COMPLEX MONGE-AMPÈRE OF A MAXIMUM

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(Communicated by Mei-Chi Shaw)

Abstract. We derive a formula for \((dd^c u)^n\) where \(u = \max_j u_j\) is a finite maximum. As an application, we compute the complex equilibrium measures of some generalized polyhedra.

Plurisubharmonic (psh) functions play a primary role in pluripotential theory. They are closely related to the operator \(dd^c = 2i\partial\bar{\partial}\) (with notation \(d = \partial + \bar{\partial}\) and \(d^c = i(\bar{\partial} - \partial)\)), which serves as a generalization of the Laplacian from \(\mathbb{C}\) to \(\mathbb{C}^{\text{dim}}\) for \(\text{dim} > 1\). If \(u\) is smooth of class \(C^2\), then for \(1 \leq n \leq \text{dim}\), the coefficients of the exterior power \((dd^c u)^n\) are given by the \(n \times n\) subdeterminants of the matrix \((\partial^2 u/\partial z_i \partial \bar{z}_j)\). The top exterior power corresponds to \(n = \text{dim}\), and in this case we have the determinant of the full matrix, which gives the complex Monge-Ampère operator. The extension of the (nonlinear) operator \((dd^c)^n\) to nonsmooth functions has been studied by several authors (see, for instance, [3]). Here it will suffice to define \((dd^c)^n\) on psh functions which are continuous.

For an open set \(E \subset \mathbb{C}^{\text{dim}}\), the set of psh functions on \(E\) forms a cone which is closed under the operations of addition and of taking finite maxima. The relation between \((dd^c)^n\) and the additive structure is given by the formula

\[
(dd^c(u_1 + u_2))^n = \sum_{n_1 + n_2 = n} \frac{n!}{n_1!n_2!} (dd^c u_1)^{n_1} \wedge (dd^c u_2)^{n_2},
\]

which holds in both the classical and generalized sense. Here we make the connection between \((dd^c)^n\) and the operation of taking a finite maximum. A few examples of this are known already: if \(u = \max(0, \log(|z_1|^2 + \cdots + |z_k|^2))\), then \((dd^c u)^k\) is a multiple of surface measure on \(\{|z_1|^2 + \cdots + |z_k|^2 = 1\}\). And if \(u = \max(0, x_1, \ldots, x_k)\), then \((dd^c u)^k\) is a multiple of surface measure on \(\{x_1 = \cdots = x_k = 0\}\). The case \((dd^c \max(u_1, u_2, u_3))^2\), \(u_j\)’s pluriharmonic, is given in [4]. A different sort of formula for \((dd^c \max(u_1, u_2))^k\) is given in [2], which, as pointed out by the referee, follows easily from the formula of this paper.

More generally, let the functions \(u_1, \ldots, u_m\) be smooth, and let

\[
u := \max(u_1, \ldots, u_m)
\]

Received by the editors June 9, 2006.

2000 Mathematics Subject Classification. Primary 32U40, 32W20; Secondary 58C35.

Key words and phrases. Current, measure, Monge-Ampère, plurisubharmonic.

The first author was supported in part by the NSF.

The second author was supported by a New Zealand Science and Technology Post-Doctoral fellowship, contract no. IDNA0401.
be their maximum. It follows that \( dd^c u_j \) and \( dd^c u \) are locally bounded below, and for the purpose of defining \((dd^c)^n\), \( u_j \) and \( u \) may be treated as \( \text{psh} \) functions. Taking the maximum stratifies \( E \) as follows: for each \( J \subseteq \{1, \ldots, m\} \) there is

\[
E_J = E_J(u_1, \ldots, u_m) := \{ z \in E : u(z) = u_j(z) \ \forall j \in J, \ u(z) > u_i(z) \ \forall i \notin J \}.
\]

The sets \( E_J \) form a partition of \( E \), and as \( J \) increases, the subsequent \( E_J \)'s lie inside the “boundaries” of the previous ones. Since \((dd^c u)^n\) is representable by integration, we may decompose it into a sum over the elements of the partition

\[
(dd^c u)^n = \sum_J (dd^c u)^n|_{E_J}.
\]

The first terms in (1) are easy to identify: \( E_j \) is open and thus \((dd^c u)^n|_{E_j} = (dd^c u_j)^n|_{E_j}\). For the rest of the terms, we write \( J = \{j_1, \ldots, j_\ell\} \), \( j_1 < \cdots < j_\ell \), so \( |J| = \ell \) is the number of elements, and we define the forms

\[
\delta^c_j = \delta^c_j(u_1, \ldots, u_m) := d^c(u_{j_1} - u_{j_2}) \wedge \cdots \wedge d^c(u_{j_\ell-1} - u_{j_\ell}),
\]

\[
\sigma^c_j = \sigma^c_j(dd^c u_1, \ldots, dd^c u_m) := \sum_{\beta_1 + \cdots + \beta_\ell = n} (dd^c u_{j_1})^{\beta_1} \wedge \cdots \wedge (dd^c u_{j_\ell})^{\beta_\ell}.
\]

If \( E_J \) is smooth, we let \([E_J]\) denote the current of integration over \( E_J \), where we orient \( E_J \) so that the current \( \delta^c_j \wedge [E_J] \) is positive (see Lemma 1). In this paper we identify the terms of the summation (1) as integrations on the strata \( E_J \):

**Theorem 1.** Let \( u_j \in C^3 \), \( 1 \leq j \leq m \), be given, and set \( u = \max(u_1, \ldots, u_m) \). If all of the sets \( E_J \) are smooth, then

\[ (dd^c u)^n = \sum_J \sigma^c_j^{n-|J|+1} \wedge \delta^c_j \wedge [E_J], \]

where the sum is taken over all \( J \subseteq \{1, \ldots, m\} \) with \( 1 \leq |J| \leq n + 1 \).

A function that arises frequently is \( v = \log(|f_1|^2 + \cdots + |f_N|^2) \), where the \( f_i \)'s are holomorphic. Set \( f = (f_1, \ldots, f_N) : \mathbb{C}^\dim \rightarrow \mathbb{C}^N \) and let \( \pi : \mathbb{C}^N - 0 \rightarrow \mathbb{P}^{N-1} \) be the projection. The powers \((dd^c v)^n\) may be determined on \( \{ v > -\infty \} \) using the fact that \( dd^c \log(|z_1|^2 + \cdots + |z_N|^2) = \pi^* \omega_{FS} \), where \( \omega_{FS} \) denotes the Fubini-Study K"ahler form on \( \mathbb{P}^{N-1} \). That is, \((dd^c v)^n\) is the pullback of \( \omega_{FS}^n \) under \((\pi \circ f)^* \). In particular, \((dd^c v)^n = 0 \) if \( n \geq N \).

Such functions and their maxima arise naturally with generalized polyhedra. For \( 1 \leq \alpha \leq A \), let \( p_{\alpha,1}, \ldots, p_{\alpha,N_\alpha} \) be polynomials. Let \( \deg\alpha = \max_{1 \leq i \leq N_\alpha} (\deg(p_{\alpha,i})) \) be the maximum of the degrees. Define

\[
u_{\alpha} = \frac{1}{2\deg\alpha} \log \sum_{i=1}^{N_\alpha} |p_{\alpha,i}|^2
\]

and

\[
K = \{ z \in \mathbb{C}^\dim : \max_{1 \leq \alpha \leq A} (|p_{\alpha,1}|^2 + \cdots + |p_{\alpha,N_\alpha}|^2) \leq 1 \}.
\]

A useful fact is \((dd^c(\max(u_{\alpha_1}, \ldots, u_{\alpha_k}))^N = 0 \) on the set where the maximum is finite, whenever \( N \geq N_{\alpha_1} + \cdots + N_{\alpha_k} \). This may be seen because (\( \ast \)) involves sums of terms of the form \((dd^c u_{\alpha_i})^{\beta_i} \wedge \cdots \wedge (dd^c u_{\alpha_k})^{\beta_k} \) with \( \beta_1 + \cdots + \beta_k = N + 1 - k \), which means that for some \( i, \beta_i \geq N_{\alpha_i} \) and thus \((dd^c u_{\alpha_i})^{\beta_i} = 0 \).
Theorem 2. Let us set \( u := \max(0, u_1, \ldots, u_A) \). Suppose that \( u \geq \log^+ |z| - C \) and that for every \( J \) such that \( E_J - K \neq \emptyset \) we have \( \sum_{j \in J} N_j \leq \dim \). Then \( u \) is the psh Green function for \( K \). If the sets \( E_J \) are smooth at points of \( K \), then the equilibrium measure \( \mu_K := (\frac{1}{2\pi} dd^c u)^{\dim} \) is given by the formula (\( * \)).

Proof. We will show first that \( u \) is the psh Green function of \( K \). It is evident that \( u \) is continuous and bounded above by \( \log^+ |z| + C \). Since we also have the lower bound \( \log^+ |z| - C \), it will suffice to show that \((dd^c u)^{\dim}\) vanishes on the complement of \( K \). Suppose that \( E_J - K \) contains a point \( z_0 \). Then \( u = \max_{j \in J} u_j \) in a neighborhood of \( z_0 \). Replacing \( u \) by \( u^\epsilon := \max(\epsilon_0, u_1 + \epsilon_1, \ldots, u_A + \epsilon_A) \) for small \( \epsilon_j \)'s we may assume that \( E^\epsilon_J = E_J(\epsilon_0, u_1 + \epsilon_1, \ldots, u_A + \epsilon_A) \) is smooth near \( z_0 \). We evaluate \((dd^c u)^{\dim}\). By the useful fact above, near \( z_0 \) we have \((dd^c u^\epsilon)^N = 0\) on \( E^\epsilon_J - K \) for all \( \bar{J} \subset J \) if \( N \geq \sum_{j \in \bar{J}} N_j \); thus \((dd^c u^\epsilon)^{\dim} = 0\). Letting \( \epsilon \to 0 \), we have \((dd^c u)^{\dim} = 0 \) near \( z_0 \), and hence on \( \mathbb{C}^{\dim} - K \). The formula for \( \mu_K \) then follows from Theorem 1. \( \square \)

Let \( D^k \) denote the space of test forms of degree \( k \). The currents of dimension \( k \) are defined as the dual of \( D^k \). Since we may decompose the \( k \)-forms into terms of bidegree \( (p, q) \), \( D^k = \bigoplus_{p+q=k} D^{p,q} \), each current may be written as a sum of currents of bideimension \( (p, q) \). We have operators \( \partial : D^{p,q} \to D^{p+1,q} \) and \( \bar{\partial} : D^{p,q} \to D^{p,q+1} \); and their adjoints, which we denote again by \( \partial \) and \( \bar{\partial} \), act on the spaces of currents by duality. If \( u \) is psh, then \( u[E] \) is a current which has the same dimension as \( E \), and \( dd^c(u[E]) \) is a positive, closed current. If \( u \) is psh and continuous, we may define \((dd^c u)^n\) by induction on \( n \) (cf. [1]). Specifically, since \((dd^c u)^1\) is positive, then it is represented by integration. It follows that \( u(dd^c u)^n\) is a well-defined current, and we set \((dd^c u)^{n+1} := dd^c(u(dd^c u)^n)\); in other words, its action on a test form \( \varphi \) is given by

\[
\langle (dd^c u)^{n+1}, \varphi \rangle := \int dd^c \varphi \wedge u \wedge (dd^c u)^n.
\]

This definition gives a continuous extension of \((dd^c)^n\) to the continuous, psh functions.

Let \( M \subset \mathbb{C}^{\dim} \) be a smooth submanifold of locally finite volume. If \( M \) has codimension \( k \), then we may orient \( M \) by choosing a simple \( k \)-form \( \nu \) of unit length which annihilates the tangent space to \( M \). We may define the current of integration \([M]\), which acts on a test form \( \varphi \) according to the formula

\[
\langle \varphi, [M] \rangle := \int \ast(\varphi \wedge \nu) \|\nu\|^{-1} dS_M,
\]

where \( \ast \) is the Hodge \( \ast \)-operator taking volume form to a scalar function, and \( dS_M \) the euclidean surface measure on \( M \). Given a \( k \)-tuple \( (\rho_1, \ldots, \rho_k) \) of defining functions, we define an orientation as follows. \( \mathbb{C}^{\dim} \) has a canonical orientation induced by its complex structure. We orient \( M_1 = \{\rho_1 = 0\} \) as the boundary of \( \{\rho_1 < 0\} \subset \mathbb{C}^{\dim} \). Thus \( \nu_1 = d\rho_1 \). We orient \( M_2 = \{\rho_1 = \rho_2 = 0\} \) as the boundary of \( \{\rho_2 < 0\} \cap M_2 \) inside \( M_1 \). Thus \( \nu_2 = d\rho_1 \wedge d\rho_2 \). Continuing this way, we orient \( M \) using \( \nu = d\rho_1 \wedge \cdots \wedge d\rho_k \).

A \((p,p)\) current \( T \) is said to be positive if \( \langle T, i\alpha_1 \wedge \bar{\alpha}_1 \wedge \cdots \wedge i\alpha_p \wedge \bar{\alpha}_p \rangle \geq 0 \) for all smooth test forms \( \alpha_j \) of type \((1,0)\). Here we choose to orient \( E_J \) so that \( \delta_J \wedge [E_J] \) is positive, a choice which is justified by the following.
Lemma 1. We may orient $E_J$ so that $\delta_j^* \wedge [E_J]$ is a positive current. If we let $E'_J$ denote $E_J$ with the orientation given (as above) by taking successive boundaries in terms of the defining functions $(\rho_1 = u_{j_1} - u_{j_2}, \ldots, \rho_{\ell-1} = u_{j_{\ell-1}} - u_{j_\ell})$, then we have $[E_J] = (-1)^{\ell(\ell-1)/2}[E'_J]$.

Proof. The $k$-form defining the orientation of $E_J$ will be given by $\pm d\rho_1 \wedge \cdots \wedge d\rho_{\ell-1}$, with the sign $\pm$ to be determined. In the notation above, we have

$$\delta_j^* \wedge [E'_J] = d^c \rho_1 \wedge \cdots \wedge d^c \rho_{\ell-1} \wedge d\rho_1 \wedge \cdots \wedge d\rho_{\ell-1} \lvert [d\rho_1 \wedge \cdots \wedge d\rho_{\ell-1}]^{-1} dS,$$

where $dS$ is the euclidean surface measure on $E_J$. Now since $d\rho = \partial \rho + \tilde{\partial} \rho$ and $d^c \rho = i(\partial \rho - \tilde{\partial} \rho)$ we see that

$$(-1)^{\ell(\ell-1)/2} 2^{1-\ell} d^c \rho_1 \wedge \cdots \wedge d^c \rho_{\ell-1} \wedge d\rho_1 \wedge \cdots \wedge d\rho_{\ell-1} = (i\partial \rho_1 \wedge \tilde{\partial} \rho_1) \wedge \cdots \wedge (i\partial \rho_{\ell-1} \wedge \tilde{\partial} \rho_{\ell-1}).$$

It follows that we may orient $E_J$ to make $\delta_j^* \wedge [E_J]$ positive, and the relation between the orientations on $E_J$ and $E'_{J}$ is as claimed. \qed

We will use the notation

$$(4) \quad d^c_j = d^c_j(u_1, \ldots, u_m) := d^c(u_{j_1} \wedge \cdots \wedge d^c u_{j_\ell}).$$

For $1 \leq t \leq \ell$, we write $J(t)$ to denote the set $J$ with the $t$-th element removed, or if $s \in J$, $J(s)$ denotes the set $J$ with $s$ removed. The meaning will be clear from the context.

Lemma 2. We have the following identities:

1. $d(d^c_J(u)) = \sum_{t=1}^\ell (-1)^{t-1} d^c_{J(t)}(u) \wedge d^c u_{j_t}$.
2. $d^c u_{j_t} \wedge \delta_j^* (u) = (-1)^{t-1} d^c_J (u)$ for any $1 \leq t \leq \ell$.
3. $\sum_{t=1}^\ell (-1)^{t-\ell} d^c_{J(t)} = \delta_j^*.$
4. $dd^c u_{j_t} \wedge \sigma^n_j + \sigma^{n+1}_J = \sigma^n_{J(sl)}.$

Proof. These identities follow from the product rule and anticommutation of 1-forms. \qed

Lemma 3. If $\alpha$ is a smooth form of type $(a, a)$ and if $\beta$ is a smooth form of type $(b, b)$, then the forms $d\alpha \wedge d^c \beta$ and $d\beta \wedge d^c \alpha$ have the same parts of type $(a + b + 1, a + b + 1)$.

Proof. Expand $d\alpha \wedge d^c \beta$ into terms of the form $\partial \alpha \wedge \tilde{\partial} \beta$, etc., and compare bidegrees. \qed

Lemma 4. $d[E_J] = [\partial E_J] = \sum_{\tilde{J}} \epsilon^J_{\tilde{J}} [E_{\tilde{J}}]$, where the sum is taken over all $\tilde{J}$ such that $J \subset \tilde{J} \subset \{1, \ldots, m\}$ and $|\tilde{J}| = |J| + 1$. For each such $\tilde{J}$, there is an $s$ such that $j_1 < \cdots < j_k < s < j_{k+1} < \cdots < j_\ell$ and $J = \tilde{J}(s)$, and we have $\epsilon^J_{\tilde{J}} = (-1)^k$.

Proof. By Stokes’ Theorem, we have $d[E_J] = [\partial E_J] = \sum [E'_{\tilde{J}}]$, where $E'_{\tilde{J}}$ denotes the manifold $E_{\tilde{J}}$ with the induced boundary orientation on $\partial E_{\tilde{J}}$. Thus we need to compare the orientations of $E'_{\tilde{J}}$ and $E_J$. As in the discussion before Lemma 1, the orientations of $E_J$ and $E'_{\tilde{J}}$ are given by the defining functions $(u_{j_2} - u_{j_1}, \ldots, u_{j_k} - u_{j_{k-1}})$ and $(u_{j_2} - u_{j_1}, \ldots, u_{j_k} - u_{j_{k-1}}, \rho)$, respectively, where we may take $\rho$ to be either $u_s - u_{j_{k+1}}$ or $u_s - u_{j_k}$. By Lemma 1, the orientation of $E_J$ is given by

$$\nu_J = (-1)^{\ell(\ell-1)/2} A \wedge d(u_{j_{k+1}} - u_{j_k}) \wedge B,$$
where $A = d(u_{j_2} - u_{j_1}) \wedge \cdots \wedge d(u_{j_k} - u_{j_{k-1}})$ and $B = d(u_{j_{k+2}} - u_{j_{k+1}}) \wedge \cdots \wedge d(u_{j_\ell} - u_{j_{\ell-1}})$. Thus the orientation of $E'_j$ is given by the form

$$\nu'_j = (-1)^{\ell(\ell-1)/2} A \wedge d(u_{j_{k+1}} - u_{j_k}) \wedge B \wedge d(u_s - u_{j_k}).$$

By Lemma 1 again, the orientation of $E_j$ is given by the form

$$\nu_j = (-1)^{\ell(\ell+1)/2} A \wedge d(u_s - u_{j_k}) \wedge d(u_{j_{k+1}} - u_s) \wedge B.$$

Since the degree of $B$ is $\ell - (k + 1)$, we find that $\nu'_j = (-1)^k \nu_j$, which completes the proof.

**Proposition 1.** Let $M$ be a smooth submanifold with boundary, and let $\chi$ be a smooth form on $E$ so that $\chi \wedge [M]$ is a current of bidimension $(p, p)$. If $v$ is a smooth function and $\phi$ is a smooth form of bidegree $(p - 1, p - 1)$, then

$$\int_M v \wedge \chi \wedge dd^c \phi = \int_M d(\delta^c v \wedge \chi) \wedge \phi - \int_{\partial M} \delta^c v \wedge \chi \wedge \phi + \int_M \chi \wedge d(v \wedge d^c \phi).$$

**Proof.** By the product rule, we have

$$\int_M v \wedge \chi \wedge dd^c \phi = \int_M \chi \wedge (d(v \wedge d^c \phi) - dv \wedge d^c \phi).$$

Note that $v$ is a $(0, 0)$-form, $\phi$ is of bidegree $(p - 1, p - 1)$, and $\chi \wedge [M]$ is a current of bidimension $(p, p)$. Thus the only nonzero terms integrated against this current can come from $(p, p)$-forms. Thus by Lemma 3 we may replace $dv \wedge d^c \phi$ by $d\phi \wedge d^c v$ in the right-hand integral. Now we integrate by parts in the right-hand integral to obtain the desired formula.

**Proposition 2.** We have

$$dd^c(u \wedge \delta^c_j \wedge [E_j]) = (-1)^{k+1} (d(d_j^c(u)) \wedge [E_j] - d_j^c(u) \wedge [\partial E_j]) - d^c(u \wedge d(\delta^c_j \wedge [E_j])).$$

**Proof.** Let us note first that $d(\delta^c_j \wedge [E_j]) = d(\delta^c_j) \wedge [E_j] + (-1)^{k-1} \delta^c_j \wedge [\partial E_j]$, so this current is represented by integration. Thus, since $u = u_j$ for all $j$ on the sets $E_j$ and $\partial E_j$, the current $u_j \wedge d(\delta^c_j \wedge [E_j]) = u \wedge d(\delta^c_j \wedge [E_j])$ is the same for all $j \in J$. Similarly, we may substitute $u_{j'}$ for $u$ in the left-hand term of the equation.

By Lemma 1, $\delta^c_j \wedge [E_j]$ is a current of bidegree $(|J|, |J|)$. Thus we evaluate the left-hand term by testing it against a form of type $(\dim - |J| - 1, \dim - |J| - 1)$:

$$\langle \phi, dd^c(u \wedge \delta^c_j \wedge [E_j]) \rangle = \langle dd^c \phi, u \wedge \delta^c_j \wedge [E_j] \rangle.$$

Now we apply Proposition 1 with $v = u_j$, $\chi = \delta^c_j$ and $M = E_j$ to obtain

$$\int_{E_j} u_j \wedge \delta^c_j \wedge dd^c \phi = \int_{E_j} d(d^c u_j \wedge \delta^c_j) \wedge \phi - \int_{\partial E_j} d^c u_j \wedge \delta^c_j \wedge \phi + \int_{E_j} \delta^c_j \wedge d(u_j \wedge d^c \phi).$$

Our formula now follows by applying Lemma 2(2) to the first and second integrals, rewriting the terms as currents, then substituting $u$ for $u_j$.

**Proof of Theorem 1.** In the case $n = 1$, we have $J = \{j\}$ and $\delta^c_j = 1$, so the statement of Proposition 2 becomes

$$dd^c(u \wedge [E_j]) = dd^c(u_j \wedge [E_j]) = dd^c u_j \wedge [E_j] - d^c u_j \wedge [\partial E_j] - d^c u \wedge [E].$$

We identify $dd^c u$ with $dd^c u \wedge [E] = dd^c(\sum [E_j])$, which gives

$$dd^c u \wedge [E] = \sum_j dd^c u_j \wedge [E] + \sum_{j_1 < j_2} \delta^c_{j_1,j_2} \wedge [E_{j_1,j_2}],$$
where the second sum is a consequence of Lemmas 4 and 2(3) applied to \( \sum d^c u_j \wedge [\partial E_j] \), and the other terms vanish since \( d(\sum [E_j]) = d[E] = 0 \).

Now we proceed by induction, assuming that Theorem 1 has been proved for \( n \). Then we have

\[
(dd^c u)^{n+1} = dd^c (u \wedge (dd^c u)^n) = dd^c \left( u \sum_j \sigma_j^{n-|J|+1} \wedge \delta_j \wedge [E_j] \right).
\]

Since \( \sigma_j^{n-|J|+1} \) is an even form which is both \( d \)- and \( d^c \)-closed, this expression is

\[
eq \sum_j \sigma_j^{n-|J|+1} \wedge \delta_j \wedge [E_j].
\]

We apply Proposition 2 to obtain

\[
= \sum_j \sigma_j^{n-|J|+1} \wedge \left( (-1)^{|J|+1} \{ d(d_j^c(u)) \wedge [E_j] - d_j^c(u) \wedge [\partial E_j] \} \right)
\]

\[
- \left\{ d^c(u \wedge d(\delta_j \wedge [E_j])) \right\} = I + II.
\]

Since \( d^c \sigma_j^{n-|J|+1} = 0 \), we have

\[
II = -d^c \sum u(d \sigma_j^{n-|J|+1} \wedge \delta_j \wedge [E_j]).
\]

Then, by our induction hypothesis,

\[
II = -d^c(u \wedge d((dd^c u)^n)) = -d^c(u \wedge 0) = 0.
\]

Now use Lemma 2(1) in the left-hand summation in \( I \) to obtain

\[
(dd^c u)^{n+1} = \sum \sigma_j^{n-|J|+1} \wedge (-1)^{|J|+1} \sum_{t=1}^{|J|} (-1)^{t-1} dd^c u_j(t) \wedge d_j^c(t) \wedge [E_j]
\]

\[
- \sum (-1)^{|J|+1} \sigma_j^{n-|J|+1} \wedge d_j^c \wedge [\partial E_j] = A + B.
\]

In the notation of Lemma 4, we have

\[
B = \sum_j \sigma_j^{n-|J|+1} \wedge (-1)^{|J|} d_j^c \wedge \sum_j c_j^J [E_j].
\]

Now let us rewrite \( B \), summing over \( \tilde{J} \) on the outside, and summing over subsets \( J = \tilde{J}(\tilde{s}) \) on the inside. By Lemma 4, \( c_{\tilde{s}}^{J(\tilde{s})} = (-1)^{s-1} \), \( 1 \leq s \leq |\tilde{J}| \). This gives

\[
B = \sum \left( \sum_{s=1}^{|\tilde{J}|} (-1)^{|\tilde{J}|-s} \sigma_j^{n-|J|+2} \wedge d_j^c(t) \wedge [E_j].
\]

Since we are summing over all subsets \( \tilde{J} \), we can remove the tilde from \( \tilde{J} \). Further, we can set \( s = t \), which lets us rewrite \( A + B \) as

\[
\sum \sum \left( \sigma_j^{n-|J|+1} \wedge dd^c u_j(s) + \sigma_j^{n-|J|+2} \right) \wedge d_j^c(s(t)) \wedge (\tilde{J})^{|J|} \wedge d_j^c(t) \wedge [E_j].
\]

Finally, by Lemma 2(3) and (4), we have

\[
(dd^c u)^{n+1} = \sum \sigma_j^{n-|J|+2} \wedge \left( \sum (-1)^{|J|-t} d_j^c(s(t)) \right) \wedge [E_j] = \sum \sigma_j^{n-|J|+2} \wedge \delta_j \wedge [E_j],
\]

which completes the proof. \( \square \)
ACKNOWLEDGEMENT

The authors would like to thank Norm Levenberg for helpful discussions on this material.

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