SIMULTANEOUS SURFACE RESOLUTION
IN CYCLIC GALOIS EXTENSIONS

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(Communicated by Ted Chinburg)

ABSTRACT. We show that simultaneous surface resolution is not always possible in a cyclic extension whose degree is greater than three and is not divisible by the characteristic. This answers a recent question of Ted Chinburg.

1. INTRODUCTION

Let $K$ be a two dimensional algebraic function field over an algebraically closed ground field $k$. Recall that $K/k$ has a minimal model means that amongst all the nonsingular projective models of $K/k$ there is one which is dominated by all others (basic reference [Ab4] or [Ab5]). Also recall that $K/k$ has a minimal model if and only if it is not a ruled function field, i.e., $K$ is not a simple transcendental field extension of a one dimensional algebraic function field over $k$ (see [Zar]). A finite algebraic field extension $L/K$ is said to have a simultaneous resolution if there exist nonsingular projective models $V$ and $W$ of $K/k$ and $L/k$, respectively, such that $W$ is the normalization of $V$ in $L$. Given any positive integer $q$ that is not divisible by the characteristic $\text{char}(K)$ of $K$ and letting $\mathbb{Z}_q$ denote a cyclic group of order $q$, in [Ab2] it was shown that if $q \leq 3$ and $L/K$ is a $\mathbb{Z}_q$ extension, i.e., a Galois extension whose Galois group is a cyclic group of order $q$, then it has a simultaneous resolution, whereas if $K/k$ has a minimal model and $q > 3$ with $q$ being a prime number, then there exists a $\mathbb{Z}_q$ extension $L/K$ that has no simultaneous resolution. Here we shall extend this second result to those nonprimes $q$ that are divisible by the square of some prime $p$. By taking $q = 4$, this answers a question raised by Ted Chinburg at the March 2006 AMS Meeting in New Hampshire to the effect whether every $\mathbb{Z}_2$ by $\mathbb{Z}_2$ extension $L/K$, i.e., a $\mathbb{Z}_2$ extension $L/J$ of a $\mathbb{Z}_2$ extension $J/K$, has a simultaneous resolution. By using a theorem of David Harbater and Florian Pop, we generalize our extended result by replacing $\mathbb{Z}_q$ by its direct sum $H \oplus \mathbb{Z}_q$ with any finite group $H$. For a related matter, see [AbK].

In Lemma (2.2) of Section 2 we shall give a consequence of the Harbater-Pop Theorem to be used in proving our generalized extended result. In Lemma (2.1) of Section 2 we shall summarize some technical results from our previous papers [Ab2] and [Ab3]. These technical results deal with the structure of the integral closure of a normal noetherian domain in a cyclic extension. They are used in the proof of Theorem (3.1) of Section 3, which gives a sufficient condition for a two dimensional
local domain to be nonregular. Theorem (3.1) is used in proving the special case of Theorem (3.2) of Section 3, which corresponds to our extended result, i.e., the $H = 1$ case of our generalized extended result. The general case of Theorem (3.2), which corresponds to our generalized extended result, then follows by using Lemma (2.2).

2. TWO LEMMAS

Let $M(R)$ denote the maximal ideal of a local ring $R$. In Lemma (2.1) we summarize some properties of the integral closure of a normal noetherian domain in a cyclic extension. In Lemma (2.2) we give a consequence of the Harbater-Pop theorem.

**Lemma 2.1.** Let $R$ be a normal noetherian domain with quotient field $K$, let $S$ be the integral closure of $R$ in a finite algebraic field extension $L$ of $K$, and let $[L : K] = q$. Assume that $q$ is a unit in $R$ and that $L$ contains a nonzero element $z$ such that $L = K(z)$ and

$$z^q = u \prod_{j=1}^{d} x_j^{a(j)}$$

where $u$ is a unit in $R$, $d$ is a nonnegative integer, $a(j)$ is an integer such that $\gcd(a(j), q) = 1$ for $1 \leq j \leq d$, and $x_1, \ldots, x_d$ are elements in $R$ such that $x_1R, \ldots, x_dR$ are pairwise distinct minimal ($= \text{height one}$) prime ideals in $R$. Let $b(i, j)$ and $c(i, j)$ be the unique integers such that

$$b(i, j) = a(j)i + c(i, j)q \quad \text{and} \quad 0 \leq b(i, j) < q.$$ 

Let

$$z_i = z^i \prod_{j=1}^{d} x_j^{c(i, j)}.$$

Then we have the following:

1. $(z_0, \ldots, z_{q-1})$ is a free $R$-basis of $S$.
2. If $R$ is a local domain and $d \geq 1$, then $S$ is a local domain and for its maximal ideal $M(S)$ we have $M(S) = M(R)S + (z_1, \ldots, z_{q-1})S$ with $S/M(S) = R/M(R)$.
3. If $R$ is a regular local domain and $d \geq 2$, then $S$ is a nonregular local domain.

**Proof.** For (1) and (2) see Theorem 7 [Ab3]. For (3) see Theorem 6 [Ab2] with the observation that, although in the context of this theorem $q$ is a prime number, the primeness of $q$ was never used in its proof. A different version of (1) and (2) can also be found in Theorems 4 and 5 [Ab2]; see Remark 2 on page 28 of [Ab3].

**Lemma 2.2.** Let $K/k$ be a two dimensional algebraic function field over an algebraically closed ground field $k$. For any finite group $H$, there exists a Galois extension $\tilde{L}/K$ with Galois group $H$.

**Proof.** It follows from Theorem 4.4 [Har] or the Corollary to Theorem A [Pop] that given any finite group $H$ and any one dimensional algebraic function field $E$ over an algebraically closed ground field $k$, there exists a Galois extension $F/E$ whose Galois group is $H$. The following argument, provided by Harbater and Pop, shows how the desired two-variable existence follows from this.
Given a two dimensional algebraic function field $K$ over $k$, choose a separating transcendence basis $x, y$ for $K$ over $k$. So $K$ is a finite separable field extension of $k(x, y)$. Let $E$ be the algebraic closure of $k(x)$ in $K$. $E$ is finite over $k(x)$, since $K$ is finite over $k(x, y)$ and since $k(x)$ is algebraically closed in $k(x, y)$. Thus $E$ is a one dimensional algebraic function field over $k$ and so, by the one-variable existence theorem, $H$ is the Galois group of a finite extension $F$ of $E$. Since $E$ is algebraically closed in $K$ and since $F$ is algebraic over $E$, it follows that $F$ and $K$ are linearly disjoint over $E$. So the compositum $\tilde{L} = KF$ (in an algebraic closure of $K$) is a Galois extension of $K$ with Galois group $H$, completing the proof. 

□

3. Two theorems

In Theorem (3.1) we give a sufficient condition for a local domain to be nonregular. In Theorem (3.2) we construct our examples of simultaneous nonresolvability.

Theorem 3.1. Let $R$ be a two dimensional regular local domain, let $(X, Y)$ be generators of its maximal ideal $M(R)$, and let $K$ be its quotient field. Let $R_0 = R$. For all $n > 0$, let $Y_n = Y/X^n$ and let $R_n$ be the localization of the ring $R_{n-1}[Y_n]$ at the maximal ideal in it generated by $(X, Y_n)$. Note that then $R_n$ is a two dimensional regular local domain with quotient field $K$ such that $R_n$ dominates $R_{n-1}$ and $(X, Y_n)$ are generators of $M(R_n)$.

Let $q$ be a positive integer that is a unit in $R$. Assume that $q = pm$ where $p$ is a prime number and $m$ is a positive integer divisible by $p$. Assume that $K$ contains $q$ distinct $q$-th roots of 1. Let $L$ be a splitting field over $K$ of the polynomial of $Z^q - XY^m$. Let $S_n$ be the integral closure of $R_n$ in $L$.

Then $L/K$ is a $\mathbb{Z}_q$ extension and for every nonnegative integer $n$, the ring $S_n$ is a two dimensional nonregular local domain.

Proof. Let $w$ be the discrete valuation whose valuation ring is the one dimensional regular local domain obtained by localizing the ring $R$ at the prime ideal in it generated by $X$. Then $w(XY^m) = 1$ and hence the polynomial $Z^q - XY^m$ is irreducible in $K[Z]$ and $L/K$ is a $\mathbb{Z}_q$ extension. Let $z \in L$ be a root of the said polynomial. Then $z^q = XY^m$ and $L = K(z)$. Let $\overline{X} = z^{1/p}Y$ and $J = K(\overline{X})$. Then $\overline{X}^m = X$ and hence $J/K$ is a $\mathbb{Z}_m$ extension. By (2.1)(2) the integral closure $T_n$ of $R_n$ in $J$ is a two dimensional regular local domain whose maximal ideal $M(T_n)$ is generated by $(\overline{X}, Y_n)$. Also $z^p = \overline{X}^pY = \overline{X}^{1+nm}Y_n$ and, since $m$ is assumed divisible by $p$, upon letting $\zeta = z/\overline{X}^{nm/p}$ we get $L = J(\zeta)$ with $\zeta^p = \overline{X}Y_n$. Now $L/J$ is a $\mathbb{Z}_p$ extension with $L = J(\zeta)$, and $S_n$ is the integral closure of $T_n$ in $L$. Therefore by (2.1)(3) we see that $S_n$ is a two dimensional nonregular local domain. □

Theorem 3.2. Let $K/k$ be a two dimensional algebraic function field over an algebraically closed ground field $k$. Assume that $K/k$ has a minimal model $V^*$. Let $q$ be a positive integer that is not divisible by $\text{char}(K)$. Assume that $q = pm$ where $p$ is a prime number and $m$ is a positive integer divisible by $p$. Then, given any finite group $H$, there exists a Galois extension $L'/K$ with Galois group $H \oplus \mathbb{Z}_q$ such that $L'/K$ has no simultaneous resolution.

Proof. By (2.2) there exists a Galois extension $\tilde{L}/K$ with Galois group $H$. Take $R$ in (3.1) to be the local ring of a point of $V^*$ that is not ramified in $\tilde{L}$. Let $L'$ be a compositum of $\tilde{L}$ and $L$. It is easy to see that $L'/K$ is a Galois extension whose Galois group is $H \oplus \mathbb{Z}_q$. 


By [Ab1, Lemma 12], there exists a unique valuation \( v \) of \( K \) dominating \( R_n \) for all \( n \geq 0 \). By construction each \( R_{n+1} \) is the immediate quadratic transform of \( R_n \) along \( v \). Let \( \tilde{v} \) be an extension of \( v \) to \( \tilde{L} \). Let, if possible, \( V \) and \( W \) be nonsingular projective models of \( K/k \) and \( L'/k \), respectively, such that \( W \) is the normalization of \( V \) in \( L' \). Then by the minimality of \( V^* \), \( V \) must dominate \( V^* \). Consequently by [Ab1, Theorem 3] the local ring of the center \( P \) of \( v \) on \( V \) must equal \( R_n \) for some nonnegative integer \( n \). Since \( R_n \) dominates \( R \) and \( R \) is not ramified in \( \tilde{L} \), \( R_n \) is not ramified in \( \tilde{L} \). Let \( \tilde{V} \) be the normalization of \( V \) in \( \tilde{L} \), and \( \tilde{R}_n \) be the local ring of the center \( \tilde{P} \) of \( \tilde{v} \) on \( \tilde{V} \). Then \( \tilde{P} \) lies above \( P \) in \( \tilde{V} \) and \( \tilde{R}_n \) is a two dimensional regular local ring whose maximal ideal \( M(\tilde{R}_n) \) is generated by \((X,Y)\). Now \( L' \) is a \( Z_q \) extension of \( \tilde{L} \) constructed from \( \tilde{L} \) in the same way as \( L \) is constructed from \( K \) in (3.1), and \( W \) is the normalization of \( \tilde{V} \) in \( L' \). By (3.1), the point of \( W \) lying above \( \tilde{P} \) is not a simple point, which is a contradiction.

**Remark 3.3.** The construction of a \( Z_q \) extension \( L/K \) having no simultaneous resolution does not use the results of Harbater and Pop. Their results plus the fact that a regular system of parameters lifts to a regular system of parameters through an unramified local ring extension allow us to mimic such a construction to get an \( H \otimes \mathbb{Z}_q \) extension. Similar arguments will show that the statement of (3.2) remains true if \( q > 3 \) is a prime number; see [Ab2, Theorem 11] for details.

**References**


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