A KHINCHIN SEQUENCE

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Abstract. We prove that the geometric and harmonic means of the sequence $Z_2$ of positive integers proposed by Bailey, Borwein, and Crandall converge to the corresponding Khinchin Constants.

1. Khinchin Sequences

One defines the Khinchin Constant $K$ by the following relation:

$$\log(K) = \frac{1}{\log(2)} \sum_{k=1}^{\infty} \log(k) \log \left( \frac{(k+1)^2}{k(k+2)} \right) = \log(2.685452001 \ldots).$$

For any sequence $A = (a_j)$, of positive integers, let us refer to $A$ as a Khinchin Sequence iff the geometric means of $A$ converge to $K$:

$$\lim_{n \to \infty} \left( \prod_{j=1}^{n} a_j \right)^{1/n} = K.$$

That is,

$$\lim_{n \to \infty} \frac{1}{n} \sum_{j=1}^{n} \log(a_j) = \log(K).$$

For any irrational number $x$ in the interval $(0, 1)$, let us refer to $x$ as a Khinchin Number iff the continued fraction expansion $A(x) = (a_j(x))$

$$A(x) : a_1(x), a_2(x), a_3(x), \ldots, a_j(x), \ldots$$

for $x$ is a Khinchin Sequence. In this paper, our objective is to prove that the particular sequence $C = (c_j)$

$$C : 2, 5, 1, 11, 1, 3, 1, 22, 2, 4, 1, 7, 1, 2, 1, 45, 2, 4, 1, 8, \ldots, c_j, \ldots$$

of positive integers proposed by Bailey, Borwein, and Crandall is a Khinchin Sequence. See reference [1].
In the paper just cited, the authors denote $C$ by $Z_2$. They define the sequence $C$ in terms of two auxiliary sequences $U=(u_j)$ and $V=(v_k)$, defined in turn as follows. The first sequence, $U$, is the van der Corput Sequence:

$$ U : \frac{1}{2}, \frac{1}{4}, \frac{3}{4}, \frac{1}{8}, \frac{5}{8}, \frac{3}{8}, \frac{7}{8}, \frac{1}{16}, \frac{9}{16}, \frac{5}{16}, \frac{13}{16}, \frac{3}{16}, \frac{11}{16}, \frac{7}{16}, \frac{15}{16}, \ldots, u_j, \ldots. $$

Specifically, for each positive integer $j$, $u_j$ is the dyadic rational number obtained by reflecting the binary representation of $j$ in the binary point. For example, $u_{12} := 0.0011 = 3/16$ because $12 = 1100_2$. See reference [2]. The second sequence, $V$, describes a particular partition of the interval $(0,1]$:  

$$ V : \ldots < v_k = \frac{1}{\log(2)} \log \left( \frac{k+1}{k} \right) < \ldots < v_3 < v_2 < v_1 = 1. $$

Now, in terms of $U$ and $V$, Bailey, Borwein, and Crandall define the sequence $C$ as follows:

$$ C : c_j = k \iff v_{k+1} < u_j \leq v_k. $$

For example, $c_{12} = 7$ because $v_8 < u_{12} \leq v_7$.

2. Motivation

To set a context for our study of the sequence $C$, let us describe a special case of the Ergodic Theorem. Let $\lambda$ stand for Lebesgue Measure, defined as usual on $\mathbb{R}$. Let $X$ be the set of all irrational numbers in the interval $(0,1)$. Let $\mu$ stand for Gauss Measure, defined on $X$ as follows:

$$ \mu(E) := \frac{1}{\log(2)} \int_E \frac{1}{1+x} \lambda(dx) $$

where $E$ is any Borel subset of $X$. Note that $\mu(E) = 0$ iff $\lambda(E) = 0$. Let $F$ be the mapping carrying $X$ to itself defined as follows:

$$ F(x) := \frac{1}{x} - \left\lfloor \frac{1}{x} \right\rfloor $$

where $x$ is any number in $X$. Of course, $\lfloor 1/x \rfloor$ denotes the largest among all integers $\ell$ for which $\ell \leq 1/x$. Note that $F$ is continuous. One may view the ordered pair $(X,F)$ as a (discrete) dynamical system. For any $x$ in $X$, one may say that if the system is in state $x$ at time 0, then the system is in state $F(x)$ one unit of time later. By an elementary argument, one can show that $\mu$ is invariant under $F$, in the sense that, for any Borel subset $E$ of $X$, $\mu(F^{-1}(E)) = \mu(E)$. By a more substantial argument, one can also show that $\mu$ is ergodic under $F$, in the sense that, for any Borel subset $E$ of $X$, if $F^{-1}(E) = E$, then either $\mu(E) = 0$ or $\mu(E) = 1$. See reference [4]. Let $h$ be the function defined on $X$ as follows:

$$ h(x) := \left\lfloor \frac{1}{x} \right\rfloor $$

where $x$ is any number in $X$. Note that $h$ is continuous and that the values of $h$ are positive integers. One may refer to $h$ as an observable for the given dynamical system.

For any $x$ in $X$, one obtains the orbit $O(x) = (x_j)$ of $x$ under $F$,

$$ O(x) : x = x_1, x_2, x_3, \ldots $$
and one obtains the corresponding (discrete) time sequence \( A(x) = (a_j(x)) \),

\[
A(x) : a_1(x), a_2(x), a_3(x), \ldots,
\]

where

\[
x_j := F^{j-1}(x), \quad a_j(x) := h(x_j).
\]

The sequence \( A(x) \) is the Continued Fraction Expansion for \( x \).

For the assembly \( X, \mu, F, \) and \( \log(h) \), the Ergodic Theorem states that, for almost every \( x \) in \( X \),

\[
\lim_{n \to \infty} \frac{1}{n} \sum_{j=1}^{n} \log(h(F^{j-1}(x))) = \int_X \log(h(y)) \mu(dy).
\]

That is, the time average of \( \log(h) \) over \( O(x) \) equals the space average of \( \log(h) \) over \( X \). See reference [5]. Hence,

\[
\lim_{n \to \infty} \frac{1}{n} \sum_{j=1}^{n} \log(a_j(x)) = \int_X \log(h(y)) \mu(dy)
\]

\[
= \sum_{k=1}^{\infty} \log(k) \mu \left( \frac{1}{k+1}, \frac{1}{k} \right)
\]

\[
= \frac{1}{\log(2)} \sum_{k=1}^{\infty} \log(k) \log \left( \frac{(k+1)^2}{k(k+2)} \right)
\]

\[
= \log(K).
\]

Now it is plain that, for almost every irrational number \( x \) in the interval \((0, 1)\), \( x \) is a Khinchin Number. However, no particular examples of such numbers are known. The beguiling cases of \( \pi - 3 \) and even of \( K - 2 \) itself have been studied energetically but to no analytic decision as yet. In reference [3], one may find the optimistic opinion that \( \pi - 3 \) is a Khinchin Number. The graphs displayed in Figures 1 and 2 suggest a more cautious, though still hopeful, opinion on \( \pi - 3 \) and on \( K - 2 \) as well. The graphs are list plots of the following data:

\[
\frac{1}{n} \sum_{j=1}^{n} \log(a_j(x)) - \log(K) \quad (1 \leq n \leq 4096)
\]

where \( x = \pi - 3 \) and \( x = K - 2 \).

Failing to identify particular Khinchin Numbers, one naturally turns to the design of particular Khinchin Sequences. One might, for instance, design sequences \( A = (a_j) \) such that, for each \( j \), \( a_j \) equals 2 or 3 and such that the 2s and 3s occur in \( A \) in correct “limiting proportions,” specifically, the proportions \( p \) and \( q \), where \( p \) and \( q \) are the positive numbers for which \( p + q = 1 \) and \( \log(K) = p \log(2) + q \log(3) \).

However, such a design would be very difficult to implement, since it depends upon the calculation of \( \log(K) \) to arbitrary accuracy. In sharp contrast, Bailey, Borwein, and Crandall have proposed a particular candidate for a Khinchin Sequence, namely, the sequence \( C \), which they have defined in constructive and rapidly computable terms. Let us prove formally that \( C \) is indeed a Khinchin Sequence.
3. The function $\phi$

Let $\phi$ be the function defined on the interval $J = (0, 1]$ as follows. For each $x$ in $J$ and for any positive integer $k$,

$$\phi(x) = \log(k) \iff v_{k+1} < x \leq v_k.$$  

In particular, for each positive integer $j$, $\phi(u_j) = \log(c_j)$. See Figure 3. Clearly,

$$\int_J \phi(x) \lambda(dx) = \frac{1}{\log(2)} \sum_{k=1}^{\infty} \log(k) \left( \log \left( \frac{k+1}{k} \right) - \log \left( \frac{k+2}{k+1} \right) \right)$$

$$= \frac{1}{\log(2)} \sum_{k=1}^{\infty} \log(k) \log \left( \frac{(k+1)^2}{k(k+2)} \right)$$

$$= \log(K).$$
Now it is plain that $C$ is a Khinchin Sequence iff

\[ \lim_{n \to \infty} \frac{1}{n} \sum_{j=1}^{n} \phi(u_j) = \int_{J} \phi(x) \lambda(dx). \]

We proceed to prove relation (1).

4. Integrating Sequences

Let $\psi$ be a real-valued Borel function defined on $J$ and integrable with respect to $\lambda$. Let us say that the sequence $U$ integrates $\psi$ iff

\[ \lim_{n \to \infty} \frac{1}{n} \sum_{j=1}^{n} \psi(u_j) = \int_{J} \psi(x) \lambda(dx). \]

To prove relation (1), we must prove that $U$ integrates $\phi$. Obviously, the functions integrated by $U$ comprise a linear space. By an elementary argument, one can show that, for each subinterval $I$ of $J$, $U$ integrates the characteristic function $\chi_I$ of $I$.

One summarizes this property of $U$ by saying that $U$ is uniformly distributed in $J$. We will prove this property in an appendix to this paper. Presuming the property, let us prove that $U$ integrates $\phi$. To that end, we require only that:

1. $\phi$ is nonnegative and decreasing;
2. for each positive integer $p$, $U$ integrates the function $\phi_p := \chi_{[1/2^p, 1]} \phi$.

Let $n$ be any positive integer. Let $\alpha_n$ be the average of the values of $\phi$ at the first $n$ terms of $U$:

\[ \alpha_n := \frac{1}{n} \sum_{j=1}^{n} \phi(u_j). \]
Let
\[ \beta := \int_J \phi(x) \lambda(dx). \]

We must prove that
\[ \lim_{n \to \infty} \alpha_n = \beta. \]

Let \( p \) be any positive integer. Let \( \phi_p \) be the function defined on \( J \) by truncation of \( \phi \), as follows:
\[ \phi_p := \chi_{[1/2^p,1]} \phi. \]

That is,
\[ \phi_p(x) := \begin{cases} 0 & \text{if } 0 < x < 1/2^p, \\ \phi(x) & \text{if } 1/2^p \leq x \leq 1. \end{cases} \]

Obviously, for each \( x \) in \( J \),
\[ \phi_1(x) \leq \phi_2(x) \leq \cdots \leq \phi_p(x) \leq \cdots \uparrow \phi(x). \]

Let \( \alpha_{n,p} \) be the corresponding average of the values of \( \phi_p \) at the first \( n \) terms of \( U \):
\[ \alpha_{n,p} := \frac{1}{n} \sum_{j=1}^{n} \phi_p(u_j). \]

Let
\[ \beta_p := \int_J \phi_p(x) \lambda(dx). \]

Clearly, \( \phi_p \) is a linear combination of characteristic functions of subintervals of \( J \). By our initial remarks, it is plain that \( U \) integrates \( \phi_p \):
\[ \lim_{n \to \infty} \alpha_{p,n} = \beta_p. \]

Now let \( \epsilon \) be any positive real number. By the Monotone Convergence Theorem, we may introduce a positive integer \( p \) such that
\[ \beta - \epsilon < \beta_p \leq \beta, \]
from which it follows that
\[ \int_{(0,1/2^p)} \phi(x) \lambda(dx) < \epsilon. \]

By relation (2), we may introduce a positive integer \( m \) such that, for every positive integer \( n \), if \( m \leq n \), then
\[ \beta_p - \epsilon < \alpha_{n,p} < \beta_p + \epsilon. \]

We may as well arrange that \( 2^p \leq m \). Let \( n \) be any positive integer for which \( m \leq n \). Let \( q \) be the positive integer for which \( 2^{q-1} - 1 < n \leq 2^q - 1 \). Clearly, \( p < q \). One may say that the first \( n \) terms of \( U \) have run through the first \( q-1 \) “cycles” of \( U \) and have at least begun (perhaps even finished) the \( q \)-th cycle. The smallest term of the \( q \)-th cycle is \( 1/2^q \). Hence, for each positive integer \( j \), if \( 1 \leq j \leq n \), then \( 1/2^q \leq u_j \). Consequently,
\[ \alpha_{n,q} = \alpha_n. \]

Now let
\[ t_1, t_2, \ldots, t_{\ell} \quad (\ell = 2^{q-p} - 1) \]
be the terms among
\[ u_1, u_2, \ldots, u_r \quad (r = 2^q - 1) \]

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that are less than $1/2^p$. Figure 4, $p = 2$, $q = 4$, and $\ell = 3$. Since $\phi$ is nonnegative and decreasing, we find that

$$\alpha_{n,q} - \alpha_{n,p} \leq \frac{1}{n} \sum_{j=1}^{\ell} \phi(t_j)$$

$$= \frac{2^q}{n} \frac{1}{2^q} \sum_{j=1}^{\ell} \phi(t_j)$$

$$\leq 4 \int_{(0,1/2^p)} \phi(x) \lambda(dx) \quad \text{(since } 2^{2^q-1} < 2(n+1))$$

$$< 4\epsilon \quad \text{(by relation (4))}.$$

Hence, by relations (3) and (5) and by the foregoing computation,

$$\beta - 2\epsilon < \beta_p - \epsilon < \alpha_{n,p} \leq \alpha_{n,q} < \alpha_{n,p} + 4\epsilon < \beta_p + 5\epsilon \leq \beta + 5\epsilon$$

so that, by relation (6), $\beta - 2\epsilon < \alpha_n < \beta + 5\epsilon$. Therefore,

$$\lim_{n \to \infty} \frac{1}{n} \sum_{j=1}^{n} \phi(u_j) = \int \phi(x) \lambda(dx).$$

5. Questions

The number $x$ in $(0,1)$ of which $C$ is the continued fraction expansion is approximately equal to 0.46107049595671951935. Of course, it is a Khinchin Number. Can one identify $x$ in “familiar” terms?
In Figures 5 and 6, we display list plots of the following data:

\[
\frac{1}{n} \sum_{j=1}^{n} \log(c_j) - \log(K) \quad (1 \leq n \leq N),
\]

where \(N = 4096\) and \(N = 8192\). Can one explain, in formally precise terms, the apparent self-similarity of the data?

### 6. Harmonic Means

Let \(r\) be any real number for which \(r < 1\) and \(r \neq 0\). With reference to Section 4, let us define the function \(\phi_r\) on \(J\) as follows. For each \(x\) in \(J\) and for any positive integer \(k\),

\[
\phi_r(x) = k^r \iff v_{k+1} < x \leq v_k.
\]

In particular, for each positive integer \(j\), \(\phi_r(u_j) = c_j^r\). If \(r < 0\), then \(1 - \phi_r\) is similar to \(\phi\), in the sense that it meets the conditions (1) and (2) stated in Section 4. If
0 < r < 1, then \( \phi_r \) itself is similar to \( \phi \). In either case, \( U \) integrates \( \phi_r \). Hence,

\[
\lim_{n \to \infty} \frac{1}{n} \sum_{j=1}^{n} c_j^r = \lim_{n \to \infty} \frac{1}{n} \sum_{j=1}^{n} \phi_r(u_j) = \int J \phi_r(x) \lambda(dx) = \frac{1}{\log(2)} \sum_{k=1}^{\infty} k^r \log \left( \frac{(k+1)^2}{k(k+2)} \right).
\]

One defines the Khinchin Constant \( K_r \) by the following relation:

\[
K_r = \frac{1}{\log(2)} \sum_{k=1}^{\infty} k^r \log \left( \frac{(k+1)^2}{k(k+2)} \right).
\]

We infer that the \( r \)-th harmonic means of \( C \) converge to \( K_r \):

\[
\lim_{n \to \infty} \left( \frac{1}{n} \sum_{j=1}^{n} c_j^r \right)^{1/r} = K_r.
\]

7. Appendix

The van der Corput Sequence \( U \) falls into cycles:

\[
U : 1, 2, 4, 1, 4, 1, 8, 2, 8, 5, 3, 7, 1, 9, 5, 13, 3, 11, 7, 15, \ldots , u_j, \ldots
\]

For each positive integer \( p \), the first term of the \( p \)-th cycle is \( 1/2^p \) and the length of the \( p \)-th cycle is \( 2^p - 1 \). The sum of the lengths of the first \( p \) cycles is \( 2^p - 1 \). Moreover,

\[
u_{2^p+j} = \frac{1}{2^{p+1}} + u_j \quad (0 < j < 2^p).
\]

By these observations, it is plain that, for any dyadic interval \( I \) of the form

\[
I = [j/2^p, (j+1)/2^p) \quad (p \in \mathbb{Z}^+, \ 0 < j < 2^p),
\]

the sequence \( U \) visits \( I \) precisely once in the course of its first \( p \) cycles. Thereafter, it visits \( I \) periodically with period \( 2^p \). Hence, for any positive integer \( n \), if \( 2^p \leq n \), then

\[
\frac{m}{n} \leq \frac{1}{n} \sum_{j=1}^{n} \chi_I(u_j) \leq \frac{m+1}{n},
\]

where \( m \) is the positive integer for which

\[
m2^p - 1 < n \leq (m + 1)2^p - 1.
\]

Consequently,

\[
\lim_{n \to \infty} \frac{1}{n} \sum_{j=1}^{n} \chi_I(u_j) = \frac{1}{2^p} = \lambda(I),
\]

which is to say that \( U \) integrates \( \chi_I \).
In turn, for any subinterval $I$ of the interval $(0, 1)$ and for any positive real number $\epsilon$, we may introduce finite disjoint unions $I'$ and $I''$ of dyadic intervals of the foregoing form such that $I' \subseteq I \subseteq I''$ and $\lambda(I'' \setminus I') < \epsilon$. Hence,

$$\limsup_{n \to \infty} \frac{1}{n} \sum_{j=1}^{n} \chi_I(u_j) \leq \lim_{j \to \infty} \frac{1}{n} \sum_{j=1}^{n} \chi_{I'}(u_j)$$

$$\leq \lambda(I') + \epsilon$$

$$= \lim_{n \to \infty} \frac{1}{n} \sum_{j=1}^{n} \chi_{I'}(u_j) + \epsilon$$

$$\leq \liminf_{n \to \infty} \frac{1}{n} \sum_{j=1}^{n} \chi_{I}(u_j) + \epsilon.$$

Consequently,

$$\lim_{n \to \infty} \frac{1}{n} \sum_{j=1}^{n} \chi_{I}(u_j) = \lambda(I),$$

which is to say that $U$ integrates $\chi_I$.

REFERENCES


