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CHARACTERIZING STRONG ESTIMATES

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Abstract. We describe necessary and sufficient conditions for square functions to map L^∞ to L^∞ for ergodic averages and Lebesgue derivatives.

1. Introduction

Square functions are extensively studied in ergodic theory. It has been proved in [5] and [6] that square functions for ergodic averages and Lebesgue derivatives are of weak type (1, 1) and of strong type (p, p) for $1 < p < \infty$. It is also proved in [6], [7], and [9] that square functions map L^∞ to BMO for Lebesgue derivatives. In particular it has been proved in [12] that the square functions for Lebesgue derivatives map BMO to BMO. This is why we concentrate on characterizing the necessary and sufficient conditions for square functions of ergodic averages or Lebesgue derivatives to map L^∞ to L^∞. We show that this condition for the square function of the successive differences of a sequence (A_{n_k}) of ergodic averages is that $\sum_{k=1}^{\infty}(\frac{n_{k+1}-n_k}{n_{k+1}})^2$ is finite. A similar necessary and sufficient condition can be given for Lebesgue derivatives.

First, we define the square functions for ergodic theory. Let $(X, \mathcal{F}, \mu, T)$ be a dynamical system, and let T be an ergodic transformation from X to X. Given $f \in L^p = L^p(X, \mathcal{F}, \mu)$, let $A_n f = \frac{1}{n}\sum_{l=1}^{n} f \circ T^l$ be the usual average in ergodic theory.

Definition 1.1. Given a sequence (n_k) of whole numbers, the square function Sf is defined as

$$Sf = (\sum_{k=1}^{\infty} |A_{n_{k+1}} f - A_{n_k} f|^2)^{\frac{1}{2}}.$$

Now, we define the square functions for Lebesgue derivatives.

Definition 1.2. Given a sequence of real numbers (ϵ_k), for any $x \in \mathbb{R}$, let

$$D_k f(x) = \frac{1}{\epsilon_k}\int_0^{\epsilon_k} f(x+t)dt.$$

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Also, for any $x \in \mathbb{R}$, let

$$Sf(x) = \left(\sum_{k=1}^{\infty} |D_k f(x) - D_{k+1} f(x)|^2 \right)^{\frac{1}{2}}.$$

We will need to use the partial sums S_M of both of these square functions in this article.

2. Characterizing the strong estimates for ergodic averages

In this section, we give necessary and sufficient conditions for the square function of differences of ergodic averages to map L^∞ to L^∞.

Theorem 2.1. *For a dynamical system (X, Σ, μ, T), if $\sum_{k=1}^{\infty} |\frac{n_{k+1}-n_k}{n_{k+1}}|^2 < \infty$, then S maps L^∞ to L^∞. Also, if*

$$C = 2 \left\{ \sum_{k=1}^{\infty} |\frac{n_{k+1} - n_k}{n_{k+1}}|^2 \right\}^{\frac{1}{2}},$$

then for all $f \in L^\infty$, $\|Sf\|_\infty \le C\|f\|_\infty$.

Proof. Let T be a measure preserving transformation from X to X and let $f \in L^\infty$. We have

$$\begin{aligned}
Sf(x) &= (\sum_{k=1}^{\infty} (A_{n_{k+1}} f(x) - A_{n_k} f(x))^2)^{\frac{1}{2}} \\
&= (\sum_{k=1}^{\infty} (\frac{1}{n_{k+1}} \sum_{j=1}^{n_{k+1}} f(T^j(x)) - \frac{1}{n_k} \sum_{j=1}^{n_k} f(T^j(x)))^2)^{\frac{1}{2}} \\
&= (\sum_{k=1}^{\infty} (\frac{1}{n_{k+1}} \sum_{j=n_k+1}^{n_{k+1}} f(T^j(x)) - (\frac{1}{n_k} - \frac{1}{n_{k+1}}) \sum_{j=1}^{n_k} f(T^j(x)))^2)^{\frac{1}{2}} \\
&\le (\sum_{k=1}^{\infty} (\frac{1}{n_{k+1}} \sum_{j=n_k+1}^{n_{k+1}} f(T^j(x)))^2)^{\frac{1}{2}} + (\sum_{k=1}^{\infty} ((\frac{1}{n_k} - \frac{1}{n_{k+1}}) \sum_{j=1}^{n_k} f(T^j(x)))^2)^{\frac{1}{2}} \\
&\le (\sum_{k=1}^{\infty} (\frac{n_{k+1} - n_k}{n_{k+1}})^2)^{\frac{1}{2}} \|f\|_\infty + (\sum_{k=1}^{\infty} ((\frac{n_{k+1} - n_k}{n_k n_{k+1}}) n_k)^2)^{\frac{1}{2}} \|f\|_\infty \\
&\le (\sum_{k=1}^{\infty} (\frac{n_{k+1} - n_k}{n_{k+1}})^2)^{\frac{1}{2}} \|f\|_\infty + (\sum_{k=1}^{\infty} (\frac{n_{k+1} - n_k}{n_{k+1}})^2)^{\frac{1}{2}} \|f\|_\infty \\
&= 2(\sum_{k=1}^{\infty} (\frac{n_{k+1} - n_k}{n_{k+1}})^2)^{\frac{1}{2}} \|f\|_\infty.
\end{aligned}$$

Since $\sum_{k=1}^{\infty} (\frac{n_{k+1}-n_k}{n_{k+1}})^2 < \infty$ and $f \in L^\infty$, we have $Sf \in L^\infty$. Moreover, $\|Sf\|_\infty \le C\|f\|_\infty$. □

Now we consider the negative results for square functions if $\sum_{k=1}^{\infty} |\frac{n_{k+1}-n_k}{n_{k+1}}|^2 = \infty$.

Theorem 2.2. *Let (n_k) be an increasing sequence in $\mathbb{N}$. If $\sum_{k=1}^{\infty} |\frac{n_{k+1}-n_k}{n_{k+1}}|^2 = \infty$, then there is an ergodic non-atomic dynamical system (X, Σ, μ, T) and an $f \in L^\infty$ such that $||Sf||_\infty = \infty$.*

In the proof of Theorem 2.2, we need some definitions from probability theory and results from [3]. Additional related material is in [1] and [4].

Proof. **First step: Construct a particular (a_k).**

Fix a sequence $(a_k : k \geq 2)$. Let $\epsilon_j = 1$ for $1 \leq j \leq n_1$ and let $\epsilon_j = a_{k+1}$ when $n_k + 1 \leq j \leq n_{k+1}$ for $k \geq 1$.

We are going to choose (a_k) inductively to take the values 0 and 1. So whatever choices we have made for $a_l, l \leq k$, we have $\frac{1}{n_k}\sum_{j=1}^{n_k} \epsilon_j \in [0,1]$. Hence, either $\frac{1}{n_k}\sum_{j=1}^{n_k} \epsilon_j \in [0,1/2)$ or $\frac{1}{n_k}\sum_{j=1}^{n_k} \epsilon_j \in [1/2,1]$. Therefore, we can choose a_{k+1} such that $|a_{k+1} - \frac{1}{n_k}\sum_{j=1}^{n_k} \epsilon_j| \geq \frac{1}{2}$.

Second step: Construct $(x_n : n \in \mathbb{Z})$ IID and a suitable non-atomic ergodic dynamical system (X, Σ, μ, T). Let $\Omega = \{0,1\}$ with probability distribution P giving $P(1) = P(0) = \frac{1}{2}$ and define a map y_0 from Ω to $\mathbb{R}$ satisfying $y_0(0) = 0$ and $y_0(1) = 1$. We construct the product space $X = \prod_{-\infty}^{\infty} \Omega$, $\Sigma = \prod_{-\infty}^{\infty} 2^{\{0,1\}}$, and $\mu = \prod_{-\infty}^{\infty} P$.

Assume that $w = (w_k) \in X$. We will also write $w(k)$ for the kth coordinate w_k. We define a shift operator T on X by $T(w)(n) = w_{n+1}$. We define a random variable $x_n(w) = x_n((w_k))$:

$$x_n((w_k)) = y_0(w_n) = \begin{cases} 0 & \text{if } w_n = 0, \\ 1 & \text{if } w_n = 1\,. \end{cases}$$

Then, $(x_n : n \in \mathbb{Z})$ is an IID sequence of $\{0,1\}$-valued random variables on (X, Σ, μ). Also, (X, Σ, μ, T) is a non-atomic ergodic dynamical system.

Third step: Prove $\|Sx_0\|_\infty = \infty$. Since T is defined as the shift operator on the above non-atomic ergodic dynamical system (X, Σ, μ, T), we have $x_j \circ T = x_{j+1}$. Also, $A_n x_0(w) = \frac{1}{n}\sum_{j=1}^{n} x_0 \circ T^j(w) = \frac{1}{n}\sum_{j=1}^{n} x_j(w)$.

Take the $\{0,1\}$-valued sequence (a_j) constructed above. Let

$$E_k = \bigcap_{j=n_k+1}^{n_{k+1}} \{w \in X : x_j(w) = a_{k+1}\},$$

and

$$F_M = \bigcap_{k=1}^{M} E_k.$$

By independence of (x_j), we have $\mu(F_M) > 0$ for all $M \geq 1$. Now we calculate the difference of $A_{n_k}x_0(w)$ and $A_{n_{k+1}}x_0(w)$ for $w \in F_M$ and $1 \leq k \leq M$. We have

$$\begin{aligned} |A_{n_k}x_0(w) - A_{n_{k+1}}x_0(w)| &= |\frac{1}{n_{k+1}} \sum_{j=n_k+1}^{n_{k+1}} x_j(w) - (\frac{1}{n_k} - \frac{1}{n_{k+1}}) \sum_{j=1}^{n_k} x_j(w)| \\ &= |\frac{n_{k+1} - n_k}{n_{k+1}} a_{k+1} - (\frac{n_{k+1} - n_k}{n_{k+1} n_k}) \sum_{j=1}^{n_k} \epsilon_j| \\ &= |\frac{n_{k+1} - n_k}{n_{k+1}} (a_{k+1} - \frac{1}{n_k} \sum_{j=1}^{n_k} \epsilon_j)|. \end{aligned}$$

We have chosen a_{k+1} to be 0 or 1 such that $|(a_{k+1} - \frac{1}{n_k}\sum_{j=1}^{n_k}\epsilon_j)| \geq \frac{1}{2}$. Therefore on the measurable set F_M with $\mu(F_M) > 0$ for any k, $1 \leq k \leq M$, we have $|A_{n_k}x_0(w) - A_{n_{k+1}}x_0(w)| \geq \frac{1}{2}\ \frac{n_{k+1}-n_k}{n_{k+1}}$.

Hence, on F_M we have the following estimate:

$$\begin{aligned} S_M x_0(w) &= (\sum_{k=1}^{M}(A_{n_k}x_0(w) - A_{n_{k+1}}x_0(w))^2)^{\frac{1}{2}} \\ &\geq (\sum_{k=1}^{M}(\frac{1}{2}|\frac{n_{k+1}-n_k}{n_{k+1}}|)^2)^{\frac{1}{2}}. \end{aligned}$$

Because $\sum_{k=1}^{\infty}|\frac{n_{k+1}-n_k}{n_{k+1}}|^2 = \infty$, for any $L > 0$ we can choose large enough M such that $S_M x_0(w) > L$ for all $w \in F_M$. Therefore, $\|Sx_0\|_\infty \geq \|S_M x_0\| \geq L$. But L is arbitrary, $\|Sx_0\|_\infty = \infty$. Therefore, $\|Sf\| = \infty$ for some $f \in L^\infty$ in the non-atomic ergodic dynamical system (X, Σ, μ, T). □

Now we apply the Calderón Transfer Principle and the Kakutani-Rokhlin Lemma to the previous result. For convenience, we state the Kakutani-Rokhlin Lemma first.

Lemma 2.3 (Kakutani-Rokhlin Lemma [8], [10], and [11])**.** *Let (X, Σ, μ, T) be an ergodic non-atomic dynamical system, n a positive integer, and $\epsilon > 0$. Then, there is a measurable set $E \subset X$ and $\mu(E) > 0$ such that E, TE,..., T^nE are pairwise disjoint and cover X up to a set of measure less than ϵ.*

Theorem 2.4. *If $\sum_{k=1}^{\infty}|\frac{n_{k+1}-n_k}{n_{k+1}}|^2 = \infty$, then for any ergodic non-atomic dynamical system (X, Σ, μ, T), there is no C constant such that $\|Sf\|_\infty \leq C\|f\|_\infty$ for any $f \in L^\infty$.*

Proof. We use contradiction to prove the theorem. Assume that there is a constant C satisfying $\|Sf\|_\infty \leq C\|f\|_\infty$ for some ergodic non-atomic dynamical system $(X, \Sigma, \mu, \widetilde{T})$. Fix $\varphi \in l^\infty(N)$. For any fixed $\epsilon > 0$ and $n \geq 1$, by the Kakutani-Rokhlin Lemma, there is a set E and pairwise disjoint $\{\widetilde{T}^k E : k = 0, \pm 1, \ldots, \pm n\}$ such that $\mu(X \setminus \bigcup_{k=-n}^{n}\widetilde{T}^k E) < \epsilon$. We define $f_n(x) \in L^\infty$ as follows:

$$f_n(x) = \begin{cases} \varphi(k) & \text{if } x \in T^k E, \text{ and } k = 0, \pm 1, \ldots, \pm n, \\ 0 & \text{otherwise.} \end{cases}$$

We have $f_n \in L^\infty$, $\|f_n\|_\infty \leq \|\varphi_n\|_\infty$. Indeed, $\|f_n\|_\infty$ increases to $\|\varphi\|_\infty$ as n tends to ∞. But in any case, for any M, we have $\|S_M f_n\|_\infty \leq C\|f_n\|_\infty \leq \|\varphi\|_\infty$. In addition, it is also clear from the construction that $\|S_M f_n\|_\infty$ tends to $\|S_M\varphi\|_\infty$ as n tends to ∞. Hence, $\|S_M\varphi\|_\infty \leq C\|\varphi\|_\infty$ for all $\varphi \in l^\infty(\mathbb{Z})$. The constant C is independent of M. So letting M go to infinity, we have for all $\varphi \in l^\infty(\mathbb{Z})$,

$$\|S\varphi\|_\infty \leq C\|\varphi\|_\infty. \tag{2.1}$$

Now, applying the Calderón Transfer Principle, we have for any dynamical system $(X, \Sigma, \mu, \widetilde{S})$ the inequality $\|Sf\|_\infty \leq C\|f\|_\infty$ for this same constant C. This contradicts Theorem 2.2 because we are assuming $\sum_{k=1}^{\infty}|\frac{n_{k+1}-n_k}{n_{k+1}}|^2 = \infty$. □

A close examination of the arguments above show that we have proved this next corollary. It will be useful in the next section.

Corollary 2.5. *Given a non-decreasing sequence (n_k), there exists $\varphi \in l^\infty(\mathbb{Z})$ such that*

$$\|S_M\varphi\|_\infty \geq \frac{1}{3}(\sum_{k=1}^{M} |\frac{n_{k+1}-n_k}{n_{k+1}}|^2)^{\frac{1}{2}} \|\varphi\|_\infty.$$

So, if

$$\sum_{k=1}^{\infty} |\frac{n_{k+1}-n_k}{n_{k+1}}|^2 = \infty,$$

then for any $C > 0$, there exist M and $\varphi \in l^\infty(\mathbb{Z})$ such that $\|S_M\varphi\|_\infty \geq C\|\varphi\|_\infty$.

Using a standard Baire Category argument in [2], we have the following theorem.

Theorem 2.6. *If $\sum_{k=1}^{\infty} |\frac{n_{k+1}-n_k}{n_{k+1}}|^2 = \infty$, then for any ergodic non-atomic dynamical system (X, Σ, μ, T) there is an $f \in L^\infty$ such that $\|Sf\| = \infty$. Furthermore, there is actually a dense G_δ set Γ in L^∞ such that for all $f \in \Gamma$ we have $\|Sf\| = \infty$.*

Proof. Let $L_K = \{f \in L^\infty : \|Sf\|_\infty \leq K\}$.

Claim 1: L_K is a closed set of L^∞.

Since $\{f \in L^\infty : \|S_M f\|_\infty \leq K\}$ is a closed set in L^∞ for each finite M, $L_K = \bigcap_{M=1}^{\infty}\{f \in L^\infty : \|S_M f\|_\infty \leq K\}$ is a closed set in L^∞.

Claim 2: L_K has no interior.

If Claim 2 is false, then there is $f_0 \in L_K$ and $\delta > 0$ such that $f_0 + \delta f \in L_K$ for any f with $\|f\|_\infty \leq 1$. We have the following estimate for $\|Sf\|_\infty$ if $\|f\|_\infty \leq 1$:

$$\begin{aligned}\|Sf\|_\infty &= \|\frac{1}{\delta}S(\delta f + f_0 - f_0)\|_\infty \\ &\leq \frac{1}{\delta}(\|S(\delta f + f_0)\|_\infty + \|S(f_0)\|_\infty) \\ &\leq \frac{1}{\delta}(K + K) = \frac{1}{\delta}(2K).\end{aligned}$$

Therefore $\|Sf\|_\infty \leq \frac{1}{\delta}(2K)\|f\|_\infty$ for all $f \in L^\infty$. This is a contradiction to Theorem 2.4. Hence L_K has empty interior.

Let $\Lambda = \bigcup_{K=1}^{\infty} L_K$. Then $\Gamma = L^\infty \backslash \Lambda$ is a dense G_δ set and not empty by the Baire Category Theorem in the complete metric space L^∞. Moreover, for any $f \in \Gamma$, we have $\|Sf\|_\infty = \infty$. □

3. Characterizing the strong estimates for Lebesgue derivatives

In this section, we are going to get the same results for Lebesgue derivatives as we did for ergodic averages.

Theorem 3.1. *If (ϵ_k) is decreasing to 0 and*

$$\sum_{k=1}^{\infty} |\frac{\epsilon_k - \epsilon_{k+1}}{\epsilon_k}|^2 < \infty,$$

then there is a constant

$$C = 2\left\{\sum_{k=1}^{\infty} |\frac{\epsilon_k - \epsilon_{k+1}}{\epsilon_k}|^2\right\}^{\frac{1}{2}} < \infty$$

such that $\|Sf\|_\infty \leq C\|f\|_\infty$ for $f \in L^\infty(\mathbb{R})$.

Proof. Let $f \in L^\infty(\mathbb{R})$. We have

$$\begin{aligned}
Sf(x) &= \left(\sum_{k=1}^{\infty}\left|\frac{1}{\epsilon_{k+1}}\int_0^{\epsilon_{k+1}} f(x+t)dt - \frac{1}{\epsilon_k}\int_0^{\epsilon_k} f(x+t)dt\right|^2\right)^{\frac{1}{2}} \\
&= \left(\sum_{k=1}^{\infty}\left|(\frac{1}{\epsilon_{k+1}}-\frac{1}{\epsilon_k})\int_0^{\epsilon_{k+1}} f(x+t)dt - \frac{1}{\epsilon_k}\int_{\epsilon_{k+1}}^{\epsilon_k} f(x+t)dt\right|^2\right)^{\frac{1}{2}} \\
&\le \left(\sum_{k=1}^{\infty}\left|(\frac{1}{\epsilon_{k+1}}-\frac{1}{\epsilon_k})\int_0^{\epsilon_{k+1}} f(x+t)dt\right|^2\right)^{\frac{1}{2}} + \left(\sum_{k=1}^{\infty}\left|\frac{1}{\epsilon_k}\int_{\epsilon_{k+1}}^{\epsilon_k} f(x+t)dt\right|^2\right)^{\frac{1}{2}} \\
&\le \left(\sum_{k=1}^{\infty}\left|\frac{\epsilon_k-\epsilon_{k+1}}{\epsilon_k\epsilon_{k+1}}\epsilon_{k+1}\right|^2\right)^{\frac{1}{2}}\|f\|_\infty + \left(\sum_{k=1}^{\infty}\left|\frac{\epsilon_k-\epsilon_{k+1}}{\epsilon_k}\right|^2\right)^{\frac{1}{2}}\|f\|_\infty \\
&\le \left(\sum_{k=1}^{\infty}\left|\frac{\epsilon_k-\epsilon_{k+1}}{\epsilon_k}\right|^2\right)^{\frac{1}{2}}\|f\|_\infty + \left(\sum_{k=1}^{\infty}\left|(\frac{\epsilon_k-\epsilon_{k+1}}{\epsilon_k})\right|^2\right)^{\frac{1}{2}}\|f\|_\infty \\
&= 2\left(\sum_{k=1}^{\infty}\left|\frac{\epsilon_k-\epsilon_{k+}}{\epsilon_k}\right|^2\right)^{\frac{1}{2}}\|f\|_\infty.
\end{aligned}$$

Because $\sum_{k=1}^{\infty}(\frac{\epsilon_k-\epsilon_{k+1}}{\epsilon_k})^2 < \infty$, we have $Sf \in L^\infty$. □

We can also prove the converse result for this bound, analogously to the arguments in the last section. First, we need a suitable variant of Corollary 2.5.

Lemma 3.2. *Given a non-decreasing sequence (n_k), there exists $\varphi \in L^\infty(\mathbb{R})$ such that $\|S_M\varphi\|_\infty \ge \frac{1}{3}(\sum_{k=1}^{M}|\frac{n_{k+1}-n_k}{n_{k+1}}|^2)^{\frac{1}{2}}\|\varphi\|_\infty$. So, if $\sum_{k=1}^{\infty}|\frac{n_{k+1}-n_k}{n_{k+1}}|^2 = \infty$, then for any $C > 0$, there exist M and $\varphi \in L^\infty(\mathbb{R})$ such that $\|S_M\varphi\|_\infty \ge C\|\varphi\|_\infty$.*

Proof. The proof of this is a standard quantization argument. A more complete version of this result allows one to transfer back and forth between norm inequalities in $l^\infty(\mathbb{Z})$ for ergodic averages for the shift $T(i) = i+1, i \in \mathbb{Z}$, to norm inequalities in $L^\infty(\mathbb{R})$ for the ergodic averages given by the regular flow of $\mathbb{R}$ on itself. We only need one direction of this here. We will be comparing values of two square functions, one from averages on $\mathbb{Z}$ and one from averages on $\mathbb{R}$. So we will denote the first square function by $\mathcal{S}$ and the second one just by S as we have been doing in this section. Let $\varphi_o \in l^\infty(\mathbb{Z})$ be as in Corollary 2.5 such that $\|\mathcal{S}_M\varphi_o\|_\infty \ge \frac{1}{3}(\sum_{k=1}^{M}|\frac{n_{k+1}-n_k}{n_{k+1}}|^2)^{\frac{1}{2}}\|\varphi_o\|_\infty$. Let $\varphi \in L^\infty(\mathbb{R})$ be given by $\varphi = \varphi_o^* \star 1_{[0,1]}$. So φ takes the value $\overline{\varphi_o}(-k)$ on $[k, k+1)$. Also, $\|\varphi\|_\infty = \|\varphi_o\|_\infty$. Consider a typical average in $\mathbb{R}$ given by $A_n\varphi(t) = \frac{1}{n}\int_0^n \varphi(t+x)dx$. This is the same as $\mathcal{A}_n\varphi_o^* \star 1_{[-1,0]} \star 1_{[0,1]}$ where $\mathcal{A}_n\varphi_o(j) = \frac{1}{n}\sum_{k=0}^{n-1}\varphi_o(j+k)$ for all $j \in \mathbb{Z}$. Now $T(t) = 1_{[-1,0]} \star 1_{[0,1]}$ is a continuous piecewise linear function that is 0 off $(-1,1)$, 1 at 0, and linear on each of $[-1,0]$ and $[0,1]$. This creates some interference between adjacent values when trying to make the estimate that we need. For example, for t near $m \in \mathbb{Z}$, $A_n\varphi(t) = \delta_t\mathcal{A}_n\varphi_o^*(m) + \epsilon_{t,1}\mathcal{A}_n\varphi_o^*(m-1) + \epsilon_{t,2}\mathcal{A}_n\varphi_o^*(m+1)$ where δ_t approaches 1 and both $\epsilon_{t,1}$ and $\epsilon_{t,2}$ approach 0 as t converges to m. Hence, for values of t

close enough to m, $S_M^{(n_k)}\varphi(t)$ is as close to $\mathcal{S}_M^{(n_k)}\varphi_o(-m)$ as we like. Therefore, $\|S_M^{(n_k)}\varphi\|_\infty \geq \|\mathcal{S}_M^{(n_k)}\varphi_o\|_\infty \geq \frac{1}{3}(\sum_{k=1}^M |\frac{n_{k+1}-n_k}{n_{k+1}}|^2)^{\frac{1}{2}}\|\varphi_o\|_\infty$. This gives the desired result. □

Theorem 3.3. *If $\{\epsilon_k\}_{k=1}^\infty$ is decreasing to 0 and $\sum_{k=1}^\infty |\frac{\epsilon_k-\epsilon_{k+1}}{\epsilon_k}|^2 = \infty$, there is an $f \in L^\infty$ such that $||Sf||_\infty = \infty$. Indeed, there is a dense G_δ set Γ in L^∞ such that $\|Sf\|_\infty = \infty$ for any $f \in \Gamma$.*

Proof. Let $C_0 = 4\sum_{n=1}^\infty \frac{1}{n^2}$.

First step: For any $C > 0$, there is an $M \in \mathbb{Z}$ and a $\varphi \in L^\infty$ such that $\|S_M^{\{r_k\}}\varphi\|_\infty \geq C\|\varphi\|_\infty$. Fix a large value $\widetilde{C}$ that we will specify later. Choose numbers H, M, and $\{r_k\}_{k=1}^M$ such that $r_k = \frac{H}{\epsilon_M}\epsilon_{M-k+1}$, there is only one r_k in the interval $[n_k, n_k+1]$ where $n_k = [r_k]$, and $\sum_{k=1}^M \left|\frac{\epsilon_k-\epsilon_{k+1}}{\epsilon_k}\right|^2 \geq \widetilde{C}$. We have $r_k = n_k + \alpha_k$ where $0 \leq \alpha_k < 1$. Since

$$\begin{aligned}\sum_{k=1}^M \left|\frac{r_{k+1}-r_k}{r_{k+1}}\right|^2 &= \sum_{k=1}^M \left|\frac{\frac{H}{\epsilon_M}\epsilon_{M-k} - \frac{H}{\epsilon_M}\epsilon_{M-k+1}}{\frac{H}{\epsilon_M}\epsilon_{M-k}}\right|^2 \\ &= \sum_{k=1}^M \left|\frac{\epsilon_{M-k}-\epsilon_{M-k+1}}{\epsilon_{M-k}}\right|^2 = \sum_{k=1}^M \left|\frac{\epsilon_k-\epsilon_{k+1}}{\epsilon_k}\right|^2,\end{aligned}$$

we have $\sum_{k=1}^M \left|\frac{r_{k+1}-r_k}{r_{k+1}}\right|^2 \geq \widetilde{C}$ as well.

We calculate the difference of the terms $\frac{n_{k+1}-n_k}{n_{k+1}}$ and $\frac{r_{k+1}-r_k}{r_{k+1}}$:

$$\begin{aligned}\left|\frac{n_{k+1}-n_k}{n_{k+1}} - \frac{r_{k+1}-r_k}{r_{k+1}}\right| &= \left|\frac{n_k r_{k+1} - n_{k+1}r_{k+1} - n_{k+1}r_k + n_{k+1}r_{k+1}}{n_{k+1}r_{k+1}}\right| \\ &= \left|\frac{n_k r_{k+1} - n_{k+1}r_k}{n_{k+1}r_{k+1}}\right| \\ &= \left|\frac{n_k(n_{k+1}+\alpha_{k+1}) - n_{k+1}(n_k+\alpha_k)}{n_{k+1}r_{k+1}}\right| \\ &= \left|\frac{n_k\alpha_{k+1} - n_{k+1}\alpha_k}{n_{k+1}r_{k+1}}\right| \\ &\leq \left|\frac{2n_{k+1}}{n_k n_{k+1}}\right| \\ &= \left|\frac{2}{n_k}\right| \leq \left|\frac{2}{k}\right|.\end{aligned}$$

Now we estimate the difference between $\left|\frac{n_{k+1}-n_k}{n_{k+1}}\right|^2$ and $\left|\frac{r_{k+1}-r_k}{r_{k+1}}\right|^2$:

$$\begin{aligned}\left|\left(\frac{n_{k+1}-n_k}{n_{k+1}}\right)^2 - \left(\frac{r_{k+1}-r_k}{r_{k+1}}\right)^2\right| &\leq \left|\frac{n_{k+1}-n_k}{n_{k+1}} - \frac{r_{k+1}-r_k}{r_{k+1}}\right|^2 \\ &\leq \left|\frac{2}{k}\right|^2 = \frac{4}{k^2}.\end{aligned}$$

Therefore, we have $\left(\frac{n_{k+1}-n_k}{n_{k+1}}\right)^2 \geq \left(\frac{r_{k+1}-r_k}{r_{k+1}}\right)^2 - \frac{4}{k^2}$. Hence,

$$\sum_{k=1}^{M}\left(\frac{n_{k+1}-n_k}{n_{k+1}}\right)^2 \geq \sum_{k=1}^{M}\left(\frac{r_{k+1}-r_k}{r_{k+1}}\right)^2 - \sum_{k=1}^{M}\frac{4}{k^2} \geq \widetilde{C} - C_0.$$

According to Lemma 3.2, we have a $\varphi \in L^\infty(\mathbb{R})$ such that the partial sum $\|S_M^{(n_k)}\varphi\|_\infty \geq \frac{1}{3}(\widetilde{C}-C_0)^{\frac{1}{2}}\|\varphi\|_\infty$.

Now we will transfer the partial sum results for the sequence $\{n_k\}_{k=1}^M$ to the sequence $\{r_k\}_{k=1}^M$ with some adjustment for the constants. We are going to prove that for the above $\varphi \in L^\infty(\mathbb{R})$ we have $\|S_M^{(r_k)}\varphi\|_\infty \geq \frac{1}{3}((\widetilde{C}-C_0)^{\frac{1}{2}} - \sqrt{C_0})\|\varphi\|_\infty$.

First we estimate the difference between $\frac{1}{r_k}\int_0^{r_k}\varphi(x+t)dt$ and $\frac{1}{n_k}\int_0^{n_k}\varphi(x+t)dt$. Notice that $0 \leq r_k - n_k < 1$. We have

$$\begin{aligned}
&\left|\frac{1}{r_k}\int_0^{r_k}\varphi(x+t)dt - \frac{1}{n_k}\int_0^{n_k}\varphi(x+t)dt\right| \\
&\qquad = \left|(\frac{1}{r_k}-\frac{1}{n_k})\int_0^{n_k}\varphi(x+t)dt - \frac{1}{r_k}\int_{n_k}^{r_k}\varphi(x+t)dt\right| \\
&\qquad = \left|\frac{r_k-n_k}{r_kn_k}\right|\|\varphi\|_\infty + \left|\frac{r_k-n_k}{r_k}\right|\|\varphi\|_\infty \\
&\qquad \leq \frac{1}{r_k}\|\varphi\|_\infty + \frac{1}{r_k}\|\varphi\|_\infty \leq \frac{2}{k}\|\varphi\|_\infty.
\end{aligned}$$

Now we are ready to estimate the difference between $S_M^{(r_k)}\varphi(x)$ and $S_M^{\{n_k\}}\varphi(x)$.

$$\begin{aligned}
\left|S_M^{\{r_k\}}\varphi(x) - S_M^{\{n_k\}}\varphi(x)\right| &= \left|\left\{\sum_{k=1}^{M}\left(\frac{1}{r_k}\int_0^{r_k}\varphi(x+t)dt - \frac{1}{r_{k+1}}\int_0^{r_{k+1}}\varphi(x+t)dt\right)^2\right\}^{\frac{1}{2}}\right. \\
&\quad \left. - \left\{\sum_{k=1}^{M}\left(\frac{1}{n_k}\int_0^{n_k}\varphi(x+t)dt - \frac{1}{n_{k+1}}\int_0^{n_{k+1}}\varphi(x+t)dt\right)^2\right\}^{\frac{1}{2}}\right| \\
&\leq \left\{\sum_{k=1}^{M}\left((\frac{1}{r_k}\int_0^{r_k}\varphi(x+t)dt - \frac{1}{n_k}\int_0^{n_k}\varphi(x+t)dt)\right.\right. \\
&\quad \left.\left. - (\frac{1}{r_{k+1}}\int_0^{r_{k+1}}\varphi(x+t)dt - \frac{1}{n_{k+1}}\int_0^{n_{k+1}}\varphi(x+t)dt)\right)^2\right\}^{\frac{1}{2}} \\
&\leq 2\left\{\sum_{k=1}^{M}(\frac{2}{k}\|\varphi\|_\infty)^2\right\}^{\frac{1}{2}} = 4\left(\sum_{k=1}^{M}\frac{1}{k^2}\right)^{\frac{1}{2}}\|\varphi\|_\infty.
\end{aligned}$$

Taking the supremum over all $x \in \mathbb{R}$, we have $\|S_M^{\{r_k\}}\varphi - S_M^{\{n_k\}}\varphi\|_\infty \leq 2\sqrt{C_0}\|\varphi\|_\infty$. So we have

$$\left| \|S_M^{(r_k)}\varphi\|_\infty - \|S_M^{(n_k)}\varphi\|_\infty \right| \leq 2\sqrt{C_0}\|\varphi\|_\infty.$$

Hence, we have $\|S_M^{(r_k)}\varphi\|_\infty \geq \|S_M^{(n_k)}\varphi\|_\infty - 2\sqrt{C_0}\|\varphi\|_\infty$. Finally, we have

$$\|S_M^{(r_k)}\varphi\|_\infty \geq \frac{1}{3}((\widetilde{C} - C_0)^{\frac{1}{2}} - 2\sqrt{C_0})\|\varphi\|_\infty.$$

But then since C_0 is an absolute constant and $\widetilde{C}$ is arbitrary, it is clear that for all C there exist M and $\varphi \in L^\infty(\mathbb{R})$ such that $\|S_M^{(r_k)}\varphi\|_\infty \geq C\|\varphi\|_\infty$

Second step: For any $C > 0$, there is a $g \in L^\infty$ such that $\|S^{(\epsilon_k)}g\|_\infty \geq C\|g\|_\infty$. In the first step, we have M and $\varphi \in L^\infty(\mathbb{R})$ such that $\|S_M^{(r_k)}\varphi\|_\infty \geq C\|\varphi\|_\infty$. We let $g(x) = \varphi(\frac{H}{\epsilon_M}x)$ and note that $\|g\|_\infty = \|\varphi\|_\infty$. We calculate

$$\begin{aligned}
\frac{1}{\epsilon_k}\int_0^{\epsilon_k} g(x+t)dt &= \frac{1}{\epsilon_k}\int_0^{\epsilon_k} \varphi(\frac{H}{\epsilon_M}(x+t))dt \\
&= \frac{1}{\epsilon_k}\int_0^{\epsilon_k} \varphi(\frac{H}{\epsilon_M}x + \frac{H}{\epsilon_M}t)dt \\
&= \frac{1}{\epsilon_k}\int_0^{\frac{H}{\epsilon_M}\epsilon_k} \varphi(\frac{H}{\epsilon_M}x + u)\frac{1}{\frac{H}{\epsilon_M}}du \\
&= \frac{1}{\frac{H}{\epsilon_M}\epsilon_k}\int_0^{\frac{H}{\epsilon_M}\epsilon_k} \varphi(\frac{H}{\epsilon_M}x + u)du \\
&= \frac{1}{r_{M-k+1}}\int_0^{r_{M-k+1}} \varphi(\frac{H}{\epsilon_M}x + u)du.
\end{aligned}$$

So we have $S_M^{\{\epsilon_k\}}g(x) = S_M^{\{r_k\}}\varphi(\frac{H}{\epsilon_M}x)$. Thus, $\|S_M^{\{\epsilon_k\}}g\|_\infty = \|S_M^{\{r_k\}}\varphi\|_\infty$. Hence, we have $\|S_M^{\{\epsilon_k\}}g\|_\infty \geq C\|g\|_\infty$. Because $\|S^{(\epsilon_k)}g\|_\infty \geq \|S_M^{\{\epsilon_k\}}g\|_\infty$, we have our results for the second step $\|S^{(\epsilon_k)}g\|_\infty \geq C\|g\|_\infty$.

Third step: There exists a generic set of $f \in L^\infty$ such that $\|S^{(\epsilon_k)}f\|_\infty = \infty$. This argument is a Baire Category argument that is analogous to the one given in Theorem 2.6. □

The same results as in Theorem 3.3 hold for R_k increasing to infinity in place of ϵ_k decreasing to zero. In this context, also see [12] where important caveats about square functions are pointed out; see the example on p. 245 for instance. In [9], for $\epsilon_n = 2^{-n}$, there is a detailed discussion for square functions in ergodic averages and Lebesgue derivatives.

4. Further results

The methods above can be used on other operators. Let $B = (B_n)$ be a sequence of linear operators on L^p and let (B_{n_k}) be a fixed subsequence. For any $r, 1 \leq r < \infty$, let

$$S_r^{(n_k)} f = \left(\sum_{k=1}^{\infty} |B_{n_{k+1}} f - B_{n_k} f|^r \right)^{1/r}$$

and let

$$O_r^{(n_k)} f = \left(\sum_{k=1}^{\infty} \sup_{n_k \leq i, j \leq n_{k+1}} |B_j f - B_i f|^r \right)^{1/r}.$$

The following result holds; the proof is similar to arguments in the previous sections.

Theorem 4.1. *If $(B_n) = (A_n)$, the standard ergodic averages for an ergodic dynamical system, or if $(B_n) = (D_n)$ for some sequence (ϵ_n) decreasing to 0, then $S_r^{(n_k)}$ and $O_r^{(n_k)}$ both map L^∞ to L^∞ if and only if $\sum_{k=1}^{\infty} (\frac{n_{k+1}-n_k}{n_k})^r < \infty$. If $\sum_{k=1}^{\infty} (\frac{n_{k+1}-n_k}{n_k})^r = \infty$, then there is a dense G_δ subset Γ of L^∞ such that for all $f \in \Gamma$, both $\|S_r^{(n_k)} f\|_\infty = \infty$ and $\|O_r^{(n_k)} f\|_\infty = \infty$.*

Remark 4.2. This remark is about general results for operators S_r, O_r, and V_r with $1 \leq r < \infty$.

(1) It is not hard to see that a given (n_k) might not allow $S_r^{(n_k)}$ or $O_r^{(n_k)}$ to map L^∞ to L^∞, but for larger values of r they would map L^∞ to L^∞.

(2) Consider

$$V_r^B f = \left(\sup_{(n_k)} \sum_{k=1}^{\infty} |B_{n_{k+1}} f - B_{n_k} f|^r \right)^{1/r}$$

where the supremum is over all non-decreasing subsequences. Because of the supremum over all non-decreasing sequences in the definition, V_r^B never maps L^∞ to L^∞.

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