MINIMAL BOUNDED INDEX SUBGROUP
FOR DEPENDENT THEORIES

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Abstract. For a dependent theory $T$, in $\mathfrak{C}_T$ for every type definable group $G$, the intersection of type definable subgroups with bounded index is a type definable subgroup with bounded index.

§0. Introduction

Assume that $T$ is a dependent (complete first-order) theory, $\mathfrak{C}$ is a $\bar{\kappa}$-saturated model of $T$ (a monster), $G$ is a type definable (in $\mathfrak{C}$) group in $\mathfrak{C}$ (of course we consider only types of cardinality $< \bar{\kappa}$).

A type definable subgroup $H$ of $G$ is call bounded if the index $(G:H)$ is $< \bar{\kappa}$. We prove that there is a minimal bounded definable subgroup. The first theorem on this line for $T$ stable is due to Baldwin and Saxl [BaSx76].

Recently Hrushovski, Peterzil and Pillay [HPP0x] investigated definable groups, o-minimality and measure. In an earlier work on definable subgroups in o-minimal $T$ in Berarducci, Otero, Peterzil and Pillay [BOPP05] the minimal type-definable bounded index theorem and more results are proved for o-minimal theories.

Hrushovski, in a lecture at the Hebrew University, mentioned that he, Peterzil and Pillay had observed the main result of the current paper, but assuming in addition the existence of an invariant measure on the group in question, and Hrushovski asked if the measure assumption could be removed. So we answer it positively. The current version of their paper [HPP0x] includes an exposition of our proof.

Recent works of the author on dependent theories are [Sh:783] (see §3,§4 on groups) [Sh:863] (e.g., the first-order theory of the $p$-adics is strongly\(^1\) dependent but not strongly\(^2\) dependent, see end of §1; on strongly\(^2\) dependent fields, see §5) and [Sh:F705]. This work is continued in [Sh:F753] (getting mainly a parallel result for $G$ abelian and $L_{\infty,\bar{\kappa}}$-definable subgroups).

I thank Aviv Tatarski and Ilay Kaplan for their very careful checking.
1.1 Theorem. Assume $T$ is a dependent (complete first-order) theory, $\mathcal{C}$ a $\bar{\kappa}$-saturated model for $T$. We consider types of cardinality $< \kappa$.

1) If $\oplus$ below holds, then:

\begin{enumerate}
\item[(a)] $q(\mathcal{C})$ is a subgroup of $p(\mathcal{C})$,
\item[(b)] $q(\mathcal{C})$ is of index $< \kappa$,
\item[(c)] $q(x)$ is of cardinality $\leq \lambda := |T|^{|\mathcal{C}|}$ (i.e., for some $q'(x) \subseteq q(x)$ of cardinality $\leq \lambda$, $q(x)$ is equivalent to $p(x) \cup q'(x)$),
\item[(d)] we can strengthen (a), (b) to $q(\mathcal{C})$ is a subgroup of index $\leq 2^\lambda$,
\end{enumerate}

where

\begin{enumerate}
\item[(a)] $p(x)$ is a type such that $p(\mathcal{C})$ is a group which we call $G$ (with some definable operations $xy, x^{-1}$ and the identity $e_G$ which is constant here),
\item[(b)] $q(x) = p(x) \cup \{r(x) : r(x) \in R\}$, where
\item[(c)] $R = \{r(x) : r(x)$ a type such that $(p \cup r)(\mathcal{C})$ is a subgroup of $p(\mathcal{C})$ of index $< \kappa\}.$
\end{enumerate}

2) There exists some $q' \subseteq q$ over $\text{Dom}(p)$, equivalent to $q$ and such that $|q'| \leq |T| + \text{Dom}(p)$. So $(p(\mathcal{C}) \cup q(\mathcal{C})) \leq 2^{\text{Dom}(p)+|T|}$.

3) If $r_i(x) \in R$ for $i < \lambda^+$, then for some $\alpha < (|T|^{|\mathcal{C}|})^+$ we have $(p(x) \cup \bigcup \{r_i(x) : i < \alpha\})(\mathcal{C}) = (p(x) \cup \bigcup \{r_i(x) : i < \lambda^+)\}(\mathcal{C})$.

Proof. 1) Note

\[\oplus_1 R\] is closed under unions of $< \kappa$. \\
\[\oplus_2 (a)\] If $r(x) \in R$, $r'(x) \subseteq r(x)$ is countable, then there is a countable $r''(x) \subseteq r(x)$ including $r'(x)$ which belongs to $R$. \\
\[\oplus_2 (b)\] If $p(x) \subseteq r(x) \in R$ and $r(x)$ is closed under conjunctions and $r'(x) \subseteq r(x)$ is countable, then we can find $\psi_n(x, b_n)$ for $n < \omega$ such that

\begin{enumerate}
\item[(a)] $\psi_n(x, b_n) \in r(x)$,
\item[(b)] $\psi_{n+1}(x, b_{n+1}) \vdash \psi_n(x, b_n)$,
\item[(c)] $\psi_{n+1}(x, b_{n+1}, \psi_{n+1}(y, b_{n+1}) \vdash \psi_n(x^{-1}y, b_n) \wedge \psi_n(x^{-1}, b_n) \wedge \psi_n(xy, b_n)$,
\item[(d)] $\psi_n(x, b_n) \vdash \varphi_n(x, a_n)$, where $\{\varphi_n(x, a_n) : n < \omega\}$ list $r'$,
\item[(e)] $\mathcal{C} \models \psi_n(e_G, b_n)$, actually follows from clause (a).
\end{enumerate}

[Why? Let $r'(x) = \{\varphi_n(x, a_n) : n < \omega\}$ (can use $\varphi_n = (x = x)$). Without loss of generality, $r(x)$ is closed under conjunctions and also $r'(x)$ is. Now we choose $\psi_n(x, b_n)$ by induction on $n < \omega$ such that $\psi_{n+1}(x, b_{n+1}) \wedge \psi_{n+1}(y, b_{n+1}) \vdash \psi_n(xy^{-1}, b_n) \wedge \varphi_n(x, a_n) \wedge \psi_n(xy, b_n)$; notice that trivially $e_G \in \varphi_n(\mathcal{C}, a_n) \cap \psi_n(\mathcal{C}, b_n)$. Such a formula exists as $(p(x) \cup r(x)) \cup (p(y) \cup r(y)) \vdash \psi_n(xy^{-1}, b_n) \wedge \varphi_n(x, a_n) \wedge \psi_n(xy, b_n)$.

Now $r''(x) = \{\varphi_n(x, a_n), \psi_n(x, b_n) : n < \omega\}$ is as required in clause (a), $\langle \psi_n(x, b_n) : n < \omega\rangle$ in (b).]

In the conclusion of Theorem 1.1, Clause (a) is obvious. Assume towards a contradiction that the conclusion $(\beta) + (\gamma)$ fails. So we can choose $(c_\alpha, r_\alpha)$ by induction on $\alpha < \lambda^+$ such that

\[\oplus_3 (a)\] $c_\alpha \in (p(x) \cup \bigcup \{r_\beta : \beta < \alpha\})(\mathcal{C})$,
\[\oplus_3 (b)\] $r_\alpha = \{\psi^*_\alpha(x, b_\alpha^\alpha) : n < \omega\} \subseteq q$ and $b_\alpha^\alpha \subseteq b_{n+1}^\alpha$,
\[\oplus_3 (c)\] $r_\alpha \in R$, and $\psi_{n+1}(x, b_\alpha^\alpha) \vdash \psi_n(x, b_\alpha^\alpha)$,
\[\oplus_3 (d)\] $c_\alpha$ does not realize $r_\alpha$, in fact $\mathcal{C} \models \neg \psi^*_\alpha(c_\alpha, b_\alpha^\alpha)$, hence $c_\alpha \notin q(\mathcal{C})$. 


Without loss of generality, we can assume \( c_8 \). If \( \langle \bar{a} \rangle \) and \( \bar{b} \) and add dummy variables). Now we define \( \psi \).

So by our assumption toward a contradiction, \( q \), \( \vartheta \) of \( \alpha \), \( \bar{y} \), \( n + 1 \), \( \beta > \alpha \), \( \alpha \) and \( \bar{G} \).

\[ \langle \bar{a} \rangle \cup \{ \bar{b} : n \leq \omega \} \}

\] \[ \bar{b}_n \]

For some \( \bar{y} \) and \( \bar{y}_n \) as we can change the order and name the free variable and add dummy variables). Now we define \( \psi_n(x, \bar{y}_n) \) by induction on \( n \) by

\[ \psi_0^\alpha(x, \bar{y}_n) = \psi_0^\alpha(x, \bar{y}_n) \]

\[ \psi_0^{\alpha,*}(x, \bar{y}_n) = \psi_0^\alpha(x, \bar{y}_n) \]

\[ \land (\forall m) \psi_m^\alpha(x, \bar{y}_m) \}

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[Why? As in the proof of \( \diamond_2 \) above, i.e., during the induction in the proof of \( \diamond_3 \), we use \( \diamond_2(\beta) \) and get \( \psi_\beta^\alpha(x, \bar{y}_n), \bar{b}_n \) and without loss of generality \( \bar{b}_n \leq \bar{b}_{n+1}, \bar{y}_n < \bar{y}_{n+1} \) (as we can change the order and name the free variable and add dummy variables). Now we define \( \psi_n(x, \bar{y}_n) \) by induction on n by

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\[ \Rightarrow \psi_\alpha^\alpha(x, \bar{y}_n) \land \psi_\alpha^\alpha(x, \bar{y}_n) \}

\] \[ \bar{b}_n \]

[Why? By the pigeon hole principle.]

\[ \langle c_\alpha \rangle \]

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[Why? By the pigeon hole principle.]
(α) it is a subgroup,
(β) \(c_{2n+1}\) belongs to it by clause (a) of \(\Theta_3\) and
(γ) \(c_{2n}\) does not belong to it by clause (e) of \(\Theta_3\).

Let \(\tilde{a}_n = a_{2n+1}, b_n^{\alpha} = b_n^{2a}\) retaining the same \(\psi\)'s. So we have obtained an example as required in \(\Theta_8\) (not losing the other demands).]

\(\Theta_9\) For some \(n < \omega\) for every \(\alpha\), we have if \(d_1, d_2 \in (p \cup r_{\alpha})(C)\), then \(d_1, c, d_2 \not\in \psi_n(C, b_1^a)\); without loss of generality, \(n = 1\).

[Why? Fix \(\alpha\). If this holds for some \(\psi_n(-, b_\alpha^a)\), by indiscernibility, renaming the \(\varphi_i\)'s, this is O.K. Otherwise for each \(n < \omega\) there are \(d_1^n, d_2^n \in (p \cup r_{\alpha})(C)\) such that \(C \models \psi_n(d_1^n, c, d_2^n, b_\alpha^n)\). By compactness for some \(d_1, d_2 \in (p \cup r_{\alpha})(C)\) we have \(\models \psi_n(d_1, c, d_2, b_\alpha^n)\) for every \(n < \omega\). So \(d_1, d_2\) belongs to the subgroup \((p \cup r_{\alpha})(C)\), but also \(d_1, d_2\) belongs to it; hence \(c_\alpha\) belongs, a contradiction. Alternatively, note that \(n = 2\) is O.K.; let \(c' = d_1c_2d_2\) and assume toward a contradiction that \(c' \in \varphi_2(C, b_2)\) and let \(d_1' = (d_1)^{-1}, d_2' = (d_2)^{-1}\), so clearly \(d_1, d_2 \in (p \cup r_{\alpha})(C) \subseteq \psi_2(C, b_2^a)\). Now by \(\Theta_4\) as \(d_1, c, d_2 \in \psi_2(C, b_\alpha^a)\) it follows that \(d_2(c')^{-1} \in \psi_1(C, b_\alpha^a)\). As \(d_1, d_2(c')^{-1} \in \psi_1(C, b_\alpha^a)\) by \(\Theta_4\) we have \(d_1(d_2(c')^{-1})^{-1} \in \psi_0(C, b_\alpha^a)\). But \(c' = d_1c_2d_2\); hence \(c_\alpha = d_1'(d_2(c')^{-1})^{-1} = d_1'(d_2)^{-1}, d_1'(d_2(c')^{-1})^{-1} \in \psi_0(C, b_\alpha^a)\) by the previous sentence, whereas \(c_\alpha \not\in \psi_0(C, b_\alpha^a)\) by \(\Theta_3(e)\), a contradiction.]

\(\Theta_{10}\) If \(w = \{i_1, \ldots, i_n\}, i_1 < \ldots < i_n < \lambda^+, \) and \(d_w := c_{i_1}c_{i_2}\ldots c_{i_n} \in G\) and \(\alpha < \lambda^+, \) then \(\varphi_1[d_w, b_\alpha^a] \iff \alpha \not\in w\).

[Why? If \(\alpha \in w\), let \(k\) be such that \(\alpha = i_k\), so \(c_{i_1}, \ldots, c_{i_{k-1}} \in (p \cup r_{\alpha})(C)\) by \(\Theta_8\) and similarly \(c_{i_{k+1}} \ldots c_{i_n} \in (p \cup r_{\alpha})(C)\); hence \(d_w = (c_{i_1} \ldots c_{i_k}) \not\in (p \cup r_{\alpha})(C)\) by \(\Theta_9\).

Second, if \(\alpha \not\in w\), this holds by \(\Theta_8\) as \(\{c_{i_\ell} : \ell < n\}\) is included in the subgroup \((p \cup r_{\alpha})(C)\).]

So we get a contradiction to “\(T\) is dependent”; hence clauses (β), (γ) hold. Also clause (δ) follows by the following observation:

**Observation:** If \(r(x) \in R\) and \(|r(x)| \leq \theta\), then \((p(C) : (p \cup r)(C)) \leq 2^\theta\) (except for being just finite when \(\theta\) is finite).

**Proof.** If \(\theta\) is finite, then the proof follows by compactness. If \(\theta\) is infinite, then without loss of generality, \(r\) is closed under conjunctions. Let \(r = \{\varphi_i(x, b) : i < \theta\}\), where \(b\) is possibly infinite.

Let \(u\) be a set of ordinals (\(< \tilde{\kappa}\) such that \(\tilde{\kappa} > |u| > (p(C) : (p \cup r)(C))\). Now for each \(i < \theta\), let \(\Gamma_{i,u} = \{p(x_\alpha) : \alpha \in u\} \cup \{\neg \varphi_i(x_\alpha x_{\beta}^{-1}, b) : \alpha < \beta\text{ from }u\}\). So for some finite \(u^*_\alpha \subseteq u\), \(\Gamma_{i,n}^\alpha\) is contradictory, so \(\Gamma_{i,n}^\alpha\) is contradictory, when \(n_i = |u_i|\).

It suffices to use \((2^\theta)^+ \rightarrow (\ldots n_i \ldots)_{i<\theta}\) [why? let \(\langle c_\alpha : \alpha < (2^\theta)^+\rangle\) exemplify the failure and let \(\zeta_{\alpha,\beta} = \text{Min}\{i : \models \neg \varphi_i(c_\alpha c_{\beta}^{-1}, b)\}\}].

This finishes the proof of part (1). We still need to prove 2), 3).

2) Let \(q'(x) \subseteq q(x)\) have cardinality \(|T|^\theta\) and be such that \(q(C) = (p \cup q')(C); q'(x)\) exists by part (1). Observe that every automorphism of \(C\) fixing \(\text{Dom}(p)\) maps \(p(C)\) onto itself and therefore maps \(q(C)\) onto itself.
It follows that if \( c_1, c_2 \in p(\mathfrak{C}) \) are such that \( \text{tp}(c_1, \text{Dom}(p)) = \text{tp}(c_2, \text{Dom}(p)) \), then \( c_1 \in q(\mathfrak{C}) \) if and only if \( c_2 \in q(\mathfrak{C}) \). Let \( P := \{ \text{tp}(b, \text{Dom}(p)) : b \in q(\mathfrak{C}) \} \), \( P(\mathfrak{C}) := \bigcup \{ r(\mathfrak{C}) : r \in P \} \). Then by the above explanation, \( P(\mathfrak{C}) \subseteq q(\mathfrak{C}) \). By definition, \( q(\mathfrak{C}) \subseteq p(\mathfrak{C}) \), so they are equal. Let \( q_{**} = \bigcap \{ r : r \in P \} \), so we have \( q(\mathfrak{C}) \subseteq q_{**}(\mathfrak{C}) \).

If they are equal, then we are done. Otherwise take \( c_1 \in q_{**}(\mathfrak{C}) \setminus q(\mathfrak{C}) \). Without loss of generality, let \( \psi(x, \bar{d}) \in q \) be such that \( \models \neg \psi(c_1, \bar{d}) \).

By definition of \( P \) and \( c_1 \), for each \( \theta(x, \bar{e}) \in \text{tp}(c_1, \text{Dom}(p)) \) there exists some \( p_{\theta(x,\bar{e})} \in P \) such that \( \theta(x, \bar{e}) \in p(x, \bar{e}) \) and therefore some \( c_{\theta(x,\bar{e})} \in q(\mathfrak{C}) \) satisfies \( \theta(x, \bar{e}) \). So \( \text{tp}(c_1, \text{Dom}(p)) \cup q'(x) \) is finitely satisfiable and is therefore realized by some \( c_2 \). Thus \( \text{tp}(c_1, \text{Dom}(p)) = \text{tp}(c_2, \text{Dom}(p)) \), but \( c_1 \notin q'(\mathfrak{C}) = q(\mathfrak{C}) \) and \( c_2 \in (p \cup q')(\mathfrak{C}) \), a contradiction.

3) By the proof of part (1). \( \square_{1.1} \)

REFERENCES


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