_{\dim}$ is given by the formula $(\ast)$.

**Proof.** We will show first that $u$ is the psh Green function of $K$. It is evident that $u$ is continuous and bounded above by $\log^+ |z| + C$. Since we also have the lower bound $\log^+ |z| - C$, it will suffice to show that $(dd^c u)_{\dim}$ vanishes on the complement of $K$. Suppose that $E_J - K$ contains a point $z_0$. Then $u = \max_{j \in J} u_j$ in a neighborhood of $z_0$. Replacing $u$ by $u^\epsilon := \max(\epsilon_0, u_1 + \epsilon_1, \ldots, u_A + \epsilon_A)$ for small $\epsilon_j$’s we may assume that $E^\epsilon_J = E_J(\epsilon_0, u_1 + \epsilon_1, \ldots, u_A + \epsilon_A)$ is smooth near $z_0$. We evaluate $(dd^c u)_{\dim}$. By the useful fact above, near $z_0$ we have $(dd^c u^\epsilon)^N = 0$ on $E^\epsilon_J - K$ for all $\epsilon \geq \sum_{j \in J} N_j$; thus $(dd^c u^\epsilon)_{\dim} = 0$. Letting $\epsilon \to 0$, we have $(dd^c u)_{\dim} = 0$ near $z_0$, and hence on $C_{\dim} - K$. The formula for $\mu_K$ then follows from Theorem 1.

Let $D^k$ denote the space of test forms of degree $k$. The currents of dimension $k$ are defined as the dual of $D^k$. Since we may decompose the $k$-forms into terms of bidegree $(p, q)$, $D^k = \bigoplus_{p+q=k} D^{p,q}$, each current may be written as a sum of currents of bidual dimension $(p, q)$. We have operators $\partial : D^{p,q} \to D^{p+1,q}$ and $\bar{\partial} : D^{p,q} \to D^{p,q+1}$; and their adjoints, which we denote again by $\partial$ and $\bar{\partial}$, act on the spaces of currents by duality. If $u$ is psh, then $u[E]$ is a current which has the same dimension as $E$, and $dd^c(u[E])$ is a positive, closed current. If $u$ is psh and continuous, we may define $(dd^c u)^n$ by induction on $n$ (cf. [1]). Specifically, since $(dd^c u)^n$ is positive, then it is represented by integration. It follows that $u(dd^c u)^n$ is a well-defined current, and we set $(dd^c u)^{n+1} := dd^c(u(dd^c u)^n)$; in other words, its action on a test form $\varphi$ is given by

$$\langle (dd^c u)^{n+1}, \varphi \rangle := \int dd^c \varphi \wedge u \wedge (dd^c u)^n.$$  

This definition gives a continuous extension of $(dd^c)^n$ to the continuous, psh functions.

Let $M \subset C_{\dim}$ be a smooth submanifold of locally finite volume. If $M$ has codimension $k$, then we may orient $M$ by choosing a simple $k$-form $\nu$ of unit length which annihilates the tangent space to $M$. We may define the current of integration $[M]$, which acts on a test form $\varphi$ according to the formula

$$\langle \varphi, [M] \rangle := \int * (\varphi \wedge \nu) ||\nu||^{-1} dS_M,$$

where $*$ is the Hodge $*$-operator taking volume form to a scalar function, and $dS_M$ the euclidean surface measure on $M$. Given a $k$-tuple $(\rho_1, \ldots, \rho_k)$ of defining functions, we define an orientation as follows. $C_{\dim}$ has a canonical orientation induced by its complex structure. We orient $M_1 = \{\rho_1 = 0\}$ as the boundary of $\{\rho_1 < 0\} \subset C_{\dim}$. Thus $\nu_1 = d\rho_1$. We orient $M_2 = \{\rho_1 = \rho_2 = 0\}$ as the boundary of $\{\rho_2 < 0\} \cap M_2$ inside $M_1$. Thus $\nu_2 = d\rho_1 \wedge d\rho_2$. Continuing this way, we orient $M$ using $\nu = d\rho_1 \wedge \cdots \wedge d\rho_k$.

A $(p, p)$ current $T$ is said to be positive if $\langle T, i\alpha_1 \wedge \bar{\alpha}_1 \wedge \cdots \wedge i\alpha_p \wedge \bar{\alpha}_p \rangle \geq 0$ for all smooth test forms $\alpha_j$ of type $(1, 0)$. Here we choose to orient $E_J$ so that $\delta^*_J \wedge [E_J]$ is positive, a choice which is justified by the following.
Lemma 1. We may orient \( E_J \) so that \( \delta_J^c \wedge [E_J] \) is a positive current. If we let \( E_J' \) denote \( E_J \) with the orientation given (as above) by taking successive boundaries in terms of the defining functions \((\rho_1 = u_{j_1} - u_{j_2}, \ldots, \rho_{\ell - 1} = u_{j_{\ell - 1}} - u_{j_{\ell}})\), then we have \([E_J] = (-1)^{\ell(\ell - 1)/2}[E_J']\).  

Proof. The \( k \)-form defining the orientation of \( E_J \) will be given by \( \pm d\rho_1 \wedge \cdots \wedge d\rho_{\ell - 1} \), with the sign \( \pm \) to be determined. In the notation above, we have

\[
\delta_J^c \wedge [E_J'] = d^c \rho_1 \wedge \cdots \wedge d^c \rho_{\ell - 1} \wedge d\rho_1 \wedge \cdots \wedge d\rho_{\ell - 1} || d\rho_1 \wedge \cdots \wedge d\rho_{\ell - 1} ||^{-1} dS,
\]

where \( dS \) is the euclidean surface measure on \( E_J \). Now since \( d\rho = \partial\rho + \bar{\partial}\rho \) and \( d^c\rho = i(\partial\rho - \bar{\partial}\rho) \) we see that

\[
(-1)^{\frac{d(\ell - 2)(\ell - 1)}{2}} 2^{1 - \ell} d^c \rho_1 \wedge \cdots \wedge d^c \rho_{\ell - 1} \wedge d\rho_1 \wedge \cdots \wedge d\rho_{\ell - 1}
= (i\partial\rho_1 \wedge \bar{\partial}\rho_1) \cdots (i\partial\rho_{\ell - 1} \wedge \bar{\partial}\rho_{\ell - 1}).
\]

It follows that we may orient \( E_J \) to make \( \delta_J^c \wedge [E_J] \) positive, and the relation between the orientations on \( E_J \) and \( E_J' \) is as claimed. \( \square \)

We will use the notation

\[
(4) \quad d^c_J = d^c_J(u_1, \ldots, u_m) := d^c u_{j_1} \wedge \cdots \wedge d^c u_{j_\ell}.
\]

For \( 1 \leq t \leq \ell \), we write \( J(t) \) to denote the set \( J \) with the \( t \)-th element removed, or if \( s \in J \), \( \hat{J}(s) \) denotes the set \( J \) with \( s \) removed. The meaning will be clear from the context.

Lemma 2. We have the following identities:

\begin{enumerate}
\item \( d(d^c_J(u)) = \sum_{t=1}^\ell (-1)^{t-1} d^c_{J(t)}(u) \wedge d^c_J u_t \).
\item \( d^c_{J(t)}(u) \wedge \delta_J^c(u) = (-1)^{t-1} d_J^c(u) \) for any \( 1 \leq t \leq \ell \).
\item \( \sum_{t=1}^\ell (-1)^{\ell-t} d^c_{J(t)} = \delta_J^c \).
\item \( dd^c_{J(t)} \wedge \sigma^n_j + \sigma^{n+1}_{J(t)} = \sigma^{n+1}_j \).
\end{enumerate}

Proof. These identities follow from the product rule and anticommutation of 1-forms. \( \square \)

Lemma 3. If \( \alpha \) is a smooth form of type \((a, a)\) and if \( \beta \) is a smooth form of type \((b, b)\), then the forms \( d\alpha \wedge d^c \beta \) and \( d\beta \wedge d^c \alpha \) have the same parts of type \((a + b + 1, a + b + 1)\).  

Proof. Expand \( d\alpha \wedge d^c \beta \) into terms of the form \( \partial\alpha \wedge \bar{\partial}\beta \), etc., and compare bidegrees. \( \square \)

Lemma 4. \( d[E_J] = [\partial E_J] = \sum_J e^c_J[E_J] \), where the sum is taken over all \( \hat{J} \) such that \( J \subset \hat{J} \subset \{1, \ldots, m\} \) and \( |\hat{J}| = |J| + 1 \). For each such \( \hat{J} \), there is an \( s \) such that \( j_1 < \cdots < j_k < s < j_{k+1} < \cdots < j_\ell \) and \( J = \hat{J}(s) \), and we have \( e^c_J = (-1)^k \).

Proof. By Stokes' Theorem, we have \( d[E_J] = [\partial E_J] = \sum[E'_J] \), where \( E'_J \) denotes the manifold \( E_J \) with the induced boundary orientation on \( \partial E_J \). Thus we need to compare the orientations of \( E'_J \) and \( E_J \). As in the discussion before Lemma 1, the orientations of \( E_J \) and \( E'_J \) are given by the defining functions \((u_{j_2} - u_{j_1}, \ldots, u_{j_\ell} - u_{j_{\ell-1}}) \) and \((u_{j_2} - u_{j_1}, \ldots, u_{j_{\ell-1}} - u_{j_{\ell-1}}, \rho) \), respectively, where we may take \( \rho \) to be either \( u_s - u_{j_{k+1}} \) or \( u_s - u_{j_k} \). By Lemma 1, the orientation of \( E_J \) is given by

\[
u_J = (-1)^{\ell(\ell - 1)/2} A \wedge d(u_{j_{k+1}} - u_{j_k}) \wedge B,
\]
where $A = d(u_{j_2} - u_{j_1}) \wedge \cdots \wedge d(u_{j_k} - u_{j_{k-1}})$ and $B = d(u_{j_{k+2}} - u_{j_{k+1}}) \wedge \cdots \wedge d(u_{j_{\ell}} - u_{j_{\ell-1}})$. Thus the orientation of $E'_j$ is given by the form

$$
\nu'_j = (-1)^{\ell+1/2}A \wedge d(u_{j_{k+1}} - u_{j_k}) \wedge B \wedge d(u_s - u_{j_k}).
$$

By Lemma 1 again, the orientation of $E_j$ is given by the form

$$
\nu_j = (-1)^{(\ell+1)}\nu_j
$$

Since the degree of $B$ is $\ell - (k + 1)$, we find that $\nu'_j = (-1)^k \nu_j$, which completes the proof. \hfill \Box

**Proposition 1.** Let $M$ be a smooth submanifold with boundary, and let $\chi$ be a smooth form on $E$ so that $\chi \wedge [M]$ is a current of bidimension $(p, p)$. If $v$ is a smooth function and $\phi$ is a smooth form of bidegree $(p-1, p-1)$, then

$$
\int_M v \wedge \chi \wedge dd^{\phi} = \int_M d(d^{\phi}v \wedge \chi) \wedge \phi - \int_{\partial M} d^{\phi}v \wedge \chi \wedge \phi + \int_M \chi \wedge d(v \wedge d^{\phi}).
$$

**Proof.** By the product rule, we have

$$
\int_M v \wedge \chi \wedge dd^{\phi} = \int_M \chi \wedge (d(v \wedge d^{\phi}) - dv \wedge d^{\phi}).
$$

Note that $v$ is a $(0, 0)$-form, $\phi$ is of bidegree $(p-1, p-1)$, and $\chi \wedge [M]$ is a current of bidimension $(p, p)$. Thus the only nonzero terms integrated against this current can come from $(p, p)$-forms. Thus by Lemma 3 we may replace $dv \wedge d^{\phi} \phi$ by $d\phi \wedge d^{\phi}v$ in the right-hand integral. Now we integrate by parts in the right-hand integral to obtain the desired formula. \hfill \Box

**Proposition 2.** We have

$$
dd^{\phi}(u \wedge \delta^c_j \wedge [E_j]) = (-1)^{k+1}(d(d^{\phi}u_j)(u) \wedge [E_j] - d^{c_j}(u) \wedge [\partial E_j])\wedge d^{\phi}(u \wedge d[\delta^c_j \wedge [E_j]]).
$$

**Proof.** Let us note first that $d(\delta^c_j \wedge [E_j]) = d(\delta^c_j) \wedge [E_j] + (-1)^{\ell-1}\delta^c_j \wedge [\partial E_j]$, so this current is represented by integration. Thus, since $u = u_j$ for all $j$ on the sets $E_j$ and $\partial E_j$, the current $u_j \wedge d(\delta^c_j \wedge [E_j]) = u \wedge d(\delta^c_j \wedge [E_j])$ is the same for all $j \in J$. Similarly, we may substitute $u_j$ for $u$ in the left-hand term of the equation.

By Lemma 1, $\delta^c_j \wedge [E_j]$ is a current of bidegree $([J], [J])$. Thus we evaluate the left-hand term by testing it against a form of type $(\dim - |J| - 1, \dim - |J| - 1)$:

$$
\langle \phi, dd^{\phi}(u \wedge \delta^c_j \wedge [E_j]) \rangle = \langle dd^{\phi}\phi, u \wedge \delta^c_j \wedge [E_j] \rangle.
$$

Now we apply Proposition 1 with $v = u_j$, $\chi = \delta^c_j$ and $M = E_j$ to obtain

$$
\int_{E_j} u_j \wedge \delta^c_j \wedge dd^{\phi} \phi = \int_{E_j} d(d^{\phi}u_j \wedge \delta^c_j) \wedge \phi - \int_{\partial E_j} d^{\phi}u_j \wedge \delta^c_j \wedge \phi + \int_{E_j} \delta^c_j \wedge d(u_j \wedge d^{\phi}).
$$

Our formula now follows by applying Lemma 2(2) to the first and second integrals, rewriting the terms as currents, then substituting $u_j$ for $u_j$. \hfill \Box

**Proof of Theorem 1.** In the case $n = 1$, we have $J = \{j\}$ and $\delta^c_j = 1$, so the statement of Proposition 2 becomes

$$
dd^{\phi}(u \wedge [E_j]) = dd^{\phi}(u_j \wedge [E_j]) = dd^{\phi}u_j \wedge [E_j] - d^{c_j}u_j \wedge [\partial E_j] - d^{\phi}(u \wedge d[E_j]).
$$

We identify $dd^{\phi}u$ with $dd^{\phi}u \wedge [E] = dd^{\phi}(\sum [E_j])$, which gives

$$
dd^{\phi}u \wedge [E] = \sum_j dd^{\phi}u_j \wedge [E_j] + \sum_{j_1 < j_2} \delta^c_{j_1,j_2} \wedge [E_{j_1,j_2}],
$$

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where the second sum is a consequence of Lemmas 4 and 2(3) applied to $\sum d^c u_j \wedge [\partial E_j]$, and the other terms vanish since $d(\sum [E_j]) = d[E] = 0$.

Now we proceed by induction, assuming that Theorem 1 has been proved for $n$. Then we have

$$(dd^c u)^{n+1} = dd^c(u \wedge (dd^c u)^n) = dd^c \left( u \sum_j \sigma_j^{n-|J|+1} \wedge \delta_j \wedge [E_j] \right).$$

Since $\sigma_j^{n-|J|+1}$ is an even form which is both $d$- and $d^c$-closed, this expression is

$$= \sum_j \sigma_j^{n-|J|+1} \wedge dd^c(u \wedge \delta_j \wedge [E_j]).$$

We apply Proposition 2 to obtain

$$= \sum_j \sigma_j^{n-|J|+1} \wedge \left( (-1)^{|J|+1} \left( d(d^c(u)) \wedge [E_j] - d^c_j \wedge [\partial E_j] \right) - \left\{ d^c(u \wedge d(\delta_j \wedge [E_j])) \right\} \right) = I + II.$$  

Since $d^c \sigma_j^{n-|J|+1} = 0$, we have

$$II = -d^c \sum u d \left( \sigma_j^{n-|J|+1} \wedge \delta_j \wedge [E_j] \right).$$

Then, by our induction hypothesis,

$$II = -d^c (u \wedge d((dd^c u)^n)) = -d^c (u \wedge 0) = 0.$$

Now use Lemma 2(1) in the left-hand summation in $I$ to obtain

$$(dd^c u)^{n+1} = \sum_j \sigma_j^{n-|J|+1} \wedge (-1)^{|J|+1} \sum_{t=1}^{|J|} (-1)^{t-1} dd^c u_j \wedge d^c_{J(t)} \wedge [E_j]$$

$$- \sum_{t=1}^{|J|} (-1)^{|J|+1} \sigma_j^{n-|J|+1} \wedge d^c_j \wedge [\partial E_j] = A + B.$$

In the notation of Lemma 4, we have

$$B = \sum_j \sigma_j^{n-|J|+1} \wedge (-1)^{|J|} d^c_j \wedge \sum_j c^J_{[E_j]}.$$

Now let us rewrite $B$, summing over $\tilde{J}$ on the outside, and summing over subsets $J = \tilde{J}(\hat{s})$ on the inside. By Lemma 4, $c^J_{[E]} = (-1)^{|J|} d^c_j \wedge \sum_j \sigma_j^{n-|J|+2} \wedge \delta_j \wedge [E_j]$. This gives

$$B = \sum_j \left( \sum_{s=1}^{\tilde{|J|}} (-1)^{|J|-s} \sigma_j^{n-|J|+2} \wedge d^c_{J(s)} \right) \wedge [E_j].$$

Since we are summing over all subsets $\tilde{J}$, we can remove the tilde from $\tilde{J}$. Further, we can set $s = t$, which lets us rewrite $A + B$ as

$$\sum_j \sum_{t} \left( \sigma_j^{n-|J|+1} \wedge dd^c u_j + \sigma_j^{n-|J|+2} \right) \wedge d^c_{J(t)} \wedge (-1)^{|J|+t} d^c_{J(t)} \wedge [E_j].$$

Finally, by Lemma 2(3) and (4), we have

$$(dd^c u)^{n+1} = \sum_j \sigma_j^{n-|J|+2} \wedge \left( \sum_{t} (-1)^{|J|+t} d^c_{J(t)} \right) \wedge [E_j] = \sum_j \sigma_j^{n-|J|+2} \wedge \delta_j \wedge [E_j],$$

which completes the proof.  \qed
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