A NOTE ON EQUILIBRIUM POINTS OF GREEN’S FUNCTION

ALEXANDER YU. SOLYNIN

(Communicated by Juha M. Heinonen)

Abstract. We answer a question raised by Ahmet Sebbar and Thérèse Falliero (2007) by showing that for every finitely connected planar domain $\Omega$ there exists a compact subset $K \subset \Omega$, independent of $w$, containing all critical points of Green’s function $G(z,w)$ of $\Omega$ with pole at $w \in \Omega$.

Let $\Omega$ be a domain on $\mathbb{C}$ bordered by $n \geq 2$ Jordan analytic curves and let $G(z,w)$ be Green’s function of $\Omega$ with pole at $w \in \Omega$. For $w \in \Omega$, let $Z_\Omega(w) = \{z \in \Omega : \frac{\partial}{\partial z} G(z,w) = 0\}$ be the set of critical (= equilibrium) points of $G(z,w)$. It is well known that for every $w \in \Omega$, $\#(Z_\Omega(w)) = n - 1$ counting multiplicity.

Theorem 1. Let $\overline{Z_\Omega}$ denote the closure of the set $Z_\Omega = \bigcup_{w \in \Omega} Z_\Omega(w)$. Then $\overline{Z_\Omega}$ is a compact subset of $\Omega$ having at most $n - 1$ connected components.

Thus this theorem answers affirmatively a question raised by Ahmet Sebbar and Thérèse Falliero in [2, p. 314]. For the case when $\Omega$ is doubly-connected, this theorem was proved in [2] by direct computation involving an explicit expression of Green’s function of a circular annulus.

To prove Theorem 1, we use the well-known Schiffer’s formula linking Green’s function with the Bergman kernel. The Bergman kernel and the adjoint Bergman kernel of $\Omega$ are defined as $K(z,w) = -\frac{2}{\pi} \frac{\partial^2}{\partial z \partial \bar{w}} G(z,w)$ and $L(z,w) = -\frac{2}{\pi} \frac{\partial^2}{\partial z \partial w} G(z,w)$, respectively. Necessary properties of the Bergman kernel can be found, for example, in [1]. It is convenient to use subscripts to denote differentiation so that $G_z(z,w) = \frac{\partial}{\partial z} G(z,w)$, $G_{\bar{w}}(z,w) = \frac{\partial^2}{\partial z \partial \bar{w}} G(z,w)$, etc. To emphasize dependence on $\Omega$ if necessary, we will write $G_\Omega(z,w)$, $K_\Omega(z,w)$, etc.

Lemma 1. Let $\Omega$ be a Dirichlet domain in the upper half-plane $\mathbb{H} = \{z : \Im z > 0\}$ having an open interval $I = (-1,1)$ on $\partial \Omega$ and such that $\hat{\Omega} = \Omega \cup \Omega^* \cup I$ is a domain. Here $\Omega^* = \{z : \bar{z} \in \Omega\}$. Then

\begin{align*}
(1) & \quad K_\Omega(z,w) = K_{\hat{\Omega}}(z,w) - L_{\hat{\Omega}}(z,\bar{w}), \quad z \in \Omega, \quad w \in \Omega \cup I, \quad z \neq w, \\
(2) & \quad \lim_{\mathbb{H} \ni w \to 0} \frac{\partial}{\partial z} \frac{\partial}{\partial \bar{w}} G_\Omega(z,w) = \pi i K_\Omega(z,0)
\end{align*}

uniformly on compact subsets of $\Omega \cup I \setminus \{0\}$.

Received by the editors December 18, 2006.

2000 Mathematics Subject Classification. Primary 30C40.

Key words and phrases. Green’s function, equilibrium point, Bergman function.

This research was supported in part by NSF grant DMS-0525339.

©2007 American Mathematical Society

Reverts to public domain 28 years from publication.

1019
Proof. Since
\[ G_{\Omega}(z, w) = G_{\Omega}(z, w) - G_{\bar{\Omega}}(z, \bar{w}) \]
for all \( z, w \in \Omega \) such that \( z \neq w \), (1) follows from (3) after differentiation.

Let \( \hat{G}(z, w) = G_{\Omega}(z, w) \). Since \( \hat{G}(z, w) \) is harmonic in each variable if \( z \neq w \), the function \( \hat{G}(z, w) \) has the following Taylor expansion at \( w = 0 \) if \( z \neq 0 \):
\[ \hat{G}_z(z, w) = G_z(z, 0) + G_{zw}(z, 0)w + G_{z\bar{w}}(z, 0)\bar{w} + \text{higher powers of } w \text{ and } \bar{w}. \]
Using (3), (4), and (1), we find that
\[ \lim_{w \to 0} \frac{\partial}{\partial w} G_{\Omega}(z, w) = \lim_{w \to 0} \frac{\hat{G}(z, w) - \hat{G}(z, \bar{w})}{3w} = -2i(\hat{G}_{z\bar{w}}(z, 0) - \hat{G}_{z\bar{w}}(z, 0)) = \pi i K(z, 0) - L(z, 0) = \pi i K(z, 0) \]
and the limit is uniform on compact subsets of \( \hat{\Omega} \setminus \{0\} \). \( \square \)

Proof of Theorem 1. Using Hurwitz’s theorem one can easily prove that the set \( Z_{\Omega}(w) \) depends continuously on \( w \in \Omega \). Since \( \Omega \) is open and connected and for every \( w \in \Omega \), \( \#(Z_{\Omega}(w)) = n - 1 \) counting multiplicity the latter implies that \( Z_{\Omega} \) has at most \( n - 1 \) connected components.
The proof of \( Z_{\Omega} \subset \Omega \) is by contradiction. Suppose there is a sequence \( w_k \in \Omega \) with \( w_k \to w_0 \) as \( k \to \infty \) such that there exists a sequence \( z_k \in \Omega \) with \( z_k \to z_0 \in \partial \Omega \) such that \( G_z(z_k, w_k) = 0 \). Now we consider two cases.
(1) If \( w_0 \in \Omega \), then let \( z_0^1, \ldots, z_0^{n-1} \) be zeros of \( G_z(z, w_0) \) counting multiplicity. Let \( \varepsilon > 0 \) be sufficiently small. By Hurwitz’s theorem, there is a positive integer \( N \) such that for all \( k \geq N \) the set \( \bigcup_{j=1}^{n-1} \{ z : |z - z_0^j| < \varepsilon \} \) contains exactly \( n - 1 \) zeros of \( G_z(z, w_k) \) counting multiplicity. Since \( \#(Z_{\Omega}(w_k)) = n - 1 \) and \( z_0^j \in \Omega \) for all \( j = 1, \ldots, n - 1 \), the latter contradicts the assumption that \( G_z(z, w_k) \) has a zero, say \( z_{1_k}^j \), such that \( z_{1_k}^j \to z_0 \in \partial \Omega \) as \( k \to \infty \).

(2) Suppose now that \( w_0 \in \partial \Omega \). Since Green’s function and the Bergman kernel are conformally invariant, we may use Koebe’s theorem on the conformal mapping onto a circular domain to reduce the problem to the case of domain \( \Omega \) in \( \mathbb{H} \) bounded by the real axis \( \mathbb{R} \) and \( n - 1 \) disjoint circles in \( \mathbb{H} \). In addition, we may assume that \( w_0 = 0 \). Let \( f_k(z) = G_z(z, w_k)/(3w_k) \). By equation (2) of Lemma 1, \( f_k(z) \to K(z, 0) \) uniformly on compact subsets of \( \Omega \).

By a theorem of N. Suita and A. Yamada in [3], if \( 0 \in \partial \Omega \) the Bergman kernel \( K_{\Omega}(z, 0) \) has exactly \( n - 1 \) zeros, say \( z_0^1, \ldots, z_0^{n-1} \), in \( \Omega \) counting multiplicity. Applying Hurwitz’s theorem to the sequence \( f_k(z) \) as in (1), we conclude that for a given sufficiently small \( \varepsilon > 0 \) there is a positive integer \( N \) such that for all \( k \geq N \) the set \( \bigcup_{j=1}^{n-1} \{ z : |z - z_0^j| < \varepsilon \} \) contains exactly \( n - 1 \) zeros of \( f_k(z) \) (which coincide with zeros of \( G_z(z, w_k) \)) counting multiplicity. Since \( \#(Z_{\Omega}(w_k)) = n - 1 \) and \( z_0^j \in \Omega \) for all \( j = 1, \ldots, n - 1 \), the latter contradicts the assumption that \( G_z(z, w_k) \) has a zero, say \( z_{1_k}^j \), such that \( z_{1_k}^j \to z_0 \in \partial \Omega \) as \( k \to \infty \). The theorem is proved. \( \square \)

Remarks. (1) If \( \Omega \) is infinitely connected, then \( Z_{\Omega}(w) = \{ z_k(w) \}_{k=1}^{\infty} \) is infinite for every \( w \in \Omega \). Since \( G_z(z, w) \) is not constant, the uniqueness theorem for analytic functions implies that \( z_k(w) \to \partial \Omega \) as \( k \to \infty \). Thus, Theorem 1 fails for every infinitely connected domain.
(2) We finish this note with a remark on the boundary of $Z_\Omega$. Let $z_0 \in \partial Z_\Omega$. If, in addition, $z_0 \in Z_\Omega$, then $G_z(z_0, w_0) = 0$ for some $w_0 \in \Omega, z_0 \neq w_0$. Let $\nabla_w G_z(z, w)$ denote the determinant of the Jacobian matrix of $G_z(z, w)$ in the variable $w$. Then
\[
\nabla_w G_z(z, w) = \frac{\pi^2}{4} \left( |L_\Omega(z, w)|^2 - |K_\Omega(z, w)|^2 \right).
\]
By the implicit function theorem, if $|K_\Omega(z_0, w_0)| \neq |L_\Omega(z_0, w_0)|$, then $z_0$ is an interior point of $Z_\Omega$. Therefore for $z_0 \in \partial Z_\Omega$, there are two possibilities: 1) either $K_\Omega(z_0, w_0) = 0$ for some $w_0 \in \partial \Omega$ or 2) $|K_\Omega(z_0, w_1)| = |L_\Omega(z_0, w_1)|$ for every $w_1 \in \Omega$ such that $G_z(z_0, w_1) = 0$. The latter possibility seems unlikely, but we could not exclude this case.

References


Department of Mathematics and Statistics, Texas Tech University, Box 41042, Lubbock, Texas 79409

E-mail address: alex.solynin@ttu.edu