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EVERY NUMERICAL SEMIGROUP IS ONE HALF OF A SYMMETRIC NUMERICAL SEMIGROUP

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Abstract. Let S be a numerical semigroup. Then there exists a symmetric numerical semigroup $\overline{S}$ such that $S = \{n \in \mathbb{N} \mid 2n \in \overline{S}\}$.

Let $\mathbb{N}$ be the set of nonnegative integers. A *submonoid* M of $\mathbb{N}$ is a subset of $\mathbb{N}$ closed under addition and such that $0 \in M$. A *numerical semigroup* S is a submonoid of $\mathbb{N}$ such that $\mathbb{N} \setminus S$ is finite. The elements of $\mathrm{H}(S) = \mathbb{N} \setminus S$ are the *gaps* of S, and the largest integer not belonging to S is known as the *Frobenius number* of S, denoted here by $\mathrm{F}(S)$. There is a huge amount of papers devoted to the Frobenius number and more particulary to the problem of computing or bounding it (see [1] and the vast list of references given there).

A numerical semigroup S is said to be *symmetric* if for every $z \in \mathbb{Z}$, either $z \in S$ or $\mathrm{F}(S) - z \in S$. Symmetric numerical semigroups earned some relevance due to a result by Kunz (see [2]), which states that the semigroup ring of a symmetric numerical semigroup is Gorenstein. Thus this class of numerical semigroups became a source of examples of Gorenstein rings.

For a numerical semigroup S, define $\mathrm{N}(S) = \{s \in S \mid s < \mathrm{F}(S)\}$. Clearly, $\#\mathrm{N}(S) + \#\mathrm{H}(S) = \mathrm{F}(S) + 1$, and S is symmetric if and only if $\#\mathrm{N}(S) = \#\mathrm{H}(S)$, where $\#A$ denotes the cardinality of A.

Let M be a submonoid of $\mathbb{N}$ and let d be a positive integer. Then

$$\frac{M}{d} = \{n \in \mathbb{N} \mid dn \in M\}$$

is again a submonoid of $\mathbb{N}$, called the *quotient of M by d*. Clearly, $M \subseteq \frac{M}{d}$.

The aim of this paper is to show that for every numerical semigroup S, there exists a symmetric numerical semigroup $\overline{S}$ such that $S = \frac{\overline{S}}{2}$.

Given A, a subset of $\mathbb{Z}$, denote by $2A = \{2a \mid a \in A\}$ (not to be confused with $A + A$). Given $x_1, \ldots, x_k \in \mathbb{Z}$, when we write $\{x_1, \ldots, x_k, \rightarrow\}$, we mean that all the integers greater than x_k belong to the set.

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Theorem 1. *Let S be a numerical semigroup. Then there exists a symmetric numerical semigroup $\overline{S}$ such that*

$$S = \frac{\overline{S}}{2}.$$

Proof. Let $g = \mathrm{F}(S)$. If $g = -1$, then $S = \mathbb{N}$. Clearly $\mathbb{N} = \frac{\mathbb{N}}{2}$ and $\mathbb{N}$ is symmetric. Now assume that $g = 1$. In this setting $S = \{0, 2, \rightarrow\}$. The reader can easily check that by taking $\overline{S} = \{0, 4, 5, 6, 8, \rightarrow\}$, one gets that $S = \frac{\overline{S}}{2}$, with $\overline{S}$ symmetric.

Now suppose that $g \geq 2$. Let $\mathrm{H}(S) = \{h_1, \ldots, h_t\}$. We claim that

$$\overline{S} = 2S \cup \{2g + (2g - 2h_1) - 1, \ldots, 2g + (2g - 2h_t) - 1\} \cup \{4g, \rightarrow\}$$

is a symmetric numerical semigroup and that $S = \frac{\overline{S}}{2}$.

First we must prove that $\overline{S}$ is a submonoid of $\mathbb{N}$. This is a consequence of the following remarks.

1. $2S$ is a submonoid of $\mathbb{N}$.
2. The result of adding any nonnegative integer to any element in $\{4g, \rightarrow\}$ remains in $\{4g, \rightarrow\}$.
3. For every $i, j \in \{1, \ldots, t\}$, we have that $2g+(2g-2h_i)-1+2g+(2g-2h_j)-1 = 2(2g+(g-h_i)+(g-h_j)-1)$. As $g = \max\{h_1, \ldots, h_t\}$, $2g+(g-h_i)+(g-h_j)-1 \geq 2g-1$. Since by hypothesis $g \geq 2$, we deduce that $2g+(g-h_i)+(g-h_j)-1 \geq g+1$. Hence $2g+(g-h_i)+(g-h_j)-1 \in S$, and $2g+(2g-2h_i)-1+2g+(2g-2h_j)-1 \in 2S \subseteq \overline{S}$.
4. Now take $s \in S \setminus \{0\}$ and $i \in \{1, \ldots, t\}$. We distinguish two cases.
   - If $s > h_i$, then $2g + (2g - 2h_i) - 1 + 2s = 4g + 2(s - h_i) - 1 \geq 4g$, whence $2g + (2g - 2h_i) - 1 + 2s$ belongs to $\overline{S}$.
   - If $s < h_i$, then $0 \leq g + (s - h_i) < g$. Thus $g + s - h_i = g - x$, with $x = h_i - s \in \{1, \ldots, g\}$. As $h_i = s + x \notin S$, and $s \in S$, we have that $x \notin S$. Hence $x = h_j$ for some $j \in \{1, \ldots, t\}$. This leads to $2g+(2g-2h_i)-1+2s = 2g + (2g - 2h_j) - 1 \in \overline{S}$.

Since $\{4g, \rightarrow\} \subseteq \overline{S}$, we have that $\mathbb{N} \setminus \overline{S}$ has finitely many elements. This proves that $\overline{S}$ is a numerical semigroup.

Observe that $4g - 1$ is odd. As $\overline{S} = 2S \cup \{2g + (2g - 2h_1) - 1, \ldots, 2g + (2g - 2h_t) - 1\} \cup \{4g, \rightarrow\}$ and the elements in $2S$ are all even, if $4g - 1 \in \overline{S}$, then $4g - 1 = 2g + (2g - 2h_i) - 1$ for some $i \in \{1, \ldots, t\}$. But this leads to $h_i = 0$, which is impossible. Hence $4g - 1 = \mathrm{F}(\overline{S})$, because $\{4g, \rightarrow\} \subseteq \overline{S}$.

In order to prove that $\overline{S}$ is symmetric, let us show that $\#\mathrm{N}(\overline{S}) = \#\mathrm{H}(\overline{S})$. By using that $\mathrm{F}(\overline{S}) = 4g - 1$, we easily deduce that

$$\begin{aligned}\mathrm{N}(\overline{S}) = 2\mathrm{N}(S) &\cup \{2(g+1), 2(g+2), \ldots, 2(g+(g-1))\} \\ &\cup \{2g + (2g - 2h_1) - 1, \ldots, 2g + (2g - 2h_t) - 1\}\end{aligned}$$

(and that these sets are disjoint). Hence $\#\mathrm{N}(\overline{S}) = \#\mathrm{N}(S) + g - 1 + \#\mathrm{H}(S)$. As we know that $g + 1 = \#\mathrm{H}(S) + \#\mathrm{N}(S)$, we obtain that $\#\mathrm{N}(\overline{S}) = 2g$. We also know that $4g - 1 + 1 = \#\mathrm{H}(\overline{S}) + \#\mathrm{N}(\overline{S})$, and consequently $\#\mathrm{H}(\overline{S}) = 2g = \#\mathrm{N}(\overline{S})$.

Finally, we show that $S = \frac{\overline{S}}{2}$. As $2S \subseteq \overline{S}$, the inclusion $S \subseteq \frac{\overline{S}}{2}$ is trivial. For the other inclusion, let $x \in \frac{\overline{S}}{2}$. Then $2x \in \overline{S}$. From the way $\overline{S}$ is defined, as $2x$ is even, either $2x \in 2S$ or $2x \geq 4g$. If $2x \in 2S$, then trivially $x \in S$. If $2x \geq 4g$, then $2x > 2g$ and thus $x > g$, which also leads to $x \in S$. $\square$

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