EVERY NUMERICAL SEMIGROUP IS ONE HALF OF A SYMMETRIC NUMERICAL SEMIGROUP

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(Communicated by Martin Lorenz)

Abstract. Let \( S \) be a numerical semigroup. Then there exists a symmetric numerical semigroup \( \overline{S} \) such that \( S = \{ n \in \mathbb{N} \mid 2n \in \overline{S} \} \).

Let \( \mathbb{N} \) be the set of nonnegative integers. A submonoid \( M \) of \( \mathbb{N} \) is a subset of \( \mathbb{N} \) closed under addition and such that \( 0 \in M \). A numerical semigroup \( S \) is a submonoid of \( \mathbb{N} \) such that \( \mathbb{N} \setminus S \) is finite. The elements of \( H(S) = \mathbb{N} \setminus S \) are the gaps of \( S \), and the largest integer not belonging to \( S \) is known as the Frobenius number of \( S \), denoted here by \( F(S) \). There is a huge amount of papers devoted to the Frobenius number and more particulary to the problem of computing or bounding it (see [1] and the vast list of references given there).

A numerical semigroup \( S \) is said to be symmetric if for every \( z \in \mathbb{Z} \), either \( z \in S \) or \( F(S) - z \in S \). Symmetric numerical semigroups earned some relevance due to a result by Kunz (see [2]), which states that the semigroup ring of a symmetric numerical semigroup is Gorenstein. Thus this class of numerical semigroups became a source of examples of Gorenstein rings.

For a numerical semigroup \( S \), define \( N(S) = \{ s \in S \mid s < F(S) \} \). Clearly, \( \#N(S) + \#H(S) = F(S) + 1 \), and \( S \) is symmetric if and only if \( \#N(S) = \#H(S) \), where \( \#A \) denotes the cardinality of \( A \).

Let \( M \) be a submonoid of \( \mathbb{N} \) and let \( d \) be a positive integer. Then

\[ \frac{M}{d} = \{ n \in \mathbb{N} \mid dn \in M \} \]

is again a submonoid of \( \mathbb{N} \), called the quotient of \( M \) by \( d \). Clearly, \( M \subseteq \frac{M}{d} \).

The aim of this paper is to show that for every numerical semigroup \( S \), there exists a symmetric numerical semigroup \( \overline{S} \) such that \( S = \frac{\overline{S}}{2} \).

Given \( A \), a subset of \( \mathbb{Z} \), denote by \( 2A = \{ 2a \mid a \in A \} \) (not to be confused with \( A + A \)). Given \( x_1, \ldots, x_k \in \mathbb{Z} \), when we write \( \{ x_1, \ldots, x_k, \rightarrow \} \), we mean that all the integers greater than \( x_k \) belong to the set.

Received by the editors January 19, 2007.

2000 Mathematics Subject Classification. Primary 20M14, 13H10.

Key words and phrases. Numerical semigroup, symmetric numerical semigroup, Frobenius number.

The authors were supported by the project MTM2004-01446 and FEDER funds.

The authors want to thank the referee for her/his comments and suggestions.
Theorem 1. Let $S$ be a numerical semigroup. Then there exists a symmetric numerical semigroup $\overline{S}$ such that

$$S = \frac{\overline{S}}{2}.$$ 

Proof. Let $g = F(S)$. If $g = -1$, then $S = \mathbb{N}$. Clearly $\mathbb{N} = \frac{\mathbb{N}}{2}$ and $\mathbb{N}$ is symmetric. Now assume that $g = 1$. In this setting $S = \{0, 2, \to\}$. The reader can easily check that by taking $\overline{S} = \{0, 4, 5, 6, 8, \to\}$, one gets that $S = \frac{\overline{S}}{2}$, with $\overline{S}$ symmetric.

Now suppose that $g \geq 2$. Let $H(S) = \{h_1, \ldots, h_t\}$. We claim that

$$\overline{S} = 2S \cup \{2g + (2g - 2h_1) - 1, \ldots, 2g + (2g - 2h_t) - 1\} \cup \{4g, \to\}$$

is a symmetric numerical semigroup and that $S = \frac{\overline{S}}{2}$.

First we must prove that $\overline{S}$ is a submonoid of $\mathbb{N}$. This is a consequence of the following remarks.

1. $2S$ is a submonoid of $\mathbb{N}$.

2. The result of adding any nonnegative integer to any element in $\{4g, \to\}$ remains in $\{4g, \to\}$.

3. For every $i, j \in \{1, \ldots, t\}$, we have that $2g + (2g - 2h_i) - 1 + 2g + (2g - 2h_j) - 1 = 2(2g + (g - h_i) + (g - h_j) - 1)$. As $g = \max\{h_1, \ldots, h_t\}$, $2g + (g - h_i) + (g - h_j) - 1 \geq 2g - 1$. Since by hypothesis $g \geq 2$, we deduce that $2g + (g - h_i) + (g - h_j) - 1 \geq g + 1$. Hence $2g + (g - h_i) + (g - h_j) - 1 \in S$, and $2g + (2g - 2h_i) - 1 + 2g + (2g - 2h_j) - 1 \in 2S \subseteq \overline{S}$.

4. Now take $s \in S \setminus \{0\}$ and $i \in \{1, \ldots, t\}$. We distinguish two cases.

   - If $s > h_i$, then $2g + (2g - 2h_i) - 1 + 2s = 4g + 2(s - h_i) - 1 \geq 4g$, whence $2g + (2g - 2h_i) - 1 + 2s$ belongs to $\overline{S}$.

   - If $s < h_i$, then $0 \leq g + (s - h_i) < g$. Thus $g + s - h_i = g - x$, with $x = h_i - s \in \{1, \ldots, g\}$. As $h_i = s + x \notin S$, and $s \in S$, we have that $x \notin S$. Hence $x = h_j$ for some $j \in \{1, \ldots, t\}$. This leads to $2g + (2g - 2h_i) - 1 + 2s = 2g + (2g - 2h_j) - 1 \in \overline{S}$.

Since $\{4g, \to\} \subseteq \overline{S}$, we have that $\mathbb{N} \setminus \overline{S}$ has finitely many elements. This proves that $\overline{S}$ is a numerical semigroup.

Observe that $4g - 1$ is odd. As $\overline{S} = 2S \cup \{2g + (2g - 2h_1) - 1, \ldots, 2g + (2g - 2h_t) - 1\} \cup \{4g, \to\}$ and the elements in $2S$ are all even, if $4g - 1 \notin \overline{S}$, then $4g - 1 = 2g + (2g - 2h_i) - 1$ for some $i \in \{1, \ldots, t\}$. But this leads to $h_i = 0$, which is impossible. Hence $4g - 1 = F(\overline{S})$, because $\{4g, \to\} \subseteq \overline{S}$.

In order to prove that $\overline{S}$ is symmetric, let us show that $\#N(\overline{S}) = \#H(\overline{S})$. By using that $F(\overline{S}) = 4g - 1$, we easily deduce that

$$N(\overline{S}) = 2N(S) \cup \{2(g + 1), 2(g + 2), \ldots, 2(g + (g - 1))\}$$

$$\cup \{2g + (2g - 2h_1) - 1, \ldots, 2g + (2g - 2h_t) - 1\}$$

(and that these sets are disjoint). Hence $\#N(\overline{S}) = \#N(S) + g - 1 + \#H(S)$. As we know that $g + 1 = \#H(S) + \#N(S)$, we obtain that $\#N(\overline{S}) = 2g$. We also know that $4g - 1 + 1 = \#H(\overline{S}) + \#N(\overline{S})$, and consequently $\#H(\overline{S}) = 2g = \#N(\overline{S})$.

Finally, we show that $S = \frac{\overline{S}}{2}$. As $2S \subseteq \overline{S}$, the inclusion $S \subseteq \frac{\overline{S}}{2}$ is trivial. For the other inclusion, let $x \in \frac{\overline{S}}{2}$. Then $2x \in \overline{S}$. From the way $\overline{S}$ is defined, as $2x$ is even, either $2x \in 2S$ or $2x \geq 4g$. If $2x \in 2S$, then trivially $x \in S$. If $2x \geq 4g$, then $2x > 2g$ and thus $x > g$, which also leads to $x \in S$.\qed
References


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