STABLE INDECOMPOSABILITY OF LOOP SPACES
ON SYMPLECTIC GROUPS

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(Communicated by Paul Goerss)

Dedicated to the memory of Professor Masahiro Sugawara

Abstract. We prove that $\Omega Sp(n)$ is stably indecomposable if $n \geq 2$ or $n = \infty$.

1. Introduction

A spectrum $X$ is said to be decomposable if $X$ is homotopy equivalent to a wedge sum $X_1 \vee X_2$ of non-trivial spectra $X_1$ and $X_2$. Otherwise $X$ is said to be indecomposable. A CW complex $X$ is said to be stably decomposable if the suspension spectrum $\Sigma^\infty X$ is decomposable as a spectrum. Otherwise it is said to be stably indecomposable. We are considering the following problem.

Question. Let $G$ be a compact connected Lie group. Is the loop space $\Omega G$ stably indecomposable?

If $G$ is not simply connected, then $\Omega G$ is not connected and, therefore, is stably decomposable. If $G = G_1 \times G_2$, then $\Omega G$ is stably homotopy equivalent to $\Omega G_1 \vee \Omega G_2$ and, therefore, is stably decomposable, too. Thus, to solve the problem above, it is sufficient to consider a simply connected, simple Lie group. Hopkins [2] proved that $\Omega Sp(2)$ and $\Omega Sp(3)$ are stably indecomposable. Later Hubbuck [3] added $\Omega G_2$ and $\Omega F_4$ to the list of such spaces. We [4] also proved that $\Omega E_6$ and $\Omega E_7$ are stably indecomposable. In contrast to these results $\Omega SU(n)$ is known to be stably decomposable [1].

In this paper we will show that $\Omega Sp(n)$ are stably indecomposable for $n \geq 2$, which was conjectured by Hubbuck.

Theorem 1.1. $\Omega Sp(n)$ is stably indecomposable if $n \geq 2$ or $n = \infty$.

Needless to say, $\Omega Sp(1) = \Omega S^3$ is stably decomposable.

To prove the theorem we will investigate $H_*(\Omega Sp(n); F_2)$. We are not showing that it is indecomposable as a module over the Steenrod algebra. For $n \geq 4$, $H_*(\Omega Sp(n); F_2)$ is a sum of three indecomposable modules over the Steenrod algebra, and these modules are linked by higher-order operations as showed by Hubbuck.
2. Proofs

We are really going to prove the following localized version of Theorem 1.1.

**Theorem 2.1.** \( \Omega Sp(n) \) is stably indecomposable at the prime 2 if \( n \geq 2 \) or \( n = \infty \).

From now on until the end of this paper all spaces and spectra are assumed to be localized at the prime 2, and \( H_*(X) \) and \( H^*(X) \) stand for \( H_*(X; \mathbb{F}_2) \) and \( H^*(X; \mathbb{F}_2) \), respectively.

The Steenrod operation acts on homology groups via the following formula:

\[
\langle \alpha, xSq^i \rangle = \langle Sq^i \alpha, x \rangle
\]
for \( \alpha \in H^*(X) \) and \( x \in H_*(X) \), where \( \langle , \rangle \) denotes the Kronecker pairing of cohomology with homology.

First we recall the ring structure of \( H_*(\Omega Sp(n)) \) and the action of the Steenrod algebra on them by Kono-Kozima [5]:

\[
H_*(\Omega Sp(n)) \cong \mathbb{F}_2[z_1, z_3, \ldots, z_{2n-1}]
\]
and \( |z_{2i-1}| = 4i - 2 \), where \( |x| \) denotes the degree of an element of \( x \). To state the action of the Steenrod algebra on \( z_{2i-1} \), for a positive integer \( i \) we define

\[
z_i = (z_{b(i)})^{2^{n(i)}},
\]
where \( a(i) \) and \( b(i) \) are unique non-negative integers such that \( i = 2^{a(i)}b(i) \) and \( b(i) \) is odd. Then we have

\[
z_{2i-1}^2 = \binom{2i - 2 - j}{j} z_{2i-1-j}.
\]
In particular, \( z_{2i-1}^2 = z_{2i-2} \). We also have

\[
z_i Sq^j = \binom{i - 1 - j}{j} z_{i-j}
\]
for any positive integer \( i \) such that \( z_i \in H_*(\Omega Sp(n)) \).

As \( Sq^2 Sq^2 = 0 \) on \( H_*(\Omega Sp(n)) \) we can define

\[
H_*(H_*(\Omega Sp(n)); Sq^2) = \text{Ker} Sq^2/\text{Im} Sq^2.
\]
To compute this group we put

\[
\tilde{z}_{2i-1} = \begin{cases} \frac{z_{2i+1}}{z_i} + \frac{z_{2i}}{z_i} z_{2i-2} & \text{if } 2i - 1 = 2^j + 1 \text{ for some } j \geq 2, \\ \frac{z_{2i-1}}{z_i} z_{2i-1} + z_3 z_{2i-2} & \text{otherwise.} \end{cases}
\]

**Lemma 2.2.** If \( 2n - 1 = 2^m + 1 \) for some \( m \geq 3 \), then

\[
H_*(H_*(\Omega Sp(n)); Sq^2) \cong \Lambda(z_1, \tilde{z}_5, \tilde{z}_9, \ldots, \tilde{z}_{2m-2+1}) \otimes \mathbb{F}_2[\tilde{z}_{2m-1+1}, \tilde{z}_{2m+1}].
\]
If \( 2^m + 1 < 2n - 1 < 2^m+1 + 1 \) for some \( m \geq 2 \), then

\[
H_*(H_*(\Omega Sp(n)); Sq^2) \cong \Lambda(z_1, \tilde{z}_5, \tilde{z}_9, \ldots, \tilde{z}_{2m-1+1}) \otimes \mathbb{F}_2[\tilde{z}_{2m+1}].
\]
For \( n = \infty \), \( H_*(H_*(\Omega Sp)); Sq^2) \cong \Lambda(z_1, \tilde{z}_5, \tilde{z}_9, \ldots, \tilde{z}_{2m+1}, \ldots). \)

**Proof.** If \( z_1 \) is inverted, then

\[
H_*(\Omega Sp(n))[z_1^{-1}] = \mathbb{F}_2[z_1, z_3, \tilde{z}_5, \ldots, \tilde{z}_{2n-1}][z_1^{-1}].
\]
As \( \tilde{z}_{2i-1} Sq^2 = 0 \) and \( z_3 Sq^2 = z_1^2 \), it is easy to see that

\[
\text{Ker}(Sq^2 : H_*(\Omega Sp(n))[z_1^{-1}] \to H_*(\Omega Sp(n))[z_1^{-1}]) = \mathbb{F}_2[z_1, z_3^2, \tilde{z}_5, \ldots, \tilde{z}_{2n-1}][z_1^{-1}]
\]
and that
\[ \text{Ker}(Sq^2 : H_*(\Omega Sp(n)) \to H_*(\Omega Sp(n))) = \mathbb{F}_2[z_1, z_3, \tilde{z}_5, \ldots, \tilde{z}_{2n-1}]. \]

Let \( m \) be the unique integer such that \( 2^m + 1 \leq 2n - 1 < 2^{m+1} + 1 \). Since 
\[ (z_3 z_{2i-1}) Sq^2 = \tilde{z}_{2i-1} \]
for \( i \) such that \( 2i - 1 \neq 2^j + 1 \) for any \( j \geq 2 \), and
\[ (z_{2i+1} + z_3^{2j+1} - 4z_7) Sq^2 = \tilde{z}_{2i+1}^2 \]
for \( j \geq 2 \), \( z_3 Sq^2 = \tilde{z}_1^2 \) and \( z_7 Sq^2 = \tilde{z}_3^2 \), there is an epimorphism
\[ \mathbb{F}_2[z_1, \tilde{z}_5, \tilde{z}_9, \ldots, \tilde{z}_{2m+1}]/I \to H_*(H_*(\Omega Sp(n)); Sq^2), \]
where \( I \) is the ideal generated by \( \tilde{z}_1^2 \) and \( \{\tilde{z}_{2i+1}^2 | 11 \leq 2i+1 + 3 \leq 2n - 1\} \). Here we remark that \( 2n - 1 \geq 7 \).

If \( 2n - 1 = 2m + 1 \) for some \( m \geq 3 \), then
\[ \mathbb{F}_2[z_1, \tilde{z}_5, \tilde{z}_9, \ldots, \tilde{z}_{2m+1}]/I \cong \Lambda(z_1, \tilde{z}_5, \tilde{z}_9, \ldots, \tilde{z}_{2m-2+1}) \otimes \mathbb{F}_2[\tilde{z}_{2m-1}, \tilde{z}_{2m+1}]. \]
Since \( z_{2i-1} Sq^2 = z_i^2 \) for \( i \leq 2m - 1 \), \( y Sq^2 \) is a sum of monomials
\[ \tilde{z}_1^{k_1} \tilde{z}_3^{k_3} \tilde{z}_5^{k_5} \cdots \tilde{z}_{2m-1}^{k_{2m-1}} \tilde{z}_{2m+1}^{k_{2m+1}} \]
with \( k_{2i-1} \geq 1 \) for some \( 2i - 1 \leq 2m - 1 - 1 \). Therefore, the epimorphism
\[ \Lambda(z_1, \tilde{z}_5, \tilde{z}_9, \ldots, \tilde{z}_{2m-2+1}) \otimes \mathbb{F}_2[\tilde{z}_{2m-1}, \tilde{z}_{2m+1}] \to H_*(H_*(\Omega Sp(n)); Sq^2) \]
is monomorphically, and therefore, isomorphic.

The other cases are proved similarly. \( \square \)

**Lemma 2.3.** Let \( n \geq 4 \) and \( x \in H_*(\Omega Sp(n)) \) with \( |x| > 2 \). If \( x Sq^i = 0 \) for all \( i > 0 \), then \( x \in (H_{|x|+2}(\Omega Sp(n))); Sq^2 \).

**Proof.** We prove the lemma only when \( 2n - 1 = 2m + 1 \) for some \( m \geq 3 \) since the other cases are proved similarly.

First we will show that without loss of generality we may assume that \( x \) is in the subring \( \mathbb{F}_2[z_1^2, z_3, \ldots, z_{2m+1}] \). We write \( x = x' + x'' \) where \( x', x'' \) are in the subring \( \mathbb{F}_2[z_1^2, z_3, \ldots, z_{2m+1}] \). If \( x Sq^i = 0 \) for all \( i > 0 \), then we have
\[ 0 = x Sq^i = z_1 (x' Sq^i + x'' Sq^i). \]
Since \( x' Sq^i, x'' Sq^i \) are in the subring \( \mathbb{F}_2[z_1^2, z_3, \ldots, z_{2m+1}] \), the equation above implies that \( x' Sq^i = x'' Sq^i = 0 \). If \( x', x'' \) are in the \( Sq^2 \) image, then so is \( x \). Thus we assume that \( x \) is in the subring \( \mathbb{F}_2[z_1^2, z_3, \ldots, z_{2m+1}] \).

For a sequence of non-negative integers \( I = (\varepsilon_2, \varepsilon_3, \ldots, \varepsilon_{m-2}, \varepsilon_{m-1}, \varepsilon_m) \) with \( \varepsilon_i = 0 \) or 1, we define
\[ \tilde{z}_I = \tilde{z}_5^{\varepsilon_2} \cdots \tilde{z}_{2m-1}^{\varepsilon_{m-2}} \tilde{z}_{2m-1}^{\varepsilon_{m-1}} \tilde{z}_{2m+1}^{\varepsilon_m}. \]

Then by Lemma 2.2 \( x \) is written as
\[ x = \sum_I a_I \tilde{z}_I + y Sq^2 \]
for some \( y \in H_{|x|+2}(\Omega Sp(n)) \), where \( a_I \in \mathbb{F}_2 \) and the sum is taken over sequences of non-negative integers \( I = (\varepsilon_2, \varepsilon_3, \ldots, \varepsilon_{m-2}, \varepsilon_{m-1}, \varepsilon_m) \) such that \( \varepsilon_i = 0 \) or 1 and \( |I| = |x| \). Now we are breaking the argument up into three steps.

Step 1). We will show that \( a_I = 0 \) for all \( I \) such that \( \varepsilon_i = 1 \) for some \( 2 \leq i \leq m - 2 \).
Let $\Lambda_1$ be the ideal of $\mathbb{F}_2[z_1, z_3, \ldots, z_{2^m+1}]$ generated by the elements $z_{2i-1}^2$ for $1 \leq 2i-1 \leq 2^m-1$. Since $y Sq^2 \in \Lambda_1$ and the ideal $\Lambda_1$ is stable under the action of the Steenrod algebra, by applying $Sq^4$ to equation (2.1) we have

$$0 = x Sq^4 = \sum_I a_I (\tilde{z}_I Sq^4)$$

in $\mathbb{F}_2[z_1, z_3, \ldots, z_{2^m+1}]/\Lambda_1 \cong \Lambda(\mathbb{F}_2[z_1, z_3, \ldots, z_{2^m+1}], \mathbb{F}_2[z_{2^m+1}, \ldots, z_{2^m+1}]).$

For an integer $\ell$ such that $2 \leq \ell \leq m - 2$, we define a map

$$\phi_\ell : \Lambda(z_1, z_3, \ldots, z_{2^m+1}) \otimes \mathbb{F}_2[z_{2^m+1}, \ldots, z_{2^m+1}]$$
$$\rightarrow \Lambda(z_1, z_3, \ldots, z_{2\ell-3}, z_{2\ell+1}, \ldots, z_{2^m-1}) \otimes \mathbb{F}_2[z_{2^m+1}, \ldots, z_{2^m+1}]$$

as follows: If $\alpha \in \Lambda(z_1, z_3, \ldots, z_{2^m-1}) \otimes \mathbb{F}_2[z_{2^m+1}, \ldots, z_{2^m+1}]$ is written as

$$\alpha = z_{2^\ell-1}^2 + \gamma$$

with $\beta, \gamma \in \Lambda(z_1, z_3, \ldots, z_{2^\ell-3}, z_{2^\ell+1}, \ldots, z_{2^m-1}) \otimes \mathbb{F}_2[z_{2^m+1}, \ldots, z_{2^m+1}]$, then we define $\phi_\ell(\alpha) = \beta$.

For $2 \leq \ell \leq m - 2$ and $I = (e_2, e_3, \ldots, e_{m-2}, j_{m-1}, j_m)$ we define

$$z_{It} = z_{e_2}^{e_2} \cdots z_{e_{\ell-1}}^{e_{\ell-1}} z_{e_{\ell+1}}^{e_{\ell+1}} \cdots z_{e_{m-2}}^{e_{m-2}} z_{j_{m-1}}^{j_{m-1}} z_{j_m}^{j_m}.$$ 

Since $\tilde{z}_{2^{k+1}} Sq^4 = \left(\binom{2^{k+1} - 1}{2^k - 1}\right) z_{2^{k-1}} = z_{2^{k-1}}$ for $k > 1$, it is easy to see that

$$\phi_\ell(\tilde{z}_I Sq^4) = \begin{cases} z_{I\ell} & \text{if } e_{\ell} = 1, \\ 0 & \text{otherwise.} \end{cases}$$

Thus equation (2.2) implies that

$$0 = \phi_\ell(x Sq^4) = \sum_{I \text{ such that } e_\ell = 1} a_I z_{I\ell}$$

in $\Lambda(z_1, z_3, \ldots, z_{2^\ell-3}, z_{2^\ell+1}, \ldots, z_{2^m-1}) \otimes \mathbb{F}_2[z_{2^m+1}, \ldots, z_{2^m+1}]$ and we proved that $a_I = 0$ for all $I$ such that $e_i = 1$ for some $2 \leq i \leq m - 2$.

Step II). We proved that $x$ is written as

$$x = \sum a_{(j_{m-1}, j_m)} z_{j_{m-1}}^{j_{m-1}} z_{2^{m-1}+1}^{j_m} y Sq^2$$

for some $y \in H_{|x|+2}(\Omega Sq^n)$. In the second step we will show that $a_{(j_{m-1}, j_m)} = 0$ if $j_m > 0$. This is done by downward induction on $j_m$. Let $k$ be a positive integer and assume that $a_{(j_{m-1}, j_m)} = 0$ if $j_m > k$.

Let $\Lambda_2$ be the ideal generated by $z_{2i-1}$ for $1 \leq 2i-1 \leq 2^m-1$. Since $y Sq^2 \in \Lambda_2$ and the ideal $\Lambda_2$ is stable under the action of the Steenrod algebra, by applying $Sq^{2^m}$ to equation (2.3) we have

$$0 = x Sq^{2^m} = \sum a_{(j_{m-1}, j_m)} (\tilde{z}_{j_{m-1}}^{j_{m-1}} z_{j_m}^{j_m}) Sq^{2^m}$$

in $\mathbb{F}_2[z_1, \ldots, z_{2^m+1}]/\Lambda_2 \cong \mathbb{F}_2[z_{2^m+1}, \ldots, z_{2^m+1}]$.

To compute $(\tilde{z}_{j_{m-1}}^{j_{m-1}} z_{j_m}^{j_m}) Sq^{2^m}$ we remark that

$$\tilde{z}_{2^{m-1}+1} Sq^i \in \Lambda_2 \quad \text{unless } i = 0$$

and that

$$\tilde{z}_{2^m+1} Sq^i = \left(\binom{2^m - i}{i}\right) z_{2^{m-1}+1} = 0 \quad \text{if } i > 2^{m-1}.$$
Thus in $\mathbb{F}_2[z_1, \ldots, z_{2m+1}]/\Lambda_2 \cong \mathbb{F}_2[z_{2m-1+1}, \ldots, z_{2m+1}]$ we have
\[
0 = xS^2m^k = \sum a_{(j_m-1,j_m)}(z_{m-1}^{j_m-1}z_{m+1}^{j_m})S^2m^k
= a_{(j,k)}z_{m-1+1}^j(z_{m+1}^kS^2m^k) + \sum_{j_m<k} a_{(j_m-1,j_m)}z_{m-1+1}^{j_m-1}(z_{m+1}^{j_m}S^2m^k)
= a_{(j,k)}z_{m-1+1}^{j+k},
\]
which implies that $a_{(j,k)} = 0$ and completes the induction argument.

Step III). We proved that $x$ is written as
\[
(2.4) \quad x = az_{m-2+1}^j + yS^2m^{-j}
\]
for some $a \in \mathbb{F}_2$, $y \in H_{|x|=2}(\Omega Sp(n))$ and $j > 0$. In the final step we will show that $a = 0$ and complete the proof.

By applying $S^2m^{-j}$ to equation (2.4) we have
\[
az_{m-2+1}^j = yS^2S^2m^{-j} = yS^2m^{-j+2}.
\]

For a sequence of integers $S = (2s_1-1, 2s_2-1, \ldots, 2s_t-1)$ such that $2m-2+1 \leq 2s_1-1 \leq 2s_2-1 \leq \cdots \leq 2s_t-1 \leq 2m+1$, we define
\[
S = z_{s_1-1}z_{s_2-1} \cdots z_{s_t-1}.
\]
Then $y$ is written as
\[
y = \sum b_S z_S + y',
\]
where $b_S \in \mathbb{F}_2$. $S$ ranges over all sequences of integers $S = (2s_1-1, 2s_2-1, \ldots, 2s_t-1)$ such that $2m-2+1 \leq 2s_1-1 \leq 2s_2-1 \leq \cdots \leq 2s_t-1 \leq 2m+1$ and $|S| = |y|$, and $y'$ is an element of the ideal generated by $z_{2i-1}$ for $1 \leq 2i-1 \leq 2m-2-1$. To prove that $a = 0$ it is sufficient to prove that the coefficient of $z_{m-2+1}^j$ in the expansion of $z_S S^2m^{-j+2}$ as a sum of the standard monomials for $\mathbb{F}_2[z_1, z_3, \ldots, z_{2m+1}]$ is zero.

Since
\[
(z_{s_1-1}z_{s_2-1} \cdots z_{s_t-1})S^2m^{-j+2} = \sum (z_{s_1-1}S^2q^{2k_1})(z_{s_2-1}S^2q^{2k_2}) \cdots (z_{s_t-1}S^2q^{2k_t}),
\]
we consider the equation
\[
z_{s_1-1}S^2q^{2k_i} = \left(\frac{2s_i - 2 - k_i}{k_i}\right)z_{s_1-1-k_i} = z_{s_1-1}^{2r_i} z_{m-2+1}^{2r_i} = z_{(2m-2+1)2r_i},
\]
Then $2s_i - 1 - k_i = (2m-2+1)2r_i$, that is, $2s_i - 1 = (2m-2+1)2r_i + k_i$. Since $2s_i - 1 = (2m-2+1)2r_i + k_i \leq 2m+1$, we have $r_i = 0$ or 1.

If $r_i = 0$ and $k_i > 0$, then
\[
\left(\frac{2s_i - 2 - k_i}{k_i}\right) = \left(\frac{2m-2}{k_i}\right) \neq 0
\]
implies that $k_i = 2m-2$ and is even. Thus $2s_i - 1 = 2m-1 + 1$.

If $r_i = 1$ and $k_i > 0$, then
\[
\left(\frac{2s_i - 2 - k_i}{k_i}\right) = \left(\frac{2m-1+1}{k_i}\right) \neq 0
\]
implies that \( k_i = 1 \) or \( 2^m - 1 + 1 \) since \( k_i = 2s_i - 1 - 2(2^{m-2} + 1) \) is odd. Thus \( 2s_i - 1 = 2^m - 1 + 3 \) or \( 2^m + 3 \). The last case is impossible since \( 2s_i - 1 \leq 2^m + 1 \). Thus \( 2s_i - 1 = 2^{m-1} + 3 \) and \( k_i = 1 \).

According to the argument above the coefficient of \( z^j_{2m-2+1} \) in \( Sq^{2m-1j+2}z_S \) is zero unless \( z_S \) is

\[
z^r_{2m-2+1}z^s_{2m-1+1}z^t_{2m-1+3}
\]

for some non-negative integers \( r, s, t \). If the coefficient of \( z^j_{2m-2+1} \) in

\[
(z^r_{2m-2+1}z^s_{2m-1+1}z^t_{2m-1+3})Sq^{2m-1j+2} = z^r_{2m-2+1}(z^s_{2m-1+1})Sq^{2m-1s}(z^t_{2m-1+3})Sq^{2t} + \cdots
\]

is non-zero, then \( 2m-1j + 2 = 2m-1s + 2t \), that is, \( t = 1 + 2m-2(j - s) \). As \( m \geq 3 \), this implies that \( j - s \geq 0 \). Since the degree of \( z^j_{2m-1+1} \) is equal to that of

\[
(z^r_{2m-2+1}z^s_{2m-1+1}z^t_{2m-1+3})Sq^2,
\]

we have

\[
j(2m-1 + 1) = r(2m-2 + 1) + s(2m-1 + 1) + t(2m-1 + 3) - 1,
\]

that is,

\[
2 \leq r + 2 = -(j - s)(2m-1 - 1) \leq 0,
\]

which is impossible. We proved that the coefficient of \( z^j_{2m-2+1} \) in \( z_S Sq^{2m-1j+2} \) is zero and, therefore, completed the proof of the lemma.

For a connected space \( X \) of finite type we associate a graph \( G(X) \) as follows. The vertices of \( G(X) \) are non-zero elements of \( \hat{H}_s(X) \) and a pair of vertices \( \{x, y\} \) is an edge of \( G(X) \) if and only if \( xSq^i = y \) or \( ySq^i = x \) for some \( i > 0 \).

**Lemma 2.4.** Let \( n \geq 4 \) or \( n = \infty \). Every vertex of \( G(\Omega Sp(n)) \) whose dimension is greater than two is connected to \( z^1_2 \) or \( z^3_3 \).

**Proof.** By induction on the dimension of a vertex we will prove the lemma. Let \( x \) be a vertex of \( G(\Omega Sp(n)) \).

If \( |x| = 4 \), then \( x = z^2_2 \). If \( |x| = 6 \), then \( x = z^3_1, z^3_1 + z_3 \) or \( z_3 \). Since \( (z^3_1 + z_3)Sq^2 = z_3Sq^2 = z^2_1 \), the assertion is true.

Let \( |x| = 2m \geq 8 \) and assume that the assertion is true for vertices whose dimensions are less than \( |x| \). If \( xSq^i \neq 0 \) for some \( i > 0 \), then the assertion is true by induction. If \( xSq^i = 0 \) for all \( i > 0 \), then by Lemma 2.3 there is a vertex \( y' \) such that \( y'Sq^2 = x \). We put

\[
y = \begin{cases} y' & \text{if } y'Sq^4 \neq 0, \\
y' + z^{m-4}_1z_5 + z^{m-2}_1z_3 & \text{if } y'Sq^4 = 0. \end{cases}
\]

Then \( ySq^2 = x \) and \( ySq^4 \neq 0 \), and \( x \leftarrow y \rightarrow ySq^4 \) is a path which connects \( x \) and a vertex whose dimension is less than \( x \). Then by induction there is a path which connects \( ySq^4 \) and \( z^2_1 \) or \( z^3_3 \). Therefore there is a path which connects \( x \) and \( z^2_1 \) or \( z^3_3 \), and we complete the proof.

We remark that Lemma 2.3 is valid for \( n = 2 \) or \( 3 \) and that Lemma 2.4 is valid for \( n = 3 \). These facts follow Proposition 2.1 of [3].
Proof of Theorem 2.1. As the theorem for $n = 2$ and 3 was proved by Hopkins and Hubbuck, we prove the theorem for $n \geq 4$ or $n = \infty$.

We give CW-decompositions for $\Omega Sp(n)$ without odd dimensional cells for all $n$. Then $(\Omega Sp(2))_8$ is homotopy equivalent to $(\Omega Sp(n))_8$, where for a CW-complex $X$ by $X_r$ we denote the $r$-skeleton of $X$. By [3] $(\Omega Sp(2))_8$ is stably homotopy equivalent to $Z \vee S^8$, where $Z$ is a stably indecomposable CW-complex. Therefore if $\Omega Sp(n)$ is stably split as $\Omega Sp(n) \simeq X(1) \vee X(2)$, where $H_2(X(1)) \cong \mathbb{F}_2$; then $X(2)$ is 7-connected. By Lemma 2.4 this implies that $X(2)$ must be trivial and completes the proof of the theorem. 

References

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