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A TEST COMPLEX FOR GORENSTEINNESS

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Abstract. Let R be a commutative noetherian ring with a dualizing complex. By recent work of Iyengar and Krause (2006), the difference between the category of acyclic complexes and its subcategory of totally acyclic complexes measures how far R is from being Gorenstein. In particular, R is Gorenstein if and only if every acyclic complex is totally acyclic.

In this note we exhibit a specific acyclic complex with the property that it is totally acyclic if and only if R is Gorenstein.

Introduction

Let R be a commutative noetherian ring. A complex X of R-modules is said to be *acyclic* if it has zero homology, i.e. $\mathrm{H}(X) = 0$. An acyclic complex of projective modules is called *totally acyclic* if the acyclicity is preserved by $\mathrm{Hom}_R(-, P)$ for every projective module P. Dually, an acyclic complex of injective modules is totally acyclic if the acyclicity is preserved by $\mathrm{Hom}_R(I, -)$ for every injective module I.

Over a Gorenstein ring, every acyclic complex of projective or of injective modules is totally acyclic. Iyengar and Krause have recently proved a converse; indeed, by [9, cor. 5.5] the following are equivalent when R has a dualizing complex:

(i) The ring R is Gorenstein.
(ii) Every acyclic complex of projective R-modules is totally acyclic.
(iii) Every acyclic complex of injective R-modules is totally acyclic.

Moreover, for a local ring $(R, \mathfrak{m})$ that is not Gorenstein and has $\mathfrak{m}^2 = 0$ there is a natural example, provided by [9, prop. 6.1(3)], of an acyclic, but not totally acyclic, complex of projective R-modules.

The purpose of this note is to prove that for every ring R with a dualizing complex D, a specific acyclic complex K, defined in 2.1, serves as a test complex for Gorensteinness in the following sense: The ring R is Gorenstein if and only if $K \otimes_R D$ is acyclic. This is achieved by Theorem 2.2. In general, K is an acyclic complex of flat R-modules. Corollary 2.6 shows that if R is an artinian local ring, then K is a complex of projective modules, and (i)–(iii) above are equivalent with

(iv) The complex K is totally acyclic.

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Test complexes of injective modules can be obtained directly from K (Corollary 2.5) or through a potentially different construction explored in Section 3. The authors of [9] have pointed out that the latter is of particular interest, as it yields a generator for $\mathbf{K}_{\mathrm{ac}}(\mathrm{Inj\ R}) / \mathbf{K}_{\mathrm{tac}}(\mathrm{Inj\ R})$, the Verdier quotient of acyclic complexes modulo totally acyclic complexes in the homotopy category of injective R-modules. This is proved in Theorem 3.5.

1. Background

Throughout this paper R is a commutative noetherian ring. The notation $(R, \mathfrak{m}, k)$ means R is local with maximal ideal $\mathfrak{m}$ and residue field k.

Complexes of R-modules (*R-complexes* for short) are graded homologically,

$$X = \cdots \to X_{i+1} \xrightarrow{\partial^X_{i+1}} X_i \xrightarrow{\partial^X_i} X_{i-1} \to \cdots .$$

The *suspension* of X is denoted ΣX; it is the complex with $(\Sigma X)_i = X_{i-1}$ and differential $\partial^{\Sigma X} = -\partial^X$. A complex X is said to be *bounded* if $X_i = 0$ for $|i| \gg 0$.

An isomorphism between R-complexes is denoted by '$\cong$'; we write $X \cong Y$ if there exists an isomorphism $X \xrightarrow{\cong} Y$.

A morphism between R-complexes is called a *quasi-isomorphism*, and denoted $X \xrightarrow{\simeq} Y$ if the induced map in homology, $\mathrm{H}(X) \to \mathrm{H}(Y)$, is an isomorphism. Following [1, sec. 1] we write $X \simeq Y$, if X and Y can be linked by a sequence of quasi-isomorphisms with arrows in alternating directions. Recall that a morphism $X \to Y$ is a quasi-isomorphism if and only if its mapping cone, written $\mathrm{Cone}\,(X \to Y)$, is acyclic.

1.1. **Resolutions.** The following facts are established in [1, sec. 1][1] and [2].

Every R-complex X has a semi-projective resolution. That is, there is a quasi-isomorphism $P \xrightarrow{\simeq} X$, where P is a complex of projective R-modules such that $\mathrm{Hom}_R(P, -)$ preserves quasi-isomorphisms. For such a complex, also the functor $- \otimes_R P$ preserves quasi-isomorphisms. In particular, for any R-complexes $Y \simeq Z$ we have $\mathrm{Hom}_R(P, Y) \simeq \mathrm{Hom}_R(P, Z)$ and $Y \otimes_R P \simeq Z \otimes_R P$.

If there is an l such that $\mathrm{H}_i(X) = 0$ for $i < l$, then X has a semi-projective resolution P with $P_i = 0$ for $i < l$. If, in addition, $\mathrm{H}_i(X)$ is finitely generated for all i, then P can be chosen with all modules P_i finitely generated.

Every R-complex X has a semi-injective resolution. That is, there is a quasi-isomorphism $X \xrightarrow{\simeq} J$, where J is a complex of injective R-modules such that $\mathrm{Hom}_R(-, J)$ preserves quasi-isomorphisms. In particular, for such a complex J and any R-complexes $Y \simeq Z$ we have $\mathrm{Hom}_R(Y, J) \simeq \mathrm{Hom}_R(Z, J)$.

1.2. **Lemma.** *Let X and Y be R-complexes such that either $X_i = 0$ for all $i \ll 0$ or $Y_i = 0$ for all $i \gg 0$. If $\mathrm{H}(X_i \otimes_R Y) = 0$ for all $i \in \mathbb{Z}$, then $\mathrm{H}(X \otimes_R Y) = 0$.*

Proof. Let E be a faithfully injective R-module. The complex $X \otimes_R Y$ is acyclic if and only if $\mathrm{Hom}_R(X \otimes_R Y, E) \cong \mathrm{Hom}_R(X, \mathrm{Hom}_R(Y, E))$ is so. The claim is now immediate from [5, lem. (2.4)]. $\square$

[1] Where semi-projective/injective resolutions are called DG-projective/injective.

1.3. **Dualizing complexes.** Following [8, V.§2], a *dualizing complex* for R is a bounded complex D of injective R-modules such that $\mathrm{H}_i(D)$ is finitely generated for all $i \in \mathbb{Z}$, and the homothety morphism

$$\chi^D \colon R \longrightarrow \mathrm{Hom}_R(D, D)$$

is a quasi-isomorphism.

Let $(R, \mathfrak{m}, k)$ be a local ring with a dualizing complex D. After suspensions we can assume D is *normalized*, cf. [8, V.§5], in which case [8, prop. V.3.4] yields

$$\mathrm{H}(\mathrm{Hom}_R(k, D)) \cong k. \tag{1.3.1}$$

If R is artinian, then $\mathrm{E}_R(k)$, the injective hull of the residue field, is a normalized dualizing complex for R.

2. A test complex of flat modules

2.1. **A distinguished complex of flat modules.** Assume that R has a dualizing complex D, and let $\pi \colon P \xrightarrow{\simeq} D$ be a semi-projective resolution. By 1.1 we can assume that P consists of finitely generated modules with $P_i = 0$ for all $i \ll 0$. The functors $\mathrm{Hom}_R(P, -)$ and $\mathrm{Hom}_R(-, D)$ preserve quasi-isomorphisms, so the commutative diagram

$$\begin{array}{ccc}
\mathrm{Hom}_R(P,P) & \xrightarrow[\simeq]{\mathrm{Hom}_R(P,\pi)} & \mathrm{Hom}_R(P,D) \\
{\scriptstyle \chi^P}\big\uparrow & & \big\uparrow{\scriptstyle \simeq\ \mathrm{Hom}_R(\pi,D)} \\
R & \xrightarrow[\simeq]{\chi^D} & \mathrm{Hom}_R(D,D)
\end{array}$$

shows that the homothety map χ^P is a quasi-isomorphism. In particular,

$$K = \mathrm{Cone}\,(R \xrightarrow{\chi^P} \mathrm{Hom}_R(P, P))$$

is acyclic. The modules in $\mathrm{Hom}_R(P, P)$ are direct products of modules of the form $\mathrm{Hom}_R(P_i, P_{i+n})$, and each such module is flat. Thus, χ^P is a quasi-isomorphism between complexes of flat R-modules, and the mapping cone K is, therefore, an acyclic complex of flat R-modules.

We can now state the main result; the proof is given at the end of the section.

2.2. **Theorem.** *Let R be a commutative noetherian ring with a dualizing complex D, and let K be the acyclic complex of flat modules defined in* 2.1. *The ring R is Gorenstein if and only if the complex $K \otimes_R D$ is acyclic.*

2.3. *Remark.* While $C = \mathrm{Cone}\,\chi^D$ is also an acyclic complex of flat R-modules, it cannot detect Gorensteinness. Indeed, C is bounded, so $C \otimes_R X$ is acyclic for every R-complex X by Lemma 1.2. If R is artinian, then C is even split exact.

2.4. *Remark.* In the theory of Gorenstein dimensions, there is a notion of a *complete flat resolution*—due to Enochs, Jenda, and Torrecillas [6]—namely an acyclic complex F of flat modules such that $F \otimes_R I$ is acyclic for every injective module I.

If R is Gorenstein, then every acyclic complex of flat R-modules is a complete flat resolution. Indeed, every injective R-module I has finite flat dimension, and then it is straightforward to verify that the functor $- \otimes_R I$ preserves acyclicity of complexes of flat modules. On the other hand, let K and D be as in Theorem 2.2. If K is a complete flat resolution, then $K \otimes_R D$ is acyclic by Lemma 1.2.

Thus, the following assertions are equivalent:

(i) The ring R is Gorenstein.
(ii) The complex K is a complete flat resolution.
(iii) Every acyclic complex of flat modules is a complete flat resolution.

The complex K defined in 2.1 appears to be a natural test object for Gorensteinness. However, it might in the context of [9] be of interest to exhibit a test complex of injective or of projective modules.

To this end, we first note that the next corollary to Theorem 2.2 is immediate in view of Remark 2.4 and [3, prop. (6.4.1)]. See Section 3 for a further discussion of test complexes of injective modules.

2.5. **Corollary.** *Let R be a commutative noetherian ring with a dualizing complex. Let K be the acyclic complex of flat modules defined in* 2.1*, and let E be a faithfully injective R-module. The complex $\operatorname{Hom}_R(K,E)$ is an acyclic complex of injective modules, and R is Gorenstein if and only if $\operatorname{Hom}_R(K,E)$ is totally acyclic.* □

For artinian local rings $(R,\mathfrak{m})$, Theorem 2.2 provides a test complex of projective modules. In particular, for R with $\mathfrak{m}^2 = 0$ the following recovers [9, prop. 6.1(3)].

2.6. **Corollary.** *Let R be an artinian local ring. The complex K defined in* 2.1 *is an acyclic complex of projective R-modules, and R is Gorenstein if and only if K is totally acyclic.*

Proof. When R is artinian and local, every flat R-module is projective. Thus, K is an acyclic complex of projective modules.

The "only if" part is well-known. To prove "if", assume K is totally acyclic and recall from 1.3 that the module $E = \mathrm{E}_R(k)$ is dualizing for R. The first of the following isomorphisms is induced by χ^E, and the second is Hom-tensor adjointness

$$\operatorname{Hom}_R(K,R) \cong \operatorname{Hom}_R(K, \operatorname{Hom}_R(E,E)) \cong \operatorname{Hom}_R(K \otimes_R E, E).$$

The complex $\operatorname{Hom}_R(K,R)$ is acyclic and E is faithfully injective, so $K \otimes_R E$ is acyclic and, therefore, R is Gorenstein by Theorem 2.2. □

For the proof of Theorem 2.2 we need a technical lemma.

2.7. **Lemma.** *Let P be an R-complex of finitely generated projective modules, X be any R-complex, and B be a bounded R-complex of finitely generated modules. There is an isomorphism of R-complexes*

$$\omega\colon \operatorname{Hom}_R(P,X) \otimes_R B \xrightarrow{\cong} \operatorname{Hom}_R(P, X \otimes_R B).$$

Proof. It is straightforward to check that the assignment

$$\omega(\phi \otimes b)(p) = (-1)^{|p||b|}\phi(p) \otimes b,$$

where $|\cdot|$ denotes the degree of an element, defines a morphism between the relevant complexes. By assumption, there exist integers $l \leqslant u$ such that $B_h = 0$ when $h < l$

or $h > u$. For every $n \in \mathbb{Z}$ we have

$$\begin{aligned}(\mathrm{Hom}_R(P,X)\otimes_R B)_n &= \bigoplus_{i=n-u}^{n-l} \mathrm{Hom}_R(P,X)_i \otimes_R B_{n-i}\\ &= \bigoplus_{i=n-u}^{n-l} \big(\prod_{j\in\mathbb{Z}} \mathrm{Hom}_R(P_j, X_{j+i})\big) \otimes_R B_{n-i}\\ &\cong \bigoplus_{i=n-u}^{n-l} \prod_{j\in\mathbb{Z}} (\mathrm{Hom}_R(P_j, X_{j+i}) \otimes_R B_{n-i})\\ &\cong \bigoplus_{i=n-u}^{n-l} \prod_{j\in\mathbb{Z}} \mathrm{Hom}_R(P_j, X_{j+i} \otimes_R B_{n-i})\\ &\cong \prod_{j\in\mathbb{Z}} \mathrm{Hom}_R(P_j, \bigoplus_{i=n-u}^{n-l} X_{j+i} \otimes_R B_{n-i})\\ &= \prod_{j\in\mathbb{Z}} \mathrm{Hom}_R(P_j, (X\otimes_R B)_{j+n})\\ &= \mathrm{Hom}_R(P, X\otimes_R B)_n.\end{aligned}$$

Since the modules B_{n-i} are finitely generated, the functors $-\otimes_R B_{n-i}$ commute with arbitrary products for every i; this explains the first isomorphism. The modules P_j are finitely generated and projective, so for all i, j, and n the homomorphism of modules

$$\mathrm{Hom}_R(P_j, X_{j+i}) \otimes_R B_{n-i} \xrightarrow{\omega_{ijn}} \mathrm{Hom}_R(P_j, X_{j+i} \otimes_R B_{n-i})$$

is invertible, and this accounts for the second isomorphism. Thus, ω is an isomorphism of graded modules, and the sign in the definition of ω ensures that it commutes with the differentials. □

Proof of Theorem 2.2. The "only if" part was settled in Remark 2.4.

For the "if" part, assume that the complex $K \otimes_R D$ is acyclic; the isomorphism $\mathrm{Cone}\,(\chi^P \otimes_R D) \cong K \otimes_R D$ implies that

$$\chi^P \otimes_R D\colon D \longrightarrow \mathrm{Hom}_R(P,P)\otimes_R D \tag{1}$$

is a quasi-isomorphism.

Choose an n such that $\mathrm{H}_i(D) = 0$ for all $i > n$, and let B be the soft truncation of P on the left at n:

$$B = 0 \longrightarrow \mathrm{Coker}\,\partial^P_{n+1} \xrightarrow{\overline{\partial^P_n}} P_{n-1} \xrightarrow{\partial^P_{n-1}} P_{n-2} \longrightarrow \cdots .$$

There are quasi-isomorphisms $B \xleftarrow{\simeq} P \xrightarrow{\simeq} D$ and, hence, a quasi-isomorphism $\beta\colon B \xrightarrow{\simeq} D$; see [1, 1.1.I.(1) and 1.4.I]. Since the mapping cone of β is a bounded acyclic complex, and $\mathrm{Hom}_R(P,P)$ is a complex of flat modules, Lemma 1.2 applies to show that also $\mathrm{Hom}_R(P,P)\otimes_R \mathrm{Cone}(\beta)$ is acyclic. Thus, the isomorphism $\mathrm{Cone}(\mathrm{Hom}_R(P,P)\otimes_R \beta) \cong \mathrm{Hom}_R(P,P)\otimes_R \mathrm{Cone}(\beta)$ implies that

$$\mathrm{Hom}_R(P,P)\otimes_R \beta\colon \mathrm{Hom}_R(P,P)\otimes_R B \longrightarrow \mathrm{Hom}_R(P,P)\otimes_R D \tag{2}$$

is a quasi-isomorphism.

By the choice of P, cf. 2.1, the bounded complex B consists of finitely generated modules, and Lemma 2.7 yields an isomorphism

$$\omega\colon \operatorname{Hom}_R(P,P)\otimes_R B \xrightarrow{\cong} \operatorname{Hom}_R(P, P\otimes_R B). \tag{3}$$

Finally, let $\iota\colon P\otimes_R B \xrightarrow{\simeq} J$ be a semi-injective resolution; the quasi-isomorphism ι is preserved by $\operatorname{Hom}_R(P,-)$, and the resulting quasi-isomorphism combined with (1), (2), and (3) yields

$$D \simeq \operatorname{Hom}_R(P,J). \tag{4}$$

It suffices to prove that $R_\mathfrak{m}$ is Gorenstein for every maximal ideal $\mathfrak{m}$ of R. Let $\mathfrak{m}$ be a maximal ideal; the complex $D_\mathfrak{m}$ is dualizing for $R_\mathfrak{m}$; see [8, cor. V.2.3]. Set $k = R_\mathfrak{m}/\mathfrak{m}R_\mathfrak{m} \cong R/\mathfrak{m}$. We may, after suspensions, assume $D_\mathfrak{m}$ is normalized, so $\operatorname{H}(\operatorname{Hom}_{R_\mathfrak{m}}(k, D_\mathfrak{m})) \cong k$; see (1.3.1). Moreover, there are isomorphisms $\operatorname{Hom}_R(k,D) \cong \operatorname{Hom}_R(k,D)\otimes_R R_\mathfrak{m} \cong \operatorname{Hom}_{R_\mathfrak{m}}(k, D_\mathfrak{m})$, so we have

$$k \cong \operatorname{H}(\operatorname{Hom}_R(k,D)). \tag{5}$$

Let $\upsilon\colon Q \xrightarrow{\simeq} k$ be a semi-projective resolution of k over R. As $\operatorname{Hom}_R(-,D)$ preserves quasi-isomorphisms, we have

$$\operatorname{Hom}_R(k,D) \xrightarrow{\simeq} \operatorname{Hom}_R(Q,D). \tag{6}$$

Also $\operatorname{Hom}_R(Q,-)$ preserves quasi-isomorphisms, and from (4) we get

$$\operatorname{Hom}_R(Q,D) \simeq \operatorname{Hom}_R(Q,\operatorname{Hom}_R(P,J)) \cong \operatorname{Hom}_R(Q\otimes_R P, J), \tag{7}$$

where the isomorphism is Hom-tensor adjointness. Finally, $\upsilon\otimes_R P$ is a quasi-isomorphism, and hence so is

$$\operatorname{Hom}_R(\upsilon\otimes_R P, J)\colon \operatorname{Hom}_R(k\otimes_R P, J) \xrightarrow{\simeq} \operatorname{Hom}_R(Q\otimes_R P, J). \tag{8}$$

Combining (5)–(8) and again using Hom-tensor adjointness, we obtain

$$\begin{aligned} k &\cong \operatorname{H}(\operatorname{Hom}_R(k\otimes_R P, J)) \\ &\cong \operatorname{H}(\operatorname{Hom}_R((k\otimes_R P)\otimes_k k, J)) \\ &\cong \operatorname{H}(\operatorname{Hom}_k(k\otimes_R P, \operatorname{Hom}_R(k,J))) \\ &\cong \operatorname{Hom}_k(\operatorname{H}(k\otimes_R P), \operatorname{H}(\operatorname{Hom}_R(k,J))). \end{aligned}$$

Thus, $\operatorname{Hom}_k(\operatorname{H}(k\otimes_R P), \operatorname{H}(\operatorname{Hom}_R(k,J)))$ is a finitely generated k-vector space; in particular, $\operatorname{H}(k\otimes_R P)$ must be finitely generated. Note that $\operatorname{H}_i(k\otimes_R P) \cong \operatorname{H}_i(k\otimes_{R_\mathfrak{m}} P_\mathfrak{m})$ for all $i\in\mathbb{Z}$; it follows that $\operatorname{H}_i(k\otimes_{R_\mathfrak{m}} P_\mathfrak{m}) = 0$ for all $i \gg 0$. By [1, prop. 5.5] the dualizing $R_\mathfrak{m}$-complex $D_\mathfrak{m}$ then has finite flat dimension, and hence $R_\mathfrak{m}$ is Gorenstein; see [7, thm. (17.23)] or [4, thm. (8.1)]. □

3. A test complex of injective modules

The next construction is another source for test complexes.

3.1. **A distinguished complex of injective modules.** Assume R has a dualizing complex D. As in 2.1, let $\pi\colon P \xrightarrow{\simeq} D$ be a semi-projective resolution of D consisting of finitely generated modules with $P_i = 0$ for all $i \ll 0$. The assignment $\varphi \otimes p \mapsto \varphi(p)$ defines a morphism of complexes, ε, such that the following diagram is commutative:

$$\begin{array}{ccc}
\operatorname{Hom}_R(P,D)\otimes_R P & \xrightarrow{\ \varepsilon\ } & D \\
{\scriptstyle \operatorname{Hom}_R(P,\pi)\otimes_R P}\Big\uparrow{\simeq} & & {\simeq}\Big\uparrow{\scriptstyle \pi} \\
\operatorname{Hom}_R(P,P)\otimes_R P & \xleftarrow[\chi^P\otimes_R P]{\simeq} & R\otimes_R P.
\end{array}$$

Thus, ε is a quasi-isomorphism between complexes of injective R-modules, and the mapping cone

$$M = \operatorname{Cone}\left(\operatorname{Hom}_R(P,D)\otimes_R P \xrightarrow{\varepsilon} D\right)$$

an acyclic complex of injective R-modules.

An argument similar to the proof of Theorem 2.2 yields the next result, which is also a corollary of Theorem 3.5.

3.2. **Theorem.** *Let R be a commutative noetherian ring with a dualizing complex, and let M be the acyclic complex of injective modules defined in* 3.1*. The ring R is Gorenstein if and only if M is totally acyclic.* □

3.3. *Remark.* If $(R,\mathfrak{m},k)$ is an artinian local ring, then there is an isomorphism

$$K \cong \Sigma\operatorname{Hom}_R(M,\mathrm{E}_R(k))$$

where K and M are the complexes from 2.1 and 3.1, and $\mathrm{E}_R(k)$ is the injective hull of k. Indeed, with $E = \mathrm{E}_R(k)$ there is a commutative diagram

$$\begin{array}{ccc}
\operatorname{Hom}_R(E,E) & \xrightarrow[\simeq]{\operatorname{Hom}_R(\varepsilon,E)} & \operatorname{Hom}_R(\operatorname{Hom}_R(P,E)\otimes_R P,E) \\
\Big\uparrow & & {\cong}\Big\uparrow \\
{\cong}\ \Big|\ {\chi^E} & & \operatorname{Hom}_R(P,\operatorname{Hom}_R(\operatorname{Hom}_R(P,E),E)) \\
\Big| & & {\cong}\Big\uparrow \\
R & \xrightarrow[\simeq]{\chi^P} & \operatorname{Hom}_R(P,P).
\end{array}$$

The vertical maps on the right are the natural isomorphisms, and because E is a module, the homothety map χ^E is also a genuine isomorphism. The diagram induces the desired isomorphism between the complexes $K = \operatorname{Cone}(\chi^P)$ and $\operatorname{Cone}(\operatorname{Hom}_R(\varepsilon,E)) \cong \Sigma\operatorname{Hom}_R(\operatorname{Cone}(\varepsilon),E) = \Sigma\operatorname{Hom}_R(M,E)$.

When R is not artinian, we do not know if the complexes K and M are related.

Using [5, prop. (5.1)] it is not hard to prove the next parallel to Corollary 2.5.

3.4. **Corollary.** *Let R be a commutative noetherian ring with a dualizing complex. Let M be the acyclic complex of injective modules defined in* 3.1*, and let E be a faithfully injective R-module. The complex $\operatorname{Hom}_R(M,E)$ is an acyclic complex of flat modules, and R is Gorenstein if and only if $\operatorname{Hom}_R(M,E)\otimes_R I$ is acyclic for every injective module I.* □

In conversations, the authors of [9] have informed us of Theorem 3.5 below; note that it contains Theorem 3.2. For notation and terminology we refer to [9].

3.5. **Theorem.** *Let R be a commutative noetherian ring with a dualizing complex. The acyclic complex M of injective modules defined in* 3.1 *generates the quotient category* $\mathbf{K}_{\mathrm{ac}}(\mathrm{Inj\ R}) \,/\, \mathbf{K}_{\mathrm{tac}}(\mathrm{Inj\ R})$.

Proof. By [9, 1.7, 5.4, and 5.9(3)] the quotient category $\mathbf{K}_{\mathrm{ac}}(\mathrm{Inj\ R}) \,/\, \mathbf{K}_{\mathrm{tac}}(\mathrm{Inj\ R})$ is generated by the image of the dualizing complex D under the equivalence $\mathbf{D}^f(R) \xrightarrow{\sim} \mathbf{K}^{\mathrm{c}}(\mathrm{Inj\ R})$; cf. [9, 2.3(2)].

Let $P \xrightarrow{\simeq} D$ be a semi-projective resolution. The functor $\mathrm{Hom}_R(P,-)$ preserves quasi-isomorphisms, so the composite

$$R \xrightarrow{\simeq} \mathrm{Hom}_R(P,P) \xrightarrow{\simeq} \mathrm{Hom}_R(P,D)$$

provides an injective resolution $R \xrightarrow{\simeq} \mathsf{i}R = \mathrm{Hom}_R(P,D)$. Since $\mathsf{i}R$ is a compact object in $\mathbf{K}(\mathrm{Inj\ R})$, the inclusion of the localizing subcategory $\mathrm{Loc}(\mathsf{i}R) \subseteq \mathbf{K}(\mathrm{Inj\ R})$ admits a right adjoint $\rho\colon \mathbf{K}(\mathrm{Inj\ R}) \to \mathrm{Loc}(\mathsf{i}R)$; see [9, 1.5.1]. By [9, 2.3(2)] the image of D in $\mathbf{K}(\mathrm{Inj\ R})$ is

$$\mathrm{Cone}(\rho(D) \xrightarrow{\xi} D),$$

where ξ is the natural map.

It remains to show that $M \cong \mathrm{Cone}(\rho(D) \xrightarrow{\xi} D)$. It suffices to establish a commutative diagram,

$$\begin{array}{ccc} \rho(D) & \xrightarrow{\quad\xi\quad} & D \\ \Big\downarrow{\cong} & \nearrow_{\varepsilon} & \\ \mathrm{Hom}_R(P,D) \otimes_R P & & \end{array}$$

The complex $\mathrm{Hom}_R(P,D) \otimes_R P = \mathsf{i}R \otimes_R P$ is in $\mathrm{Loc}(\mathsf{i}R)$, and since ε is a quasi-isomorphism, $\mathrm{Hom}_{\mathbf{K}(\mathrm{Inj\ R})}(\mathsf{i}R, \varepsilon)$ is an isomorphism; cf. [9, 2.2]. The existence of the desired isomorphism $\rho(D) \cong \mathrm{Hom}_R(P,D) \otimes_R P$ now follows from [9, 1.4]. $\square$

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