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NOTE ON ARITHMETIC CONVOLUTION EQUATIONS

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ABSTRACT. We investigate the solvability of polynomial equations on the $\mathbb{C}$-algebra of arithmetic functions $g : \mathbb{N} \to \mathbb{C}$.

1. Introduction

Let $\mathcal{A}$ be the set of all arithmetic functions $g : \mathbb{N} \to \mathbb{C}$ under point-wise addition and scalar multiplication [5]. The null element is the null function. The Dirichlet convolution $* : \mathcal{A} \times \mathcal{A} \to \mathcal{A}$ is defined by

$$(g * h)(n) = \sum_{n_1 n_2 = n} g(n_1)g(n_2) \quad \forall n \in \mathbb{N}\,.$$

Then $\mathcal{A}$ is considered as a $\mathbb{C}$-algebra under linear operations and the Dirichlet convolution.

Let $r \in \mathbb{R}$ and $\mathcal{A}_r$ be the complex Banach algebra of arithmetic functions $g \in \mathcal{A}$ endowed with the norm

$$\|g\|_r := \sum_{n=1}^{\infty} |g(n)| n^{-r}\,.$$

For abbreviation we write $g^{*d} := g * \cdots * g$. The following result is proved by Glöckner, Lucht and Porubský in [5]:

Theorem 1.1. *For $d \in \mathbb{N}$ and $g \in \mathcal{A}$, let $T : \mathcal{A} \to \mathcal{A}$ be defined by*

$$Tg := a_d * g^{*d} + a_{d-1} * g^{*(d-1)} + \cdots + a_1 * g + a_0$$

with $a_d, a_{d-1}, \ldots, a_1, a_0 \in \mathcal{A}$ and $a_d \neq 0$. If z_0 is a simple zero of the polynomial

$$f(z) := a_d(1)z^d + a_{d-1}(1)z^{(d-1)} + \cdots + a_1(1)z + a_0(1), \tag{1.1}$$

then there exists a uniquely determined solution $g \in \mathcal{A}$ of the equation

$$Tg = 0 \tag{1.2}$$

satisfying $g(1) = z_0$. Moreover, if $a_d, a_{d-1}, \ldots, a_1, a_0 \in \mathcal{A}_\rho$ for some $\rho \geq 0$, then there is an $r \geq \rho$ such that $g \in \mathcal{A}_r$.

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We refer the reader to papers of Carroll and Gioia [2], Cohen [3], Dehaye [4], Glöckner, Lucht and Porubský [5], Haukkanen [6], Porubský [8] and Subbarao [9] for more details and known results about arithmetic convolution equations.

Theorem 1.1 is proved in [5] by using a deep result from the theory of topological vector spaces: the implicit function theorem in a version of Biller [1, Theorem 7.2]. The main purpose of this note is to present an elementary proof of Theorem 1.1. But we also extend it to any $\rho \in \mathbb{R}$ along with finding a lower bound r_0 for the number r and with estimating from above the norm $\|g\|_{r_0}$ of the solution g. At the end of this note, we also solve a simple quadratic convolution equation to apply our general result.

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2. Main results

Let z_0 be a simple zero of (1.1). As in [5], we start from (1.2) with $g(1) = z_0$, which has the form

$$\sum_{j=0}^{d} \sum_{l n_1 \cdots n_j = n} a_j(l) g(n_1) \cdots g(n_j) = 0 \quad \forall n \in \mathbb{N}\,. \tag{2.1}$$

Next, we rewrite (2.1) as

$$f'(g(1)) g(n) = -\sum_{j=0}^{d} \sum_{\substack{l n_1 \cdots n_j = n \\ n_1, \cdots, n_j < n}} a_j(l) g(n_1) \cdots g(n_j)\,, \quad n \geq 2\,. \tag{2.2}$$

Since $f'(g(1)) \neq 0$, we can solve (2.2) to get the solution $g \in \mathcal{A}$ of (1.2). If in addition $a_d, a_{d-1}, \ldots, a_1, a_0 \in \mathcal{A}_\rho$ for some $\rho \in \mathbb{R}$, then from (2.2) for $r \geq \rho$ we derive

$$\begin{aligned} f'(g(1)) g(n) n^{-r} = & -\sum_{j=0}^{d} a_j(1) \sum_{\substack{n_1 \cdots n_j = n \\ n_1, \ldots, n_j < n}} g(n_1) n_1^{-r} \cdots g(n_j) n_j^{-r} \\ & -\sum_{j=0}^{d} \sum_{\substack{l n_1 \cdots n_j = n \\ 2 \leq l; n_1, \ldots, n_j < n}} \\ & \times l^{\rho - r} a_j(l) l^{-\rho} g(n_1) n_1^{-r} \cdots g(n_j) n_j^{-r}\,, \quad n \geq 2\,. \end{aligned}$$

Hence for $m \geq 3$ we obtain

$$
\begin{aligned}
&|f'(g(1))| \sum_{n=2}^{m} |g(n)| n^{-r} \\
&\quad \leq \sum_{n=2}^{m} \sum_{j=2}^{d} |a_j(1)| \sum_{\substack{n_1 \cdots n_j = n \\ n_1, \ldots, n_j < n}} |g(n_1)| n_1^{-r} \cdots |g(n_j)| n_j^{-r} \\
&\quad\quad + 2^{\rho - r} \sum_{n=2}^{m} \sum_{j=0}^{d} \sum_{\substack{l n_1 \cdots n_j = n \\ 2 \leq l; n_1, \ldots, n_j < n}} |a_j(l)| l^{-\rho} |g(n_1)| n_1^{-r} \cdots |g(n_j)| n_j^{-r} \\
&\quad \leq \sum_{n=2}^{m} \sum_{j=2}^{d} |a_j(1)| \sum_{k=0}^{j-2} \binom{j}{k} |g(1)|^k \sum_{\substack{n_1 \cdots n_{j-k} = n \\ 2 \leq n_1, \ldots, n_{j-k} < n}} \\
&\quad\quad \times |g(n_1)| n_1^{-r} \cdots |g(n_{j-k})| n_{j-k}^{-r} \\
&\quad\quad + 2^{\rho - r} \sum_{j=0}^{d} \left(\sum_{l=1}^{m} |a_j(l)| l^{-\rho} \right) \left(\sum_{n=1}^{m-1} |g(n)| n^{-r} \right)^j \\
&\quad \leq \sum_{j=2}^{d} |a_j(1)| \sum_{k=0}^{j-2} \binom{j}{k} |g(1)|^k \left(\sum_{n=2}^{m-1} |g(n)| n^{-r} \right)^{j-k} \\
&\quad\quad + 2^{\rho - r} \sum_{j=0}^{d} \|a_j\|_\rho \left(\sum_{n=1}^{m-1} |g(n)| n^{-r} \right)^j \\
&\quad = \left(\sum_{n=2}^{m-1} |g(n)| n^{-r} \right)^2 \left(\sum_{j=2}^{d} |a_j(1)| \sum_{k=0}^{j-2} \binom{j}{k} |g(1)|^k \left(\sum_{n=2}^{m-1} |g(n)| n^{-r} \right)^{j-2-k} \right) \\
&\quad\quad + 2^{\rho - r} \sum_{j=0}^{d} \|a_j\|_\rho \left(\sum_{n=1}^{m-1} |g(n)| n^{-r} \right)^j .
\end{aligned}
$$

Recalling $g(1) = z_0$ and setting

$$
\begin{aligned}
S_m &:= \sum_{n=2}^{m} |g(n)| n^{-r}, \quad m \geq 2, \\
P(z) &:= \frac{1}{|f'(z_0)|} z^2 \left(\sum_{j=2}^{d} |a_j(1)| \sum_{k=0}^{j-2} \binom{j}{k} |z_0|^k z^{j-2-k} \right), \\
Q(z) &:= \frac{1}{|f'(z_0)|} \sum_{j=0}^{d} \|a_j\|_\rho z^j ,
\end{aligned}
$$

we obtain

$$
\begin{aligned}
&S_2 = |g(2)| 2^{-r}, \\
&S_m \leq P(S_{m-1}) + 2^{\rho - r} Q\left(|z_0| + S_{m-1}\right), \quad m \geq 3.
\end{aligned} \tag{2.3}
$$

If there are $x_0 > 0$ and $r \geq \rho$ such that

$$
|g(2)| 2^{-r} \leq x_0 , \tag{2.4}
$$

$$
P(x_0) + 2^{\rho - r} Q\left(|z_0| + x_0\right) \leq x_0 , \tag{2.5}
$$

then clearly

$$
S_m \leq x_0 \quad \forall m \geq 2 .
$$

From (2.5) we derive

$$2^{\rho-r} \leq \frac{x_0 - P(x_0)}{Q\left(|z_0| + x_0\right)} . \tag{2.6}$$

Since $P(0) = P'(0) = 0$ and $P(z) \geq 0$ for $z \geq 0$, we can easily check that for $d \geq 2$ there are positive constants M_0 and x_0 such that

$$M_0 := \max_{z \in [0,\infty)} \frac{z - P(z)}{Q\left(|z_0| + z\right)} = \frac{x_0 - P(x_0)}{Q\left(|z_0| + x_0\right)} . \tag{2.7}$$

Then conditions (2.4) and (2.5) hold with these positive constants M_0 and x_0 of (2.7) for any $r \geq r_0$ where r_0 is given by

$$r_0 := \max\left\{\frac{\log\left(|g(2)|/x_0\right)}{\log 2}, \rho - \min\left\{0, \frac{\log M_0}{\log 2}\right\}\right\} . \tag{2.8}$$

Here we set $\log 0 = -\infty$. We note

$$g(2) = -\frac{1}{f'(z_0)} \sum_{j=0}^{d} a_j(2) z_0^j . \tag{2.9}$$

Summarizing, we obtain the following result.

Theorem 2.1. *Let $d \geq 2$ and z_0 be a simple zero of the polynomial* (1.1) *with $a_d, a_{d-1}, \ldots, a_1, a_0 \in \mathcal{A}_\rho$ for some $\rho \in \mathbb{R}$ and $a_d \neq 0$. Then a unique solution $g \in \mathcal{A}$ of* (1.2) *satisfying $g(1) = z_0$ belongs to $g \in \mathcal{A}_{r_0}$ for $r_0 \geq \rho$ given by* (2.8)*, while the involved constants x_0, M_0 and $g(2)$ of the formula* (2.8) *are determined by* (2.7) *and* (2.9)*, respectively. Moreover, we have $\|g\|_{r_0} \leq |z_0| + x_0$.*

To proceed with the investigation of (1.2), we need the following definition.

Definition 2.1. An $a \in \mathcal{A}$ is said to be a *constant coefficient* if $a(j) = 0 \, \forall j \geq 2$.

Remark 2.1. If not all $a_j \in \mathcal{A}$, $j = 0, 1, \cdots, d-1, d$, are constant coefficients, then the polynomial $Q(z)$ can be replaced in Theorem 2.1 with the following:

$$Q(z) := \frac{1}{|f'(z_0)|} \sum_{j=0}^{d} \left(\|a_j\|_\rho - |a_j(1)|\right) z^j .$$

If $d = 1$ and $a_1, a_0 \in \mathcal{A}_\rho$ for some $\rho \in \mathbb{R}$, then $P(z) = 0$ and

$$Q(z) = \frac{1}{|a_1(1)|} \left(\|a_0\|_\rho - |a_0(1)| + \left(\|a_1\|_\rho - |a_1(1)|\right) z\right) ,$$

where we suppose that a_1 is not a constant coefficient, because the constant coefficient case is trivial. So (2.6) becomes

$$2^{\rho-r} \leq \frac{x_0 |a_1(1)|}{\|a_0\|_\rho - |a_0(1)| + \left(\|a_1\|_\rho - |a_1(1)|\right)\left(|z_0| + x_0\right)} , \tag{2.10}$$

where $z_0 = -a_0(1)/a_1(1)$. Analyzing (2.10), we consider two cases:

First case: $2|a_1(1)| \leq \|a_1\|_\rho$. Then we fix any r satisfying

$$r > \rho - \frac{1}{\log 2} \log\left(\frac{|a_1(1)|}{\|a_1\|_\rho - |a_1(1)|}\right) , \tag{2.11}$$

and take

$$x_0 := \max\left\{\frac{|g(2)|}{2^r}, \frac{2^{\rho-r}\left(\|a_0\|_\rho - |a_0(1)| + \left(\|a_1\|_\rho - |a_1(1)|\right)|z_0|\right)}{|a_1(1)| - 2^{\rho-r}\left(\|a_1\|_\rho - |a_1(1)|\right)}\right\}, \tag{2.12}$$

where $g(2) = \frac{a_1(2)a_0(1)-a_0(2)a_1(1)}{a_1(1)^2}$ is defined by (2.9).

Second case: $2|a_1(1)| > \|a_1\|_\rho$. Then

$$r = \rho, \quad x_0 := \max\left\{\frac{|g(2)|}{2^\rho}, \frac{\|a_0\|_\rho - |a_0(1)| + \left(\|a_1\|_\rho - |a_1(1)|\right)|z_0|}{2|a_1(1)| - \|a_1\|_\rho}\right\}. \tag{2.13}$$

Note that $r = \rho$ in (2.13) can also be derived as follows: Let $A_1(s) = \sum_{n=1}^\infty a_1(n)n^{-s}$ be the Dirichlet series for the complex variable $s \in \mathbb{C}$ with $\mathrm{Re} \geq \rho$. Then $|A_1(s)| \geq 2|a_1(1)| - \|a_1\|_{\mathrm{Re}} \geq 2|a_1(1)| - \|a_1\|_\rho > 0$. Hence [7] implies that $a_1^{-1} \in \mathcal{A}_\rho$ and then $g = -a_1^{-1} * a_0 \in \mathcal{A}_\rho$.

Summarizing, we arrive at the following result.

Theorem 2.2. *Let $a_1, a_0 \in \mathcal{A}_\rho$ for some $\rho \in \mathbb{R}$ and let a_1 be a nonconstant coefficient satisfying $a_1(1) \neq 0$. Then a unique solution $g \in \mathcal{A}$ of $a_1 * g + a_0 = 0$ belongs to $g \in \mathcal{A}_r$ and $\|g\|_r \leq \frac{|a_0(1)|}{|a_1(1)|} + x_0$, where r, x_0 are given either by* (2.11) *or* (2.12) *when $2|a_1(1)| \leq \|a_1\|_\rho$, or by* (2.13) *when $2|a_1(1)| > \|a_1\|_\rho$.*

More general systems of convolution equations are studied in [5], which need not be polynomial. But since computations like the above seem to be rather awkward for such systems, we omit them in this note. We give a rather simple example in the next section.

3. Quadratic equations

In this section, we study the quadratic equation

$$g * g = a, \quad a(1) = 1, \quad \|a\|_\infty := \sup_{n\in\mathbb{N}} |a(n)| < \infty. \tag{3.1}$$

We also suppose that a is not a constant coefficient, because the constant coefficient case is trivial. Since $a(1) = 1 \neq 0$, we can use Theorem 2.1 and Remark 2.1 for (3.1). An interesting discussion on the solvability of (3.1) and the more general polynomial equations $g^{*d} = a$, $a \in \mathcal{A}$, is given in [5].

For any $\rho > 1$ we compute

$$\|a\|_\rho \leq \|a\|_\infty \sum_{n=1}^\infty n^{-\rho} \leq \|a\|_\infty \left(1 + \int_1^\infty x^{-\rho}\, dx\right) = \frac{\rho}{\rho-1}\|a\|_\infty. \tag{3.2}$$

Also using Remark 2.1 we get

$$|z_0| = |g(1)| = \sqrt{|a(1)|} = 1, \quad |g(2)| = \tfrac{|a(2)|}{2|g(1)|} \leq \|a\|_\infty/2,$$
$$P(z) = z^2/2, \quad Q(z) = \left(\|a\|_\rho - 1\right)/2.$$

So (2.7) gives

$$M_0 = \max_{z\in[0,\infty)} \frac{(2z - z^2)}{\|a\|_\rho - 1} = \frac{1}{\|a\|_\rho - 1}, \quad x_0 = 1,$$

and by (2.8) we can take any r_0 such that

$$r_0 \geq \max\left\{\frac{\log\left(|a(2)|/2\right)}{\log 2}, \rho + \max\left\{0, \frac{\log\left(\|a\|_\rho - 1\right)}{\log 2}\right\}\right\}.$$

Hence according to (3.2) we take

$$r_0 = \max\left\{\frac{\log(\|a\|_\infty/2)}{\log 2}, \rho + \frac{1}{\log 2}\max\left\{0, \log\left(\frac{\rho(\|a\|_\infty - 1) + 1}{\rho - 1}\right)\right\}\right\}. \tag{3.3}$$

Now fixing $\|a\|_\infty \geq 1$ and varying $\rho > 1$, we intend to find the smallest value of r_0 in (3.3). We consider two cases:

First case: $\|a\|_\infty \geq 2$. Then it holds that

$$\frac{\rho(\|a\|_\infty - 1) + 1}{\rho - 1} \geq \|a\|_\infty - 1 \geq \|a\|_\infty/2 \geq 1\,.$$

Consequently, (3.3) possesses the form

$$r_0 = F_{\|a\|_\infty}(\rho) := \rho + \frac{1}{\log 2}\log\left(\frac{\rho(\|a\|_\infty - 1) + 1}{\rho - 1}\right).$$

It is easy to verify that $F_{\|a\|_\infty}(\rho)$ is strictly convex on $(1,\infty)$, i.e. $F''_{\|a\|_\infty}(\rho) > 0$ $\forall \rho > 1$. So the function $F_{\|a\|_\infty}(\rho)$ has the unique global minimum on $(1,\infty)$ achieved at a unique $\rho_0 > 1$ solving $F'_{\|a\|_\infty}(\rho_0) = 0$. But for a general $\|a\|_\infty$, the explicit formula for this global minimum $F_{\|a\|_\infty}(\rho_0)$ is ugly and long, so we do not present it here. On the other hand, we computed for $\|a\|_\infty = 10$ that $\rho_0 \doteq 1.82706$ and $F_{10}(\rho_0) \doteq 6.22562$.

Second case: $\|a\|_\infty < 2$. Then (3.3) possesses the form

$$r_0 = \begin{cases} \rho & \text{for} \quad \rho \geq \frac{2}{2-\|a\|_\infty}\,, \\ F_{\|a\|_\infty}(\rho) & \text{for} \quad 1 < \rho \leq \frac{2}{2-\|a\|_\infty}\,. \end{cases} \tag{3.4}$$

Since again for a general $\|a\|_\infty$ the explicit formula for the global minimum of (3.4) is ugly and long, we do not present it here. We concentrate on the case $\|a\|_\infty = 1$. Then (3.4) becomes

$$r_0 = \begin{cases} \rho & \text{for} \quad \rho \geq 2\,, \\ F_1(\rho) = \rho - \frac{\log(\rho-1)}{\log 2} & \text{for} \quad 1 < \rho \leq 2\,. \end{cases} \tag{3.5}$$

We can easily check that the piecewise smooth function in (3.5) has a unique global minimum 2 on $(1,\infty)$ at $\rho_0 = 2$.

Summarizing from Theorem 2.1 and Remark 2.1 we obtain the following result.

Corollary 3.1. *Let a be a nonconstant coefficient. If $\|a\|_\infty = 1$ then the only two solutions $g_\pm \in \mathcal{A}$ of* (3.1) *with $g_\pm(1) = \pm 1$, respectively, belong to $\mathcal{A}_2$ and $\|g_\pm\|_2 \leq 2$. If $\|a\|_\infty = 10$, then $g_\pm \in \mathcal{A}_{r_0}$ and $\|g_\pm\|_{r_0} \leq 2$ for $r_0 \doteq 6.22562$.*

According to [5, Corollary 2], we obtain $g_- = -g_+$ in Corollary 3.1. Similar numerical results hold for other numerical values of $\|a\|_\infty$.

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