EVERY NUMERICAL SEMIGROUP IS ONE HALF OF A SYMMETRIC NUMERICAL SEMIGROUP

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Abstract. Let $S$ be a numerical semigroup. Then there exists a symmetric numerical semigroup $\overline{S}$ such that $S = \{ n \in \mathbb{N} \mid 2n \in \overline{S} \}$.

Let $\mathbb{N}$ be the set of nonnegative integers. A submonoid $M$ of $\mathbb{N}$ is a subset of $\mathbb{N}$ closed under addition and such that $0 \in M$. A numerical semigroup $S$ is a submonoid of $\mathbb{N}$ such that $\mathbb{N} \setminus S$ is finite. The elements of $H(S) = \mathbb{N} \setminus S$ are the gaps of $S$, and the largest integer not belonging to $S$ is known as the Frobenius number of $S$, denoted here by $F(S)$. There is a huge amount of papers devoted to the Frobenius number and more particularly to the problem of computing or bounding it (see [1] and the vast list of references given there).

A numerical semigroup $S$ is said to be symmetric if for every $z \in \mathbb{Z}$, either $z \in S$ or $F(S) - z \in S$. Symmetric numerical semigroups earned some relevance due to a result by Kunz (see [2]), which states that the semigroup ring of a symmetric numerical semigroup is Gorenstein. Thus this class of numerical semigroups became a source of examples of Gorenstein rings.

For a numerical semigroup $S$, define $N(S) = \{ s \in S \mid s < F(S) \}$. Clearly, $\#N(S) + \#H(S) = F(S) + 1$, and $S$ is symmetric if and only if $\#N(S) = \#H(S)$, where $\#A$ denotes the cardinality of $A$.

Let $M$ be a submonoid of $\mathbb{N}$ and let $d$ be a positive integer. Then

$$M_d = \{ n \in \mathbb{N} \mid dn \in M \}$$

is again a submonoid of $\mathbb{N}$, called the quotient of $M$ by $d$. Clearly, $M \subseteq M_d$.

The aim of this paper is to show that for every numerical semigroup $S$, there exists a symmetric numerical semigroup $\overline{S}$ such that $S = \frac{\overline{S}}{2}$.

Given $A$, a subset of $\mathbb{Z}$, denote by $2A = \{ 2a \mid a \in A \}$ (not to be confused with $A + A$). Given $x_1, \ldots, x_k \in \mathbb{Z}$, when we write $\{ x_1, \ldots, x_k, \rightarrow \}$, we mean that all the integers greater than $x_k$ belong to the set.

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Theorem 1. Let $S$ be a numerical semigroup. Then there exists a symmetric numerical semigroup $\overline{S}$ such that

$$S = \overline{S}/2.$$  

Proof. Let $g = F(S)$. If $g = -1$, then $S = \mathbb{N}$. Clearly $\mathbb{N} = \frac{\mathbb{N}}{2}$ and $\mathbb{N}$ is symmetric. Now assume that $g = 1$. In this setting $S = \{0, 2, \rightarrow\}$. The reader can easily check that by taking $\overline{S} = \{0, 4, 5, 6, 8, \rightarrow\}$, one gets that $S = \frac{\overline{S}}{2}$, with $\overline{S}$ symmetric.

Now suppose that $g \geq 2$. Let $H(S) = \{h_1, \ldots, h_t\}$. We claim that

$$\overline{S} = 2S \cup \{2g + (2g - 2h_1) - 1, \ldots, 2g + (2g - 2h_t) - 1\} \cup \{4g, \rightarrow\}$$

is a symmetric numerical semigroup and that $S = \frac{\overline{S}}{2}$.

First we must prove that $\overline{S}$ is a submonoid of $\mathbb{N}$. This is a consequence of the following remarks.

1. $2S$ is a submonoid of $\mathbb{N}$.

2. The result of adding any nonnegative integer to any element in $\{4g, \rightarrow\}$ remains in $\{4g, \rightarrow\}$.

3. For every $i, j \in \{1, \ldots, t\}$, we have that $2g + (2g - 2h_1) - 1 + 2g + (2g - 2h_2) - 1 = 2(2g + (g - h_i) + (g - h_j) - 1)$. As $g = \max\{h_1, \ldots, h_t\}$, $2g + (g - h_i) + (g - h_j) - 1 \geq 2g - 1$. Since by hypothesis $g \geq 2$, we deduce that $2g + (g - h_i) + (g - h_j) - 1 \geq g + 1$.

Hence $2g + (g - h_i) + (g - h_j) - 1 \in S$, and $2g + (2g - 2h_1) - 1 + 2g + (2g - 2h_2) - 1 \in 2S \subseteq \overline{S}$.

4. Now take $s \in S \setminus \{0\}$ and $i \in \{1, \ldots, t\}$. We distinguish two cases. 
   - If $s > h_i$, then $2g + (2g - 2h_1) - 1 + 2s = 4g + 2(s - h_i) - 1 \geq 4g$, whence $2g + (2g - 2h_1) - 1 + 2g$ belongs to $\overline{S}$.
   - If $s < h_i$, then $0 \leq g + (s - h_i) < g$. Thus $g + s - h_i = g - x$, with $x = h_i - s \in \{1, \ldots, g\}$. As $h_i = s + x \notin S$, and $s \in S$, we have that $x \notin S$. Hence $x = h_j$ for some $j \in \{1, \ldots, t\}$. This leads to $2g + (2g - 2h_1) - 1 + 2s = 2g + (2g - 2h_j) - 1 \in \overline{S}$.

Since $\{4g, \rightarrow\} \subseteq \overline{S}$, we have that $\mathbb{N} \setminus \overline{S}$ has finitely many elements. This proves that $\overline{S}$ is a numerical semigroup.

Observe that $4g - 1$ is odd. As $\overline{S} = 2S \cup \{2g + (2g - 2h_1) - 1, \ldots, 2g + (2g - 2h_t) - 1\} \cup \{4g, \rightarrow\}$ and the elements in $2S$ are all even, if $4g - 1 \in \overline{S}$, then $4g - 1 = 2g + (2g - 2h_i) - 1$ for some $i \in \{1, \ldots, t\}$. But this leads to $h_i = 0$, which is impossible. Hence $4g - 1 = F(\overline{S})$, because $\{4g, \rightarrow\} \subseteq \overline{S}$.

In order to prove that $\overline{S}$ is symmetric, let us show that $\#N(\overline{S}) = \#H(\overline{S})$. By using that $F(\overline{S}) = 4g - 1$, we easily deduce that

$$N(\overline{S}) = 2N(S) \cup \{2(g + 1), 2(g + 2), \ldots, 2(g + (g - 1))\}$$

$$\cup \{2g + (2g - 2h_1) - 1, \ldots, 2g + (2g - 2h_t) - 1\}$$

(and that these sets are disjoint). Hence $\#N(\overline{S}) = \#N(S) + g - 1 + \#H(S)$. As we know that $g + 1 = \#H(S) + \#N(S)$, we obtain that $\#N(\overline{S}) = 2g$. We also know that $4g - 1 + 1 = \#H(\overline{S}) + \#N(\overline{S})$, and consequently $\#H(\overline{S}) = 2g = \#N(\overline{S})$.

Finally, we show that $S = \frac{\overline{S}}{2}$. As $2S \subseteq \overline{S}$, the inclusion $S \subseteq \frac{\overline{S}}{2}$ is trivial. For the other inclusion, let $x \in \frac{\overline{S}}{2}$. Then $2x \in \overline{S}$. From the way $\overline{S}$ is defined, as $2x$ is even, either $2x \in 2S$ or $2x \geq 4g$. If $2x \in 2S$, then trivially $x \in S$. If $2x \geq 4g$, then $2x > 2g$ and thus $x > g$, which also leads to $x \in S$. □
References


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