PRINCIPAL GROUPOID $C^*$-ALGEBRAS WITH BOUNDED TRACE

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ABSTRACT. Suppose $G$ is a second countable, locally compact, Hausdorff, principal groupoid with a fixed left Haar system. We define a notion of integrability for groupoids and show $G$ is integrable if and only if the groupoid $C^*$-algebra $C^*(G)$ has bounded trace.

1. INTRODUCTION

Let $H$ be a locally compact, Hausdorff group acting continuously on a locally compact, Hausdorff space $X$, so that $(H, X)$ is a transformation group. A lovely theorem of Green says that if $H$ acts freely on $X$, then the associated transformation-group $C^*$-algebra $C_0(X) \rtimes H$ has continuous trace if and only if the action of $H$ on $X$ is proper [5, Theorem 17]. Muhly and Williams defined a notion of proper groupoid and proved that for principal groupoids $G$, the groupoid $C^*$-algebra $C^*(G)$ has continuous trace if and only if the groupoid is proper [8, Theorem 2.3]. Of course, when $G = H \times X$ is the transformation-group groupoid, then $G$ is proper if and only if the action of $H$ on $X$ is proper.

In [13] Rieffel introduced a notion of an integrable action of a group $H$ on a $C^*$-algebra $A$. This notion of integrability for $A = C_0(X)$ turned out to characterize when $C_0(X) \rtimes H$, arising from a free action of $H$ on $X$, has bounded trace [6, Theorem 4.8]. In this paper we define a notion of integrability for groupoids (see Definition 3.1) which, when $G = H \times X$ is the transformation-group groupoid, reduces to an integrable action of $H$ on $X$ (see Example 3.3). We then prove that for principal groupoids $G$, $C^*(G)$ has bounded trace if and only if the groupoid is integrable (see Theorem 4.4). This theorem is thus very much in the spirit of [8, Theorem 2.4], [4, Theorem 7.9], [4, Theorem 4.1] (see also [3, Corollary 5.9]) and [4, Theorem 5.3], which characterize when principal-groupoid $C^*$-algebras are, respectively, continuous-trace, Fell, CCR and GCR $C^*$-algebras. The key technical tools used to prove Theorem 4.4 are, first, a homeomorphism of the spectrum of $C^*(G)$ onto the orbit space [4, Proposition 5.1] and, second, a generalisation to groupoids of the notion of $k$-times convergence in the orbit space of a transformation group from [4].

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2. Preliminaries

Let $A$ be a $C^*$-algebra. An element $a$ of the positive cone $A^+$ of $A$ is called a bounded-trace element if the map $\pi \mapsto \text{tr}(\pi(a))$ is bounded on the spectrum $\Lambda$ of $A$; the linear span of the bounded-trace elements is a two-sided $*$-ideal in $A$. We say $A$ has bounded trace if the ideal of (the span of) the bounded-trace elements is dense in $A$.

Throughout, $G$ is a locally compact, Hausdorff groupoid; in our main results $G$ is assumed to be second-countable and principal. We denote the unit space of $G$, and the range and source maps $r, s : G \to G^0$ are $r(\gamma) = \gamma\gamma^{-1}$ and $s(\gamma) = \gamma^{-1}\gamma$, respectively. We let $\pi : G \to G^0 \times G^0$ be the map $\pi(\gamma) = (r(\gamma), s(\gamma))$; recall that $G$ is principal if $\pi$ is injective. In order to define the groupoid $C^*$-algebra, we also assume that $G$ is equipped with a fixed left Haar system: a set $\{\lambda^x : x \in G^0\}$ of non-negative Radon measures on $G$ such that

1. $\text{supp} \lambda^x = r^{-1}(\{x\})$;
2. for $f \in C_c(G)$, the function $x \mapsto \int f \, d\lambda^x$ on $G^0$ is in $C_c(G^0)$; and
3. for $f \in C_c(G)$ and $\gamma \in G$, the following equation holds:
   $$\int f(\gamma a) \, d\lambda^{s(\gamma)}(\alpha) = \int f(\alpha) \, d\lambda^{\gamma(\gamma)}(\alpha).$$

Condition (3) implies that $\lambda^{s(\gamma)}(\gamma^{-1}E) = \lambda^{r(\gamma)}(E)$ for measureable sets $E$. The collection $\{\lambda_x : x \in G^0\}$, where $\lambda_x(E) := \lambda^x(E^{-1})$, gives a fixed right Haar system such that the measures are supported on $s^{-1}(\{x\})$ and

$$\int f(\gamma a) \, d\lambda_{r(\alpha)} = \int f(\gamma) \, d\lambda_{s(\alpha)}$$

for $f \in C_c(G)$ and $\gamma \in G$. We will move freely between these two Haar systems.

If $N \subseteq G^0$, then the saturation of $N$ is $r(s^{-1}(N)) = s(r^{-1}(N))$. In particular, we call the saturation of $\{x\}$ the orbit of $x \in G^0$ and denote it by $[x]$.

If $G$ is principal and all the orbits are locally closed, then by [4, Proposition 5.1] the orbit space $G^0/G = \{[x] : x \in G^0\}$ and the spectrum $C^*(G)^w$ of the groupoid $C^*$-algebra $C^*(G)$ are homeomorphic. This homeomorphism is induced by the map $x \mapsto L^x : G^0 \to C^*(G)^w$, where $L^x : C^*(G) \to B(L^2(G, \lambda_x))$ is given by

$$L^x(f)\xi(\gamma) = \int f(\gamma a)\xi(\alpha^{-1})d\lambda^x(\alpha)$$

for $f \in C_c(G)$ and $\xi \in L^2(G, \lambda_x)$.

3. Integrable Groupoids and Convergence

In the Orbit Space of a Groupoid

The following definition is motivated by the notion of an integrable action of a locally compact, Hausdorff group on a space from [6, Definition 3.2].

**Definition 3.1.** A locally compact, Hausdorff groupoid $G$ is integrable if for every compact subset $N$ of $G^0$,

$$\sup_{x \in N} \{\lambda^x(s^{-1}(N))\} < \infty,$$

or, equivalently, $\sup_{x \in N} \{\lambda_x(r^{-1}(N))\} < \infty$.
Remark 3.2. (1) Suppose that $G$ is a principal groupoid. Then $\lambda^x(s^{-1}(E)) = \lambda^y(s^{-1}(E))$ for all $x, y \in G^0$ such that $y \in [x]$. The map $\lambda^x \mapsto s * \lambda^x$, where $s * \lambda^x(E) = \lambda^x(s^{-1}(E))$, gives a family of measures $\{\alpha_{[x]} : [x] \in G^0 / G\}$ such that $\alpha_{[x]}$ is a measure on $[x]$ supported on $[x]$, and, for any $f \in C_c(G)$, the function

$$x \mapsto \int_{y \in [x]} f(\pi^{-1}(x,y)) \, d\alpha_{[x]}(y)$$

is continuous. (Recall that $\pi : \gamma \mapsto (r(\gamma), s(\gamma))$ is injective by definition of principality.) In fact, the existence of the Haar system $\{\lambda^x\}$ is equivalent to the existence of the family $\{\alpha_{[x]}\}$ [12] Examples 2.5(c)]. Thus a principal groupoid $G$ is integrable if and only if for every compact subset $M$ of $G^0 / G$, the function $[x] \mapsto \alpha_{[x]}(M)$ is bounded.

(2) We could have taken the supremum in (5.1) over the whole unit space, that is,

$$\sup_{x \in G^0} \{\lambda^x(s^{-1}(N))\} = \sup_{x \in N} \{\lambda^x(s^{-1}(N))\}.$$ To see this, first note that if $y$ is not in the saturation $r(s^{-1}(N)) = s(r^{-1}(N))$ of $N$, then $s^{-1}(N) \cap r^{-1}([y]) = \emptyset$, and hence $\lambda^y(s^{-1}(N)) = 0$. Second, if $y$ is in the saturation of $N$, then there exists a $\gamma \in G$ such that $s(\gamma) = y$ and $r(\gamma) \in N$. Then $r^{-1}([y]) \cap s^{-1}(N) = \gamma^{-1}r^{-1}([y]) \cap s^{-1}(N) = \gamma^{-1}(r^{-1}([r(\gamma)]) \cap s^{-1}(N))$, and now

$$\lambda^y(s^{-1}(N)) = \lambda^y(r^{-1}([y]) \cap s^{-1}(N)) = \lambda^y(\gamma^{-1}r^{-1}([r(\gamma)]) \cap s^{-1}(N)) = \lambda^{s(\gamma)}(r^{-1}([r(\gamma)]) \cap s^{-1}(N)) = \lambda^{s(\gamma)}(s^{-1}(N))$$

with $r(\gamma) \in N$.

Example 3.3. Let $(H, X)$ be a locally compact, Hausdorff transformation group with $H$ acting on the left of the space $X$. Then $G = H \times X$ with

$$G^2 = \{(h, x), (k, y) \in G \times G : y = h^{-1} \cdot x\}$$

and operations $(h, x)(k, h^{-1} \cdot x) = (hk, x)$ and $(h, x)^{-1} = (h^{-1}, h^{-1} \cdot x)$ is called the transformation-group groupoid. We identify the unit space $\{e\} \times X$ with $X$, and then the range and source maps $r, s : G \to X$ are $s(h, x) = h^{-1} \cdot x$ and $r(h, x) = x$. If $\delta_x$ is the point-mass measure on $X$ and $\mu$ is a left Haar measure on $H$, then $\{\lambda^x := \mu \times \delta_x : x \in X\}$ is a left Haar system for $G$. Now

$$\lambda^x(s^{-1}(N)) = \mu(\{h \in H : h^{-1} \cdot x \in N\})$$

and hence

$$\sup_{x \in N} \{\lambda^x(s^{-1}(N))\} = \sup_{x \in N} \{\mu(\{h \in H : h^{-1} \cdot x \in N\})\};$$

that is, Definition [5.1] reduces to [6] Definition 3.2].

Example 3.4. In [5] pp. 95-96] Green describes an action as follows: the space $X$ is a closed subset of $\mathbb{R}^3$ and consists of countably many orbits, with orbit representatives $x_0 = (0, 0, 0)$ and $x_n = (2^{-2n}, 0, 0)$ for $n = 1, 2, \ldots$. The action of the group $H = \mathbb{R}$ on $X$ is given by $s \cdot x_0 = (0, s, 0)$ for all $s$; and for $n \geq 1$,

$$s \cdot x_n = \begin{cases} (2^{-2n}, s, 0) & \text{if } s \leq n; \\
(2^{-2n} - \frac{n-s}{\pi}, 2^{-2n-1}, n \cos(s-n), n \sin(s-n)) & \text{if } n < s < n + \pi; \\
(2^{-2n-1}, s - \pi - 2n, 0) & \text{if } s \geq n + \pi. \end{cases}$$
So the orbit of each \( x_n \) (\( n \geq 1 \)) consists of two vertical lines joined by an arc of a helix situated on a cylinder of radius \( n \); the action moves \( x_n \) along the vertical lines at unit speed and along the arc at radial speed. This action is free, non-proper and integrable (see [13] Example 1.18 or [6] Example 3.3). So the associated transformation-group groupoid \( G = H \times X \) is principal and integrable by Example 3.3.

The following characterization of integrability will be important later. In the case of a transformation-group groupoid, Lemma 3.5 reduces to a special case of [1] Lemma 3.5.

**Lemma 3.5.** Let \( G \) be a locally compact, Hausdorff groupoid. Then \( G \) is integrable if and only if, for each \( z \in G^0 \), there exists an open neighborhood \( U \) of \( z \) in \( G^0 \) such that

\[
\sup_{x \in U} \{ \lambda^x(s^{-1}(U)) \} < \infty.
\]

**Proof.** The proof is exactly the same as the proof of [1] Lemma 3.5. \( \square \)

If a groupoid fails to be integrable, there exists a \( z \in G^0 \) such that

\[
\sup_{x \in U} \{ \lambda^x(s^{-1}(U)) \} = \infty
\]

for every open neighborhood \( U \) of \( z \); we then say that the groupoid fails to be integrable at \( z \).

It is evident from [1] 2 that integrability and \( k \)-times convergence in the orbit space of a transformation group are closely related. Moreover, Lemma 2.6 of [8] says that, if a principal groupoid fails to be proper and the orbit space \( G^0/G \) is Hausdorff, then there exists a sequence that converges 2-times in \( G^0/G \) in the sense of Definition 3.6.

**Definition 3.6.** A sequence \( \{x_n\} \) in the unit space of a groupoid \( G \) converges \( k \)-times in \( G^0/G \) to \( z \in G^0 \) if there exist \( k \) sequences

\[
\{\gamma_n^{(1)}\}, \{\gamma_n^{(2)}\}, \ldots, \{\gamma_n^{(k)}\} \subseteq G
\]

such that

1. \( r(\gamma_n^{(i)}) \to z \) as \( n \to \infty \) for \( 1 \leq i \leq k \);
2. \( s(\gamma_n^{(i)}) = x_n \) for \( 1 \leq i \leq k \);
3. if \( 1 \leq i < j \leq k \), then \( \gamma_n^{(j)}(\gamma_n^{(i)})^{-1} \to \infty \) as \( n \to \infty \), in the sense that \( \{\gamma_n^{(j)}(\gamma_n^{(i)})^{-1}\} \) admits no convergent subsequence.

**Remarks 3.7.** (a) Condition (2) in Definition 3.6 is needed so that the composition in (3) makes sense.

(b) Definition 3.6 does not require that \( x_n \to z \), but as in the transformation-group case ([2] Definition 2.2]), this can be arranged by changing the sequence which converges \( k \)-times: replace \( x_n \) by \( r(\gamma_n^{(1)}) \) and replace \( \gamma_n^{(j)} \) by \( \gamma_n^{(j)}(\gamma_n^{(1)})^{-1} \).

(c) Part (3) of Definition 3.6 means \( \gamma_n^{(j)}(\gamma_n^{(i)})^{-1} \) is eventually outside every compact set. In particular, if \( LL^{-1} \) is compact, \( L\gamma_n^{(i)} \cap L\gamma_n^{(j)} = \emptyset \) eventually.

**Example 3.8.** Let \( G = H \times X \) be a transformation-group groupoid (see Example 3.3) and suppose that \( \{x_n\} \subseteq G^0 \) is a sequence converging 2-times in \( G^0/G \) to \( z \in G^0 \). Then there exist two sequences

\[
\{\gamma_n^{(1)}\} = \{(s_n, y_n)\} \quad \text{and} \quad \{\gamma_n^{(2)}\} = \{(t_n, z_n)\}
\]
in $G$ such that (1) $y_n \to z$ and $z_n \to z$; (2) $s_n^{-1} \cdot y_n = x_n$ and $t_n^{-1} \cdot z_n = x_n$; and (3) $(t_n s_n^{-1} \cdot z_n) \to \infty$ as $n \to \infty$. To see that the sequence $\{x_n\}$ converges 2-times in $X/H$ to $z$ in the sense of [2, §4], consider the two sequences $\{s_n\}$ and $\{t_n\}$ in $H$.

We have $s_n \cdot x_n \to z$ and $t_n \cdot x_n \to z$ using (1) and (2). Also, since $z_n \to z$ by (1), (3) implies that $t_n s_n^{-1} \to \infty$ in $H$.

Conversely, if $\{x_n\} \subseteq X$ converges 2-times in $X/H$ to $z$, then there exist two sequences $\{s_n\}$, $\{t_n\}$ in $H$ such that (1) $s_n \cdot x_n \to z$ and $t_n \cdot x_n \to z$ and (2) $t_n s_n^{-1} \to \infty$. It is easy to check that

\[
\{\gamma_n^{(1)}\} = \{(s_n, s_n \cdot x_n)\} \quad \text{and} \quad \{\gamma_n^{(2)}\} = \{(t_n, t_n \cdot x_n)\}
\]

witness the 2-times convergence in $G^0/G$ of $\{x_n\} \subset G^0$ to $z \in G^0$.

In the transformation-group groupoid of Example [3.4] the sequence $\{x_n = (2^{-2n}, 0, 0)\}$ converges 2-times in $G^0/G$ to $z_0 = (0, 0, 0)$; to see this, just take $s_n = e$ and $t_n = 2n + \pi$ for each $n$.

In [4] we will prove that a principal groupoid $G$ is integrable if and only if $C^*(G)$ has bounded trace. For the “only if” direction we will need to know that the orbits are locally closed so that [4, Proposition 5.1] applies and $x \mapsto L^x$ induces a homeomorphism of $G^0/G$ onto $C^*(G)^\wedge$; Lemma [3.9] below establishes that if $G$ is integrable, then the orbits are in fact closed, hence locally closed. We will prove the contrapositive of the “if” direction, and a key observation for the proof is Proposition [3.11] if a groupoid fails to be integrable at some $z$, then there is a non-trivial sequence $\{x_n\}$ which converges $k$-times in $G^0/G$ to $z$, for every $k \in \mathbb{N} \setminus \{0\}$.

We thank an anonymous referee for providing the proof of Lemma [3.9].

**Lemma 3.9.** Let $G$ be a second countable, locally compact, Hausdorff, principal groupoid. If $G$ is integrable, then all orbits are closed.

**Proof.** Let $\{\alpha_{[x]} : [x] \in G^0/G\}$ be the family of measures from Remark [3.21]. We claim that, for fixed $h \in C_c(G^0/G)$, the function $[x] \mapsto \int_{y \in [x]} h(y) \, d\alpha_{[x]}(y)$ is continuous. To see this, choose $g_n \in C_c(G^0 \times G^0)$ such that, for all $u \in G^0$, the function $g_n(u, \cdot)$ increases to the function $v \mapsto 1$. Then

\[
\int_{y \in [x]} h(y) \, d\alpha_{[x]}(y) = \lim_n \int_{y \in [x]} g_n(x, y) h(y) \, d\alpha_{[x]}(y) = \lim_n \int_{G} f_n(\gamma) \, d\lambda^x(\gamma),
\]

where $f_n(\gamma) = g_n(\pi(\gamma)) h(s(\gamma))$. Since $f_n \in C_c(G)$, the function

\[
x \mapsto \int_{G} f_n(\gamma) \, d\lambda^x(\gamma)
\]

is continuous for each $n$. Note that $x \mapsto \int_{y \in [x]} g_n(x, y) h(y) \, d\alpha_{[x]}(y)$ is compactly supported for each $n$. Since limits of uniformly continuous functions are continuous, $x \mapsto \int_{y \in [x]} h(y) \, d\alpha_{[x]}(y)$ is continuous; this function is constant on orbits, which proves the claim.

Fix $x_0 \in G^0$ and suppose that $G$ is integrable. Since $G$ is principal, for each compact subset $M$ of $G^0/G$, the function $[x] \mapsto \alpha_{[x]}(M)$ is bounded. In particular, for each $h \in C_c(G^0/G)^+$, $\int h \, d\alpha_{[x_0]} \in \mathbb{R}$. Since the support of $\alpha_{[x]}$ is $[x]$, we have

\[
(3.2) \quad \{x_0\} = \bigcap_{h \in C_c(G^0/G)^+} \left\{ x : \int h \, d\alpha_{[x]} \leq \int h \, d\alpha_{[x_0]} \right\}.
\]
But the function \( \{x \mapsto \int_{y \in [x]} h(y) \, d\mu([x]) \} \) is continuous, hence lower semi-continuous, so the left-hand side of \( \text{(3.2)} \) is an intersection of closed sets. Thus \( \{x_0\} \) is closed in \( G^0/G \), and hence \( [x_0] \) is closed in \( G^0 \).

The transformation group of \([13]\) Example 1.18 provides an example of a non-integrable free action with closed orbits (by choosing repetition numbers with infinite supremum). Thus there are non-integrable principal groupoids with closed orbits.

Recall that a neighborhood \( W \) of \( G^0 \) is called \textit{conditionally compact} if the sets \( WV \) and \( VW \) are relatively compact for every compact set \( V \) in \( G \). The following lemma will be used repeatedly.

**Lemma 3.10.** Let \( G \) be a second countable, locally compact, Hausdorff groupoid. 

1. Let \( z \in G^0 \) and let \( K \) be a relatively compact neighborhood of \( z \) in \( G \). There exist \( a \in \mathbb{R} \) and a neighborhood \( U \) of \( z \) in \( G^0 \) such that \( 0 < a \leq \lambda_x(K) \) for all \( x \in U \).

2. Let \( Q \) be a conditionally compact neighborhood in \( G \). Given any relatively compact neighborhood \( V \) in \( G^0 \) such that \( QV \neq \emptyset \), there exists \( c \in \mathbb{R} \) such that \( c > 0 \) and \( \lambda_x(Q) \leq c \) for all \( x \in V \).

**Proof.** (1) Suppose not. Let \( \{U_i\} \) be a decreasing sequence of open neighborhoods of \( z \) in \( G^0 \). There exists an increasing sequence \( i_1 < i_2 < \cdots < i_n < \cdots \) and \( x_n \in U_{i_n} \) such that \( \lambda_{x_n}(K) < 1/n \) for each \( n \geq 1 \). Note that \( x_n \to z \).

Let \( f \in C_c(G) \) such that \( 0 \leq f \leq 1 \), \( f(z) = 1 \) and \( \text{sup} \ f \subseteq K \); note that \( \int f(\gamma) \, d\lambda_x(\gamma) > 0 \). By the continuity of the Haar system,

\[
\frac{1}{n} > \lambda_{x_n}(K) \geq \int f(\gamma) \, d\lambda_{x_n}(\gamma) = \int f(\gamma) \, d\lambda_x(\gamma) \quad \text{as} \quad n \to \infty,
\]

which is impossible since the left-hand side converges to \( 0 \) and \( \int f(\gamma) \, d\lambda_x(\gamma) > 0 \).

(2) Let \( V \) be any relatively compact neighborhood in \( G^0 \) such that \( QV \neq \emptyset \). Let \( f \in C_c(G) \) such that \( 0 \leq f \leq 1 \) and \( f \) is identically one on the relatively compact subset \( QV \). The function \( w \mapsto \int f(\gamma) \, d\lambda_w(\gamma) \) is in \( C_c(G^0) \), so it achieves a maximum \( c > 0 \). Then, for \( x \in V \),

\[
\lambda_x(Q) = \lambda_x(Qx) \leq \int f(\gamma) \, d\lambda_x(\gamma) \leq c.
\]

**Proposition 3.11.** Let \( G \) be a locally compact, Hausdorff groupoid. Let \( z \in G^0 \) and suppose that \( G \) fails to be integrable at \( z \). Then there exists a sequence \( \{x_n\} \) in \( G^0 \) such that \( x_n \to z \), and \( \{x_n\} \) converges \( k \)-times in \( G^0/G \) to \( z \), for every \( k \in \mathbb{N} \setminus \{0\} \). In addition, if \( G \) is second countable, principal and the orbits are locally closed, then \( x_n \neq z \) eventually.

**Proof.** Suppose the groupoid fails to be integrable at \( z \). Fix \( k \in \mathbb{N} \setminus \{0\} \). Let \( \{U_n\} \) be a decreasing sequence of open relatively compact neighborhoods of \( z \) in \( G^0 \). By Lemma 3.3

\[
\sup_{y \in U_n} \{\lambda^y(s^{-1}(U_n))\} = \infty
\]

for each \( n \). So we can choose a sequence \( \{x_n\} \) such that \( x_n \in U_n \) and \( \lambda^{x_n}(s^{-1}(U_n)) > n \). Note that \( x_n \to z \) as \( n \to \infty \).

Let \( Q \) be an open symmetric conditionally compact neighborhood of \( z \) in \( G \) and let \( V \) be an open relatively compact neighborhood of \( z \) in \( G^0 \). By Lemma 3.10(2)
there exists $c > 0$ such that $\lambda_v(Q^2) \leq c$ whenever $v \in V$. Choose $n_0$ such that $n_0 > (k-1)c$ and $U_{n_0} \subseteq V$. Temporarily fix $n > n_0$. Set $\gamma^{(1)}_n = x_n$. For $k \geq 2$ choose $k-1$ elements $\gamma^{(2)}_n, \ldots, \gamma^{(k)}_n$ as follows. Note that since $x_n = r(\gamma^{(1)}_n) \in V$, we have

$$\lambda_{x_n}(r^{-1}(U_n) \setminus Q^2 \gamma^{(1)}_n) \geq \lambda_{x_n}(r^{-1}(U_n)) - \lambda_{x_n}(Q^2 \gamma^{(1)}_n) = \lambda_{x_n}(r^{-1}(U_n) \cap s^{-1}\{x_n\}) - \lambda_{r(\gamma^{(1)}_n)}(Q^2) > (k-1)c - c = (k-2)c \geq 0.$$ 

So there exists $\gamma^{(2)}_n \in (r^{-1}(U_n) \cap s^{-1}\{x_n\}) \setminus Q^2 \gamma^{(1)}_n$, note that $r(\gamma^{(2)}_n) \in U_n \subseteq V$ and $s(\gamma^{(2)}_n) = x_n$. Next,

$$\lambda_{x_n}(r^{-1}(U_n) \setminus (Q^2 \gamma^{(1)}_n \cup Q^2 \gamma^{(2)}_n)) \geq \lambda_{x_n}(r^{-1}(U_n)) - \lambda_{x_n}(Q^2 \gamma^{(1)}_n) - \lambda_{x_n}(Q^2 \gamma^{(2)}_n) \geq \lambda_{x_n}(r^{-1}(U_n) \cap s^{-1}\{x_n\}) - \lambda_{r(\gamma^{(1)}_n)}(Q^2) - \lambda_{r(\gamma^{(2)}_n)}(Q^2) > (k-3)c \geq 0.$$ 

Continue until $\gamma^{(1)}_n, \ldots, \gamma^{(k)}_n$ have been chosen in this way.

If $n > n_0$, then by construction $s(\gamma^{(i)}_n) = x_n$ and $r(\gamma^{(i)}_n) \in U_n$ for each $n$; so $r(\gamma^{(i)}_n) \to z$ as $n \to \infty$ for $1 \leq i \leq k$. Moreover $\gamma^{(j)}_n(\gamma^{(i)}_n)^{-1} \notin Q^2$ for $1 \leq i < j \leq k$ and $n > n_0$. To see that $\{\gamma^{(j)}_n(\gamma^{(i)}_n)^{-1}\}$ tends to infinity, suppose that it doesn’t. Then, $\gamma^{(j)}_n(\gamma^{(i)}_n)^{-1} \to \gamma$ by passing to a subsequence and relabelling. But then $s(\gamma^{(j)}_n(\gamma^{(i)}_n)^{-1}) = r(\gamma^{(i)}_n) \to z$ and $r(\gamma^{(j)}_n(\gamma^{(i)}_n)^{-1}) = r(\gamma^{(j)}_n) \to z$ implies $\gamma = z$, which is impossible because $\gamma^{(j)}_n(\gamma^{(i)}_n)^{-1} \notin Q^2$ and $Q$ contains $G^0$. Hence $\{x_n\}$ converges $k$-times in $G^0/G$ to $z$.

We claim that if $G$ is second countable and principal, then $x_n \neq z$ eventually. To see this, suppose $x_n = z$ frequently. Then $\lambda^z(s^{-1}(U_n)) > n$ frequently, and hence

$$\lambda^z(s^{-1}(U_1)) = \infty.$$ 

The orbits are locally closed and $G$ is second countable and principal, so the source map restricts to a homeomorphism $s: r^{-1}(\{z\}) \to [z]$. Since $U_1$ is relatively compact, $s^{-1}(\{z\} \cap U_1)$ is relatively compact in $r^{-1}(\{z\})$ because $s: r^{-1}(\{z\}) \to [z]$ is a homeomorphism. But now $\lambda^z(s^{-1}(\{z\} \cap U_1)) = \lambda^z(s^{-1}(U_1)) < \infty$, contradicting (3.3).

4. Integrability of $G$ and Trace Properties of $C^*(G)$

**Proposition 4.1.** Let $G$ be a second-countable, locally compact, Hausdorff principal groupoid. If $C^*(G)$ has bounded trace, then $G$ is integrable.

The proof of Proposition 4.1 is based on that of [8] Theorem 2.3. There, Muhly and Williams choose a sequence $\{x_n\} \subseteq G^0$ with $x_n \to z$ which witnesses the failure of the groupoid to be proper. They then carefully construct a function $f \in C_c(G)$ to obtain an element $d$ of the Pedersen ideal of $C^*(G)$ such that $\text{tr}(L^2(d))$ does not converge to $\text{tr}(L^2(d))$. Since the Pedersen ideal is the minimal dense ideal [9] Theorem 5.6.1, the ideal of continuous-trace elements cannot be dense, so $C^*(G)$ does not have continuous trace. We adopt the same strategy, use exactly the
same function \( f \), but adapt the proof of [8] Theorem 2.3 using ideas from [6] Proposition 3.5).

**Proof of Proposition 4.1.** Fix \( M \in \mathbb{N} \setminus \{0\} \). We will show that there is an element \( d \) of the Pedersen ideal of \( C^*(G) \), a sequence of representations \( \{L^x_n\} \) and \( n_0 > 0 \) such that \( \text{tr}(L^x_n(d)) > M \) whenever \( n > n_0 \). Since \( M \) is arbitrary, \( C^*(G) \) cannot have bounded trace.

If \( G \) is not integrable, then the integrability fails at some \( z \in G^0 \) by Lemma 3.5 if the orbits are not closed, then \( C^*(G) \) cannot be CCR by [4] Theorem 4.1 and hence cannot have bounded trace. So from now on we may assume that the orbits are closed. By Proposition 3.1.11 there exists a sequence \( \{x_n\} \) such that \( x_n \not\to z \), \( x_n \to z \), and \( \{x_n\} \) converges \( k \)-times in \( G^0/G \) to \( z \), for every \( k \in \mathbb{N} \setminus \{0\} \).

Since we will use exactly the same function \( f \) that was used in the proof of [8] Theorem 2.3, our first task is to briefly outline its construction. Fix a function \( g \in C_c(G^0) \) such that \( 0 \leq g \leq 1 \) and \( g \) is identically one on a neighborhood \( U \) of \( z \). Let \( N = \text{supp} g \) and let

\[
F^N_z := s^{-1}(\{z\}) \cap r^{-1}(z \cap N) = s^{-1}(\{z\}) \cap r^{-1}(N),
\]

\[
F^N_N := r^{-1}(\{z\}) \cap s^{-1}(z \cap N) = r^{-1}(\{z\}) \cap s^{-1}(N).
\]

There exist symmetric, open, conditionally compact neighborhoods \( W_0 \) and \( W_1 \) in \( G \) such that

\[
G^0 \subseteq W_0 \subseteq \overline{W_0} \subseteq W_1 \quad \text{and} \quad F^N_N \cup F^N_z \subseteq W_0.
\]

Thus \( \overline{W^7}z \setminus W_0 z \subseteq r^{-1}(G^0 \setminus N) \). (The reason for using \( \overline{W^7} \) becomes clear at (4.4) below.) By a compactness argument, there exist open, symmetric, relatively compact neighborhoods \( V_0 \subseteq G^0 \) and \( V_1 \) of \( z \) in \( G \) such that \( V_0 \subset V_1 \) and

\[
(4.1) \quad \overline{W^7}V_0 \setminus W_0 V_0 \subseteq r^{-1}(G^0 \setminus N).
\]

Now note that if \( \gamma \in \overline{W^7}V_1 \setminus W_0 V_0 W_0 \), then \( r(\gamma) \in r(\overline{W^7}V_0 \setminus W_0 V_0) \subseteq G^0 \setminus N \). It follows that the function \( g^{(1)} : G \to [0,1] \) defined by

\[
g^{(1)}(\gamma) = \begin{cases} g(r(\gamma)) & \text{if } \gamma \in \overline{W^7}V_1 \setminus W^7, \\ 0 & \text{if } \gamma \not\in W_0 V_0 W_0 \end{cases}
\]

is well-defined and continuous with compact support in \( G \). By construction

\[
(W_0 V_0 W_0)^2 = W_0 V_0 W^2_0 W_0 \subseteq W^4_0 W^4_0 \subseteq \overline{W^4_0} \overline{W^4_0} \subseteq W^4_1 V^4_1 \subseteq \overline{W^4_1} \overline{W^4_1}.
\]

So there exists a function \( b \in C_c(G) \) such that \( 0 \leq b \leq 1 \), \( b \) is identically one on \( W_0 V_0 W^2_0 V_0 \) and it is identically zero on the complement of \( \overline{W^7}V_1 \setminus W^7 \). Further, we can replace \( b \) with \( (b + b^*)/2 \) to ensure that \( b \) is self-adjoint. Set

\[
f(\gamma) = g(\gamma)g(s(\gamma))b(\gamma);
\]

note that \( f \in C_c(G) \) is self-adjoint.
For $\xi \in L^2(G, \lambda_n)$ and $\gamma \in G$ we have

$$L^{\alpha}(f)\xi(\gamma) = \int f(\gamma\alpha)\xi(\alpha^{-1}) \, d\lambda^\alpha(\alpha)$$

$$= \int g(r(\gamma))g(s(\alpha))b(\gamma\alpha)\xi(\alpha^{-1}) \, d\lambda^\alpha(\alpha)$$

$$= g(r(\gamma)) \int g(s(\alpha))b(\gamma\alpha)\xi(\alpha^{-1}) \, d\lambda^\alpha(\alpha)$$

$$(4.2)$$

$$= g(r(\gamma)) \int g(r(\alpha))b(\gamma\alpha^{-1})\xi(\alpha) \, d\lambda_n(\alpha).$$

By [8, Lemma 2.8], $g^{(1)}$ is an eigenvector for $L^x(f)$ with eigenvalue

$$\mu_{xn}^{(1)} = \int g(r(\alpha))g^{(1)}(\alpha) \, d\lambda_{xn}(\alpha) = \int_{\gamma V^\alpha_{n0}} g(r(\alpha)) \, d\lambda_{xn}(\alpha).$$

By [8, Lemma 2.9], there exist an open $V_2 \subseteq V_0$ and a conditionally compact neighborhood $Y$ of $G^0$ so that $Y \subseteq W_0$ and if $v \in V_2$, then $r(Yv) \subseteq U$. Notice that $YV_2Y$ is a relatively compact subset of $W_0V_2W_0$. By Lemma [8,10](1) there exist an open neighborhood $V_3$ of $z$ and $a > 0$ such that

$$(4.3) \quad \lambda_n(YV_2Y) \geq a \text{ whenever } v \in V_3.$$

Now, if $\alpha \in YV_2Y$, then $r(\alpha) \in U$ and hence $g(r(\alpha)) = 1$; it follows that

$$\mu_{xn}^{(1)} \geq \int_{YV_2Y} g(r(\alpha))^2 \, d\lambda_{xn}(\alpha) = \lambda_{xn}(YV_2Y) \geq a > 0$$

whenever $x_n \in V_3$.

So far our set-up is the one from [8]. Now choose $l \in N \setminus \{0\}$ such that $la^2 > M$. (Note that $a$ is independent of $l$!) The sequence $\{x_n\}$ converges $k$-times in $G/G^0$ to $z$ for every $k \in N \setminus \{0\}$, so it certainly converges $l$ times. So there exist $l$ sequences

$$\{\gamma^{(1)}_n\}, \{\gamma^{(2)}_n\}, \ldots, \{\gamma^{(l)}_n\} \subseteq G$$

such that

1. $r(\alpha) \to z$ as $n \to \infty$ for $1 \leq i \leq l$;
2. $s(\alpha) = x_n$ for $1 \leq i \leq k$;
3. if $1 \leq i < j \leq l$, then $\gamma^{(i)}_n \gamma^{(i)}_n^{-1} \to \infty$.

Moreover, by construction (see Proposition [8,11]), we may take $\gamma^{(1)}_n = x_n$. Temporarily fix $n$. Set $y^{(1)}_n := g^{(1)}$, and for $2 \leq j \leq l$ set

$$g^{(j)}_n(\gamma) := \begin{cases} g^{(1)}(\gamma \gamma^{(1)}_n)^{-1}, & \text{if } s(\gamma) = s(\gamma^{(1)}_n); \\ 0, & \text{otherwise} \end{cases}$$

$$= \begin{cases} g(r(\gamma)), & \text{if } \gamma \in W_0V_1W_1^{-1} \gamma^{(1)}_n; \\ 0, & \text{otherwise} \end{cases}$$

$$= \begin{cases} g(r(\gamma)), & \text{if } \gamma \in W_0V_1W_1^{-1} \gamma^{(1)}_n; \\ 0, & \text{if } \gamma \notin W_0V_0W_0 \gamma^{(1)}_n. \end{cases}$$

Each $g^{(j)}_n$ ($1 \leq j \leq l$) is a well-defined function in $C_c(G)$ with support contained in $W_0V_0W_0 \gamma^{(j)}_n$. For $1 \leq i < j \leq l$, $\gamma^{(j)}_n \gamma^{(i)}_n^{-1} \notin (W_0V_0W_0)^2$ eventually, so there
exists \( n_0 > 0 \) such that, for every \( 0 \leq i, j \leq l, \ i \neq j \),

\[
W_0V_0\gamma_n^{(j)} \cap W_0V_0\gamma_n^{(i)} = \emptyset
\]

whenever \( n > n_0 \).

We now prove a generalization of [8, Lemma 2.8] which, together with (4.2),
immediately implies that each \( g_n^{(j)} \) is an eigenvector of \( L^n(f) \) for \( 1 \leq j \leq l \).

**Lemma 4.2.** With the choices made above, for all \( \alpha, \gamma \in G \) and \( 1 \leq j \leq l \),

\[
g(r(\gamma))g(r(\alpha))b(\gamma\alpha^{-1})g_n^{(j)}(\alpha) = g_n^{(j)}(\gamma)g(r(\alpha))g_n^{(j)}(\alpha).
\]

**Proof.** If \( \alpha \notin W_0V_0\gamma_n^{(j)} \), then both sides are zero. So we may assume throughout

that \( \alpha \in W_0V_0\gamma_n^{(j)} \).

If \( \gamma \in W_0V_0\gamma_n^{(j)} \), then \( g_n^{(j)}(\gamma) = g(r(\gamma)) \) and \( \gamma\alpha^{-1} \in W_0V_0W_0W_0V_0 \), so

\[
b(\gamma\alpha^{-1}) = 1
\]

and both sides agree.

If \( \gamma \notin W_1V_1W_1\gamma_n^{(j)} \), then \( g_n^{(j)}(\gamma) = 0 \), so both sides are zero.

Finally, if \( \gamma \notin W_1V_1W_1\gamma_n^{(j)} \), then \( g_n^{(j)}(\gamma) = 0 \), so both right-hand sides are zero.

On the other hand, if \( \gamma\alpha^{-1} \in W_1V_1W_1(= \text{supp } b) \), then

\[
(4.4) \quad \gamma \in W_1V_1W_1\gamma_n^{(j)} \subseteq W_1V_1W_1\gamma_n^{(j)}.
\]

So \( \gamma \notin W_1V_1W_1\gamma_n^{(j)} \) implies \( \gamma\alpha^{-1} \notin \text{supp } b \), so the left-hand side is zero as well. \( \square \)

Let \( \mu_n^{(j)} \) be the eigenvalue corresponding to the eigenvector \( g_n^{(j)} \). Using (4.3),

\[
\mu_n^{(j)} = \int_{W_0V_0\gamma_n^{(j)}} g(r(\alpha))^2 d\lambda x_n(\alpha) \geq \lambda x_n(YV_2Y \gamma_n^{(j)}) = \lambda_{r(\gamma_n^{(j)})}(YV_2Y) \geq a
\]

whenever \( r(\gamma_n^{(j)}) \in V_3 \). Choose \( n_1 > n_0 \) such that \( n > n_1 \) implies \( x_n \in V_3 \) and

\( r(\gamma_n^{(j)}) \in V_3 \) for \( 1 \leq j \leq l \). Then \( L^n(f * f) \) is a positive compact operator with

\( l \) eigenvalues \( \mu_n^{(j)} \geq a^2 \) for \( 1 \leq j \leq l \). To push \( f * f \) into the Pedersen ideal, let \( r \in C_c(0, \infty) \) be any function

satisfying

\[
r(t) = \begin{cases} 
0, & \text{if } t < \frac{a^2}{3}; \\
2t - \frac{2a^2}{3}, & \text{if } \frac{a^2}{3} \leq t < \frac{2a^2}{3}; \\
t, & \text{if } \frac{2a^2}{3} \leq t \leq \|f * f\|.
\end{cases}
\]

Set \( d := r(f * f) \). Now \( d \) is a positive element of the Pedersen ideal of \( C^*(G) \) with

\( \text{tr}(L^n(d)) \geq la^2 > M \) whenever \( n > n_1 \). Since \( M \) was arbitrary, \( L^n \mapsto \text{tr}(L^n(d)) \) is

unbounded on \( C^*(G) \). Thus \( C^*(G) \) does not have bounded trace. \( \square \)

**Proposition 4.3.** Suppose \( G \) is a second countable, locally compact, Hausdorff, principal groupoid.
If \( G \) is integrable, then \( C^*(G) \) has bounded trace.

**Proof.** Since \( G \) is principal and integrable, the orbits are closed by Lemma [3,8] and

\( x \mapsto L^n \) induces a homeomorphism of \( G^0/G \) onto \( C^*(G) \) by [3, Proposition 5.1].

To show that \( C^*(G) \) has bounded trace, it suffices to see that for a fixed \( u \in G^0 \)

and all \( f \in C_c(G) \), \( \text{tr}(L^n(f * f)) \) is bounded independent of \( u \).

Fix \( u \in G^0 \) and let \( \xi \in L^2(G, \lambda_u) \). Since

\[
L^n(f)\xi(\gamma) = \int f(\gamma\alpha^{-1})\xi(\alpha) \, d\lambda_u(\alpha),
\]

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Suppose \( G \) is a second countable, locally compact, Hausdorff, principal groupoid. Then \( G \) has bounded trace.

References


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