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CONSTRUCTING UNITS IN PRODUCT SYSTEMS

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ABSTRACT. We prove a criterion that allows us to construct units in product systems of correspondences with prescribed infinitesimal characterizations. This criterion summarizes proofs of known results and new applications. It also frees the hypotheses from the assumption that the units are contained in a product system of time ordered Fock modules.

1. INTRODUCTION

An *Arveson system* is a (measurable) family $H^\otimes = \big(H_t\big)_{t\in\mathbb{R}_+}$ of (infinite dimensional when $t > 0$ and separable) Hilbert spaces with a (measurable) associative identification $H_s \otimes H_t = H_{s+t}$. A *unit* is a (measurable nontrivial) section $u^\otimes = \big(u_t\big)_{t\in\mathbb{R}_+}$ that factors according to that identification: $u_s \otimes u_t = u_{s+t}$. Already in his trailblazing paper [Arv89] Arveson considered a sort of *Trotter product* composing two units $u^\otimes, v^\otimes$ to give a new one $w^\otimes$, whose elements are defined as

$$w_t \;:=\; \mathrm{w-}\lim_{n\to\infty}\big(u_{\frac{t}{2n}} \otimes v_{\frac{t}{2n}}\big)^{\otimes n}. \tag{1.1}$$

The new unit is characterized infinitesimally by the values of the so-called *covariance function* defined as $\gamma^{u,v} := \frac{d}{dt}\big|_{t=0}\langle u_t, v_t\rangle$, namely,

$$\gamma^{x,w} \;=\; \tfrac{1}{2}\gamma^{x,u} + \tfrac{1}{2}\gamma^{x,v} \tag{1.2}$$

for every unit $x^\otimes$.

However, when we wish to apply the idea for such types of constructions of new units to product systems of Hilbert modules, then we meet two obstacles. Firstly, the limit in (1.1) is only weak. In Hilbert spaces this is not a big problem, because weak and norm closures of subspaces coincide and proofs of the *Mazur theorem* even show how to transform a weakly convergent sequence into a norm convergent sequence. In Hilbert modules having only convergence of inner products is deadly. Secondly, the best modification of the limit in (1.1) allows us to have convex combinations on the infinitesimal level in (1.2). (For positive $\varkappa, \lambda$ with $\varkappa + \lambda = 1$ consider $w_t = \mathrm{w–}\lim_{n\to\infty}\big(u_{\frac{\varkappa t}{n}} \otimes v_{\frac{\lambda t}{n}}\big)^{\otimes n}$. Then $\gamma^{x,w} = \varkappa\gamma^{x,u} + \lambda\gamma^{x,v}$.) In critical applications like, for instance, the *Trotter product* of units in *spatial* product systems as defined in Skeide [Ske01], convex combination is not enough. We need affine combinations, that is, $\varkappa, \lambda$ complex with $\varkappa + \lambda = 1$.

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In these notes we prove a powerful criterion (split into two propositions) that allows us to summarize all existing constructions of units in a general abstract scheme. In that way we avoid having to repeat the same sort of argument in every individual case just because the formulation is not sufficiently general. The limits will be in norm. In Example 4.1 we shall show that the unit $w^{\otimes}$ in (1.1) can be obtained also as

$$w_t \; := \; \|\cdot\|\!-\!\lim_{n\to\infty}\bigl(\tfrac{1}{2}u_{\frac{t}{n}}+\tfrac{1}{2}v_{\frac{t}{n}}\bigr)^{\otimes n}. \tag{1.3}$$

Further, we do not use the hypothesis (either by explicit assumption as in [Ske01] or by referring to results like in Barreto, Bhat, Liebscher and Skeide [BBLS04], which establish equivalence to deep results like Christensen-Evans [CE79]) that the product system is contained in a subsystem of time ordered Fock modules. This makes, in fact, a couple of results proved in [BBLS04] independent of [CE79].

2. Units and CPD-semigroups

In this section we repeat the definitions and results (mainly from [BBLS04]) we need to formulate and prove the main proposition. Throughout $\mathcal{B}$ denotes a unital C^*–algebra. The concept of product systems and their relation to E_0–semigroups makes sense also for nonunital C^*–algebras (see Muhly, Skeide and Solel [Ske02, MSS06]), but not the concept of units; see the discussions in Skeide [Ske04, Ske05].

A ***correspondence*** over $\mathcal{B}$ is a Hilbert $\mathcal{B}$–module with a nondegenerate (that is, unital in our case) left action of $\mathcal{B}$. A ***product system*** of Hilbert modules (or of correspondences) is a family $E^{\odot}=\bigl(E_t\bigr)_{t\in\mathbb{R}_+}$ of correspondences E_t over a (unital) C^*–algebra $\mathcal{B}$ with an associative identification

$$E_s\odot E_t \;=\; E_{s+t}$$

and $E_0=\mathcal{B}$ (where $\mathcal{B}$ is the ***trivial*** correspondence over $\mathcal{B}$ with the natural bimodule operation and inner product $\langle b,b'\rangle=b^*b'$) and $E_0\odot E_t=E_t=E_t\odot E_0$ via $b\odot x_t=bx_t$, $x_t\odot b=x_tb$. We do not pose any continuity or measurability condition. (Those will be encoded into continuity properties of the units. For a definition of *continuous* product system see Skeide [Ske03].)

A ***unit*** in a product system $E^{\odot}$ is a family $\xi^{\odot}=\bigl(\xi_t\bigr)_{t\in\mathbb{R}_+}$ of elements $\xi_t\in E_t$ that factors as

$$\xi_s\odot\xi_t \;=\; \xi_{s+t}$$

and $\xi_0=\mathbf{1}\in\mathcal{B}=E_0$. The ***trivial*** unit is 0 for all $t>0$. For every $t>0$ we define $\mathbb{J}_t=\{(t_n,\ldots,t_1)\colon |\mathfrak{t}|=t\ (n\in\mathbb{N},t_i>0)\}$, where the ***length*** of a tuple $\mathfrak{t}=(t_n,\ldots,t_1)$ is $|\mathfrak{t}|:=t_n+\ldots+t_1$, while the ***norm*** of $\mathfrak{t}$ is $\|\mathfrak{t}\|:=\max(t_n,\ldots,t_1)$. We note that $\mathbb{J}_t$ becomes a lattice when we consider $\mathfrak{t}$ as the interval partition $\bigl(\sum_{i=1}^n t_i,\ldots,\sum_{i=1}^1 t_i\bigr)$; see [BS00, Proposition 4.1]. Clearly, for an arbitrary set S of units the spaces

$$E_t^S \;:=\; \overline{\mathsf{span}}\bigl\{b_n\xi^n_{t_n}\odot\ldots\odot b_1\xi^1_{t_1}b_0\colon n\in\mathbb{N},\mathfrak{t}\in\mathbb{J}_t,\xi^{i\odot}\in S,b_i\in\mathcal{B}\bigr\}$$

and $E_0^S=\mathcal{B}$ form a product subsystem of $E^{\odot}$, the subsystem ***generated*** by S.

The covariance function of an Arveson system, actually, is the generator of a semigroup (under pointwise multiplication) of $\mathbb{C}$–valued *positive definite kernels* on the set of units and, therefore, a *conditionally positive definite kernel.* For product systems of Hilbert modules the concept of $\mathcal{B}$–valued positive definite kernels makes sense but is no longer sufficient if we wish to speak about semigroups. Our kernels assume values in $\mathcal{B}(\mathcal{B})$, the bounded linear mappings on $\mathcal{B}$. A kernel $\mathfrak{K}\colon S \times S \to \mathcal{B}(\mathcal{B})$ on a set S is ***completely positive definite***, if

$$\sum_{i,j=1}^{n} b_i^* \mathfrak{K}^{\sigma_i,\sigma_j}(a_i^* a_j) b_j \;\geq\; 0 \text{ for all } n \in \mathbb{N}; a_i, b_i \in \mathcal{B}, \sigma_i \in S. \tag{2.1}$$

$\mathfrak{K}$ is ***conditionally completely positive definite***, if (2.1) holds at least if the condition $\sum_{i=1}^{n} a_i b_i = 0$ is satisfied. Considering a number $z \in \mathbb{C}$ as mapping $w \mapsto zw$ in $\mathcal{B}(\mathbb{C})$, the notions of positive definite and conditionally positive definite $\mathbb{C}$–valued kernels are included. A family $\mathfrak{K} = \big(\mathfrak{K}_t\big)_{t \in \mathbb{R}_+}$ of completely positive definite kernels on S is a ***CPD-semigroup*** if for all $\sigma, \sigma' \in S$ the mappings $\mathfrak{K}_t^{\sigma,\sigma'}$ form a semigroup (with identity) in $\mathcal{B}(\mathcal{B})$. The CPD-semigroup $\mathfrak{K}$ is ***uniformly continuous***, if every semigroup $\mathfrak{K}^{\sigma,\sigma'}$ is uniformly continuous. By [BBLS04, Theorem 3.4.7] the formula $\mathfrak{K}_t = e^{t\mathfrak{L}}$ establishes a one-to-one correspondence between uniformly continuous CPD-semigroups $\mathfrak{K}$ and conditionally completely positive definite kernels $\mathfrak{L}$. This unifies and generalizes the well-known results on the generators of semigroups of positive definite kernels and of CP-semigroups.

Every subset S of units in a product system $E^{\odot}$ gives rise to a CPD-semigroup $\mathfrak{U}$ on S defined by setting

$$\mathfrak{U}_t^{\xi,\xi'} \;=\; \langle \xi_t, \bullet \xi'_t \rangle. \tag{2.2}$$

We say the set S of units is ***continuous***, if $\mathfrak{U}$ is uniformly continuous. We denote the values of the generator $\mathfrak{L}$ of $\mathfrak{U}$ by $\mathfrak{L}^{\xi,\xi'}$. A product system is ***type I***, if it is generated by a continuous subset of units. Recall that type I product systems need not be *spatial* in the sense of [Ske01] (generalizing the definition of Powers [Pow88]). In fact, it is a major achievement of Section 3 that the results hold for general type I systems. Only type I product systems of von Neumann modules are spatial automatically; see [BBLS04].

Finally, by [BBLS04, Theorem 4.3.5] every (uniformly continuous) CPD-semigroup $\mathfrak{K}$ arises as the CPD-semigroup associated by (2.2) with a (continuous) set of units in a product system. (This is a generalization of Bhat and Skeide [BS00, Theorem 4.8] for CP-semigroups.) The unique minimal version among these product systems we call the ***GNS-system*** of $\mathfrak{K}$. It is the proof of this result where unitality of $\mathcal{B}$ is required.

3. The criterion

Let $E^{\odot}$ be a product system and suppose that $t \mapsto y_t \in E_t$ is differentiable at $t = 0$ in a suitable sense. It is our goal to find a criterion to check when $y_{t_n} \odot \ldots \odot y_{t_1}$ converges over the net $\mathbb{J}_t$ to ζ_t (necessarily forming a unit). In a first step (Proposition 3.1) we show (by rather algebraic means like GNS-construction) that there exists a product system, possibly bigger than the original one, that contains a unit $\zeta^{\odot}$ which fits the infinitesimal characterization of y_t. Then (Proposition 3.4) we show that convergence is equivalent to that the new product system matches the old one, and give an applicable criterion.

3.1. **Proposition.** *Let $E^{\odot}$ be a product system that is generated by a continuous subset S of units. Suppose there is a function $t \mapsto y_t \in E_t$ and there are mappings*

K, K_ξ ($\xi^\odot \in S$) in $\mathcal{B}(\mathcal{B})$ such that

$$\langle y_t, \bullet y_t \rangle = \mathrm{id}_{\mathcal{B}} + tK + O(t^2), \tag{3.1}$$

$$\langle y_t, \bullet \xi_t \rangle = \mathrm{id}_{\mathcal{B}} + tK_\xi + O(t^2). \tag{3.2}$$

Then there exist a product system $F^\odot$ that contains $E^\odot$ and a unit $\zeta^\odot$ such that $S \cup \{\zeta^\odot\}$ is continuous and

$$\mathfrak{L}^{\zeta,\zeta} = K \text{ and } \mathfrak{L}^{\zeta,\xi} = K_\xi. \tag{3.3}$$

Proof. We are done, if we show that the kernel $\mathfrak{L}$ on S extended to $S \cup \{\zeta^\odot\}$ in the way stated in the proposition (and $\mathfrak{L}^{\xi,\zeta} = * \circ K_\xi \circ *$) is still conditionally completely positive definite. In this case, $e^{t\mathfrak{L}}$ is a uniformly continuous CPD-semigroup on $S \cup \{\zeta^\odot\}$ and for the product system $F^\odot$ we may choose the GNS-system of that CPD-semigroup.

We define the family of completely positive definite kernels $\mathfrak{K}_t$ on $S \cup \{\zeta^\odot\}$ (think of $S \cup \{\zeta^\odot\}$ as disjoint union), by setting

$$\begin{aligned}
\mathfrak{K}_t^{\xi,\xi'} &= \langle \xi_t, \bullet \xi'_t \rangle, \\
\mathfrak{K}_t^{\zeta,\xi'} &= \langle y_t, \bullet \xi'_t \rangle, \\
\mathfrak{K}_t^{\xi,\zeta} &= \langle \xi_t, \bullet y_t \rangle, \\
\mathfrak{K}_t^{\zeta,\zeta} &= \langle y_t, \bullet y_t \rangle,
\end{aligned}$$

for $\xi^\odot, \xi'^\odot \in S$. These kernels are CPD but need not form a semigroup. Nevertheless, as for CPD-semigroups one easily shows that the kernel $\mathfrak{L}^{\xi,\xi'} = \lim_{t\to 0} \frac{\mathfrak{K}_t^{\xi,\xi'} - \mathrm{id}_{\mathcal{B}}}{t}$, as a limit of conditionally completely positive definite kernels, is a conditionally completely positive definite kernel, too. □

3.2. *Remark.* Of course, $F^\odot$ is unique, if we require that $S \cup \{\zeta^\odot\}$ is generating.

In Proposition 3.4 we will show that the elements of the unit $\zeta^\odot$ can be obtained as a norm limit of

$$y_{\mathfrak{t}} := y_{t_n} \odot \ldots \odot y_{t_1} \tag{3.4}$$

over the net $\mathbb{J}_t$, if and only if $\zeta^\odot \in E^\odot$, and we will provide an easily applicable necessary and sufficient criterion. First we draw some consequences from Proposition 3.1.

3.3. **Proposition.** *Under the hypotheses of Proposition* 3.1*:*

(1) $\lim_{\mathfrak{t}\in\mathbb{J}_t} \langle y_{\mathfrak{t}}, \bullet y_{\mathfrak{t}} \rangle = \langle \zeta_t, \bullet \zeta_t \rangle$.

(2) $\lim_{\mathfrak{t}\in\mathbb{J}_t} \langle \xi_t, \bullet y_{\mathfrak{t}} \rangle = \langle \xi_t, \bullet \zeta_t \rangle$ *for all* $\xi^\odot \in S$

All limits are uniformly in $t \in [0,T]$ for every $T \in \mathbb{R}_+$ as $\|\mathfrak{t}\| \to 0$.

Proof. By assumption there is a constant $M > 0$ such that $\|\langle y_s, \bullet y_s \rangle - \mathrm{id}_{\mathcal{B}} - sK\| \leq \frac{1}{2}s^2M^2$ for all sufficiently small s. Therefore, $\|\langle y_s, \bullet y_s \rangle\| \leq 1 + s\|K\| + \frac{1}{2}s^2M^2 \leq e^{s\max(\|K\|,M)}$, and further

$$\begin{aligned}
\big\|\langle y_{\mathfrak{t}}, \bullet y_{\mathfrak{t}} \rangle\big\| &= \|\langle y_{t_1}, \bullet y_{t_1} \rangle \circ \ldots \circ \langle y_{t_n}, \bullet y_{t_n} \rangle\| \\
&\leq e^{t_1\max(\|K\|,M)} \ldots e^{t_n\max(\|K\|,M)} = e^{t\max(\|K\|,M)}
\end{aligned}$$

for all t and all $\mathfrak{t}$ with $\|\mathfrak{t}\|$ sufficiently small. In other words, the net $\big(y_{\mathfrak{t}}\big)_{\mathfrak{t}\in\mathbb{J}_t}$ is eventually bounded.

Let us write $Y_s := \langle y_s, \bullet y_s\rangle$ and $Z_s := \langle \zeta_s, \bullet\zeta_s\rangle$. We compute

$$\begin{aligned}\langle y_{\mathfrak{t}}, \bullet y_{\mathfrak{t}}\rangle - \langle \zeta_t, \bullet\zeta_t\rangle \;&=\; Y_{t_1}\circ\ldots\circ Y_{t_n} - Z_{t_1}\circ\ldots\circ Z_{t_n}\\ &=\; \sum_{k=1}^{n} Y_{t_1}\circ\ldots\circ Y_{t_{k-1}}\circ(Y_{t_k}-Z_{t_k})\circ Z_{t_{k+1}}\circ\ldots\circ Z_{t_n}.\end{aligned}$$

We have

$$\begin{aligned}\|Y_{t_1}\circ\ldots\circ Y_{t_{k-1}}\| \;&\le\; e^{(t_1+\ldots+t_{k-1})\max(\|K\|,M)},\\ \|Z_{t_{k+1}}\circ\ldots\circ Z_{t_n}\| \;&\le\; e^{(t_{k+1}+\ldots+t_n)\|K\|},\\ \|Y_{t_k}-Z_{t_k}\| \;&\le\; \tfrac{1}{2}t_k^2(M^2+\|K\|^2 e^{t_k\|K\|}) \;\le\; \tfrac{1}{2}t_k^2(M^2+\|K\|^2)e^{t_k\|K\|}.\end{aligned}$$

Altogether, there is a constant $M' = \frac{1}{2}(M^2+\|K\|^2)$ such that

$$\begin{aligned}\|\langle y_{\mathfrak{t}}, \bullet y_{\mathfrak{t}}\rangle - \langle \zeta_t, \bullet\zeta_t\rangle\| \;&\le e^{t\max(\|K\|,M)}M'\sum_{k=1}^{n} t_k^2\\ &\le\; \|\mathfrak{t}\|\, e^{t\max(\|K\|,M)}M'\sum_{k=1}^{n} t_k \;=\; \|\mathfrak{t}\|\, t e^{t\max(\|K\|,M)}M' \qquad (3.5)\end{aligned}$$

for all $t\in[0,T]$. This shows (1).

Similarly, we compute

$$\begin{aligned}&\langle y_{\mathfrak{t}}, \bullet\xi_t\rangle - \langle \zeta_t, \bullet\xi_t\rangle\\ &=\sum_{k=1}^{n}\langle y_{t_1}, \bullet\xi_{t_1}\rangle\circ\ldots\circ\langle y_{t_{k-1}}, \bullet\xi_{t_{k-1}}\rangle\circ\langle y_{t_k}-\zeta_{t_k}, \bullet\xi_{t_k}\rangle\circ\langle \zeta_{t_{k+1}}, \bullet\xi_{t_{k+1}}\rangle\circ\ldots\circ\langle \zeta_{t_n}, \bullet\xi_{t_n}\rangle.\end{aligned}$$

By hypothesis for every $\xi^\odot\in S$ there is a constant $M_\xi>0$ such that $\|\langle y_s-\zeta_s, \bullet\xi_s\rangle\| \le \frac{1}{2}s^2M_\xi^2$ for all sufficiently small s. And, of course, $\|\langle y_{\mathfrak{t}}, \bullet\xi_t\rangle\| \le \|y_{\mathfrak{t}}\|\,\|\xi_t\|$. By an estimate very similar to (3.5) we show also (2). □

3.4. **Proposition.** *Let $F^\odot$ and $\zeta^\odot$ be the product system and the unit, respectively, constructed as in Proposition 3.1 from a product system $E^\odot$ and a function $t\mapsto y_t$ fulfilling the hypotheses of Proposition 3.1. Then the following conditions are equivalent:*

(1) $\lim_{\mathfrak{t}\in\mathbb{J}_t} y_{t_n}\odot\ldots\odot y_{t_1} = \zeta_t$ *for all* $t\in\mathbb{R}_+$.
(2) $\zeta^\odot\in E^\odot\subset F^\odot$, *that is, if $F^\odot$ is minimal, then $F^\odot = E^\odot$.*
(3) $\lim_{\mathfrak{t}\in\mathbb{J}_t}\langle \zeta_t, y_{t_n}\odot\ldots\odot y_{t_1}\rangle = \langle\zeta_t,\zeta_t\rangle$ *for all* $t\in\mathbb{R}_+$.

Proof. E_t is complete. So, if ζ_t is the norm limit of elements in E_t, then $\zeta_t\in E_t$. Therefore, (1) implies (2).

As elements of the form $b_n\xi^n_{t_n}\odot\ldots\odot b_1\xi^1_{t_1}b_0$ are total in E_t and the $y_{\mathfrak{t}}$ are bounded for small $\|\mathfrak{t}\|$, by Proposition 3.3(2) $\lim_{\mathfrak{t}\in\mathbb{J}_t}\langle y_{\mathfrak{t}}, \bullet x\rangle = \langle\zeta_t, \bullet x\rangle$ for all $x\in E_t$. Now suppose that $\zeta^\odot\in E^\odot$, that is, $\zeta_t\in E_t$ for all t. Therefore, (2) implies (3).

By Proposition 3.3(1) we have $\lim_{\mathfrak{t}\in\mathbb{J}_t}\langle y_{\mathfrak{t}}, \bullet y_{\mathfrak{t}}\rangle = \langle\zeta_t, \bullet\zeta_t\rangle$. Therefore, if (3) holds, then we find

$$\langle y_{\mathfrak{t}}-\zeta_t, y_{\mathfrak{t}}-\zeta_t\rangle \;=\; \langle y_{\mathfrak{t}}, y_{\mathfrak{t}}\rangle - \langle y_{\mathfrak{t}}, \zeta_t\rangle - \langle\zeta_t, y_{\mathfrak{t}}\rangle + \langle\zeta_t,\zeta_t\rangle \;\longrightarrow\; 0. \qquad (3.6)$$

Therefore, (3) implies (1). □

3.5. *Observation.* If in (3) convergence is $O(t^2)$, then all estimates in (3.6) are in $\|\mathfrak{t}\|$, uniformly in $t \in [0,T]$ for all $T \in \mathbb{R}_+$. Therefore, we may pass to sequences $\big(\mathfrak{t}^m\big)_{m\in\mathbb{N}}$ in $\mathbb{J}_t$ with $\|\mathfrak{t}^m\| \to 0$ in this case. The most common example is $\mathfrak{t}^m = (t^m_m, \ldots, t^m_1)$ with $t^m_k = \frac{t}{m}$.

3.6. *Remark.* The implication (3) $\Rightarrow$ (1) in Proposition 3.4 corresponds to the well-known method to conclude in Hilbert spaces from weak convergence and convergence of norms to convergence in norm. But, prior to its application we have to use Proposition 3.1 to construct a space which is sufficiently big to contain a candidate for the limit. Of course, we would like to give a one-step criterion allowing us to check immediately norm convergence of $y_{\mathfrak{t}}$ just by looking at inner products of the y_t with elements in E_t. The failure to be able to do so underlines once more the importance of the possibility to examine properties of a CPD-semigroup in terms of its GNS-systems. Also in [BBLS04, Theorem 4.4.12] we proved an intrinsic result about CPD-semigroups by passing to the GNS-system of the CPD-semigroup.

We close with a sufficient criterion for that the construction of the bigger product system $F^\odot$ can be avoided. This criterion applies, for instance, to Example 4.4.

3.7. **Corollary.** *Under the hypotheses of Proposition* 3.1*: If in S there is already a unit $\zeta^\odot$ such that* (3.3) *holds for all $\xi^\odot \in S$, then* $\lim_{\mathfrak{t}\in\mathbb{J}_t} y_{t_n} \odot \ldots \odot y_{t_1} = \zeta_t$ *for all $t \in \mathbb{R}_+$.*

Proof. This is a simple consequence of Remark 3.2, by which we conclude that $F^\odot = E^\odot$ for the minimal $F^\odot$, and Proposition 3.4(2). □

4. Applications

4.1. **A counterexample.** We consider Arveson's Trotter product given by the limit in (1.1). Specifically, as the product system we consider Fock spaces $H_t = \Gamma\big(L^2([0,t])\big)$, we choose $u^\otimes$ to be the vacuum unit $u_t = \omega_t$, and for $v^\otimes$ we choose the exponential vectors $v_t = \psi(\mathbb{I}_{[0,t]})$ to the indicator function of the interval $[0,t]$. By Parthasarathy and Sunder [PS98] or Skeide [Ske00], these two units generate the whole product system.

As indicated in Observation 3.5, to show that the conditions in Proposition 3.4 are not valid, it is sufficient to consider sequences. For the section y we read from (1.1) that $y_t = u_{\frac{t}{2}} \otimes v_{\frac{t}{2}}$ and it is clear that the assumptions from Proposition 3.1 are fulfilled. We easily compute

$$\lim_{n\to\infty} \langle y^{\otimes n}_{\frac{t}{n}}, y^{\otimes n}_{\frac{t}{n}} \rangle \;=\; e^{\frac{t}{2}} \text{ while } \lim_{n\to\infty} \langle y^{\otimes n}_{\frac{t}{n}}, e^{t\alpha}\psi(c\mathbb{I}_{[0,t]}) \rangle \;=\; e^{t(\alpha+\frac{c}{2})}$$

for every other unit $\big(e^{t\alpha}\psi(c\mathbb{I}_{[0,t]})\big)_{t\in\mathbb{R}_+}$. We easily check that the product system $H^\otimes$ contains the unit $w^\otimes = \big(\psi(\frac{1}{2}\mathbb{I}_{[0,t]})\big)_{t\in\mathbb{R}_+}$ such that

$$\langle w_t, e^{t\alpha}\psi(c\mathbb{I}_{[0,t]}) \rangle \;=\; e^{t(\alpha+\frac{c}{2})}$$

to which, therefore, the limit in (1.1) converges weakly. However, $\langle w_t, w_t \rangle = e^{\frac{t}{4}}$ is strictly smaller than the limit $e^{\frac{t}{2}}$ of the norm square of $y^{\otimes n}_{\frac{t}{n}}$, so that the limit is not a norm limit. In fact, it is easy to check that the (minimal) product system from Proposition 3.1 is $F^\otimes$ with $F_t = \Gamma\big(L^2([0,t], \mathbb{C}\oplus\mathbb{C})\big)$ that contains H_t as subsystem

$\Gamma\big(L^2([0,t],\mathbb{C}\oplus 0)\big)$ while the unit $\zeta^\otimes$ is given by $\zeta_t = \psi\big((\frac{1}{2}\oplus\frac{1}{2})\mathbb{I}_{[0,t]}\big)$. Clearly, the criterion Proposition 3.4(3) is violated.

4.2. **Examples from** [BBLS04] **and** [Ske01] **(now without embedding into Fock modules).** We discuss two constructions of units that have been proved in [BBLS04] explicitly assuming units in a time ordered product system and in [Ske01] by first constructing an embedding into a time ordered product system, and then using arguments as in the proof of Corollary 3.7. Here we give a proof based on Propositions 3.1 and 3.4 without any reference to a time ordered product system. (In fact, it is Proposition 3.1 that gives the construction of a type I product system that contains a suitable unit by rather algebraic means and it is Proposition 3.4 that helps to find out whether this unit is contained in the original system.)

For the first construction we study immediately a multi-summand version, instead of the two summand versions considered inside a time ordered product system in [BBLS04, Ske01]. Suppose that $\xi^{\ell\odot}$ $(\ell = 1,\ldots,k)$ are units in a continuous generating subset S of units of a product system $E^\odot$. Let $\varkappa_\ell$ be complex numbers such that $\varkappa_1+\ldots+\varkappa_k = 1$. Put $y_t = \varkappa_1\xi^1_t+\ldots+\varkappa_k\xi^k_t$. Then the section y fulfills the assumptions of Proposition 3.1 with

$$K \;=\; \sum_{\ell,\ell'=1}^k \bar{\varkappa}_\ell\varkappa_{\ell'}\mathfrak{L}^{\xi^\ell,\xi^{\ell'}} \text{ and } K_\xi \;=\; \sum_{\ell=1}^k \bar{\varkappa}_\ell\mathfrak{L}^{\xi^\ell,\xi}.$$

Therefore,

$$\langle\zeta_t, y_t\rangle \;=\; \sum_{\ell'=1}^k \varkappa_{\ell'}\langle\zeta_t,\xi^{\ell'}_t\rangle \;=\; \langle\zeta_t,\zeta_t\rangle + O(t^2).$$

From this it follows as in (3.5) that the criterion in Proposition 3.4(3) is fulfilled.

The other construction allows us to "normalize" a given (continuous) unit $\xi^\odot$ suitably within the product subsystem generated by $\xi^\odot$. Let $\beta\in\mathcal{B}$ and put $y_t = \xi_t e^{t\beta}$. It follows that y fulfills the assumptions of Proposition 3.1 with

$$K(b) \;=\; \mathfrak{L}^{\xi,\xi}(b)+\beta^*b+b\beta \text{ and } K_{\xi'}(b) \;=\; \mathfrak{L}^{\xi,\xi'}(b)+\beta^*b.$$

Also here one checks easily that Proposition 3.4(3) holds. Choosing $\beta = \frac{\mathfrak{L}^{\xi,\xi}(\mathbf{1})}{2}+ih$ ($h\in\mathcal{B}$ selfadjoint but otherwise arbitrary), the unit $\zeta^\odot$ we obtain in that way determines a unital CP-semigroup $\langle\zeta_t,\bullet\zeta_t\rangle$ that has a generator with the same CP-part as $\mathfrak{L}^{\xi,\xi}$. Obviously we obtain the same unit $\zeta^\odot$, if we start with $y_t = e^{t\beta}\xi_t$.

We discuss a further construction, so far not yet considered elsewhere. Suppose that $\xi^{\ell\odot}$ $(\ell = 0,\ldots,k)$ form a continuous set of units and choose $a_\ell, b_\ell\in\mathcal{B}$ $(\ell = 1,\ldots,k)$ such that $\sum_{\ell=1}^k a_\ell b_\ell = 0$. We put

$$y_t \;=\; \xi^0_t + \sum_{\ell=1}^k a_\ell\xi^\ell_t b_\ell.$$

Then y_t converges in norm to the elements ζ_t of a unit $\zeta^\odot$ that fulfills

$$\begin{aligned}\mathfrak{L}^{\zeta,\xi}(b) &= \mathfrak{L}^{\xi^0,\xi}(b) + \sum_{\ell=1}^{k} b_\ell^* \mathfrak{L}^{\xi^\ell,\xi}(a_\ell^* b),\\ \mathfrak{L}^{\zeta,\zeta}(b) &= \mathfrak{L}^{\xi^0,\xi^0}(b) + (\mathfrak{L}^{\xi^0,\zeta}(b) - \mathfrak{L}^{\xi^0,\xi^0}(b)) + (\mathfrak{L}^{\zeta,\xi^0}(b) - \mathfrak{L}^{\xi^0,\xi^0}(b))\\ &\qquad + \sum_{\ell,\ell'=1}^{k} b_\ell^* \mathfrak{L}^{\xi^\ell,\xi^{\ell'}}(a_\ell^* b a_{\ell'}) b_{\ell'}.\end{aligned}$$

This allows us to modify the conditionally positive definite kernel $\mathfrak{L}$ rather arbitrarily by a completely positive definite kernel associated canonically with every uniformly continuous CPD-semigroup. We refer to [BBLS04, Section 5.3], in particular Theorem 5.3.2, for details.

4.3. *Remark.* In all applications of Example 4.2 for checking the condition in Proposition 3.4(3) it is crucial that y_t is expressed very simply in terms of units as a finite $\mathcal{B}$–linear combination (cf. also Remark 3.6). In Example 4.1 we have encountered problems already in the case where $y_t = u_{\frac{t}{2}} \otimes v_{\frac{t}{2}}$ is the tensor product of two pieces from different units at smaller times, while according to Example 4.2 for the finite linear combination $y_t = \frac{1}{2}u_{\frac{t}{n}} + \frac{1}{2}v_{\frac{t}{n}}$ the limit in (1.3) is in norm. We see that while in the case of the two uniformly continuous semigroups $T_t^i = \langle \xi_t^i, \bullet \xi_t^i \rangle$ $(i = 1, 2)$ the Trotter limit over $(T^1_{\frac{t}{2n}} T^2_{\frac{t}{2n}})^n$ is as good as the limit over $(\frac{1}{2}T^1_{\frac{t}{n}} + \frac{1}{2}T^2_{\frac{t}{n}})^n$, on the product system side, which corresponds somehow to a simultaneous "square root" of the semigroups, linear combinations $(\frac{1}{2}\xi^1_{\frac{t}{n}} + \frac{1}{2}\xi^2_{\frac{t}{n}})^{\odot n}$ are much better than tensor products $(\xi^1_{\frac{t}{2n}} \odot \xi^2_{\frac{t}{2n}})^{\odot n}$.

4.4. **Quantum Lévy processes.** Every quantum Lévy process (see Schürmann [Sch93]) possesses a realization on a symmetric (or time ordered) Fock space as the solution of a quantum stochastic differential equation. Only recently Franz, Schürmann and Skeide (in preparation) have shown that the vacuum vector of the Fock space is cyclic for the minimal version of the process. Here is not the place to repeat all the somewhat heavy definitions. We refer the reader to the lecture notes of Franz [Fra06] where a sketch of a proof can also be found. The proof uses Corollary 3.7 to construct all exponential units by applying the process to the vacuum.

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