NORMS OF ELEMENTARY OPERATORS

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Abstract. Let $A_i$ and $B_i$, $1 \leq i \leq n$, be bounded linear operators acting on a separable Hilbert space $\mathcal{H}$. In this note, we prove that $\sup \{ \| \sum_{i=1}^{n} A_i X B_i \| : X \in \mathcal{B}(\mathcal{H}), \| X \| \leq 1 \} = \sup \{ \| \sum_{i=1}^{n} A_i U B_i \| : U U^* = U^* U = I, U \in \mathcal{B}(\mathcal{H}) \}$. Moreover, we prove that there exists an operator $X_0$ with $\| X_0 \| = 1$ such that $\| \sum_{i=1}^{n} A_i X_0 B_i \| = \sup \{ \| \sum_{i=1}^{n} A_i X B_i \| : X \in \mathcal{B}(\mathcal{H}), \| X \| \leq 1 \}$ if and only if there exists a unitary $U_0 \in \mathcal{B}(\mathcal{H})$ such that $\| \sum_{i=1}^{n} A_i U_0 B_i \| = \sup \{ \| \sum_{i=1}^{n} A_i X B_i \| : X \in \mathcal{B}(\mathcal{H}), \| X \| \leq 1 \}$.

1. Introduction and statement of the main theorem

Let $\mathcal{B}(\mathcal{H})$ be the $C^*$-algebra of all bounded linear operators on a Hilbert space $\mathcal{H}$. The unit ball $\{ A : A \in \mathcal{B}(\mathcal{H}) \text{ and } \| A \| \leq 1 \}$ and the unitary group $\{ U : U \in \mathcal{B}(\mathcal{H}) \text{ and } U U^* = U^* U = I \}$ of $\mathcal{B}(\mathcal{H})$ are denoted, respectively, by $\mathcal{B}(\mathcal{H})_1$ and $\mathcal{U}(\mathcal{H})$. For $A_i, B_i \in \mathcal{B}(\mathcal{H})$, $i = 1, 2, \ldots, n$, the $n$-tuples $\tilde{A}$ and $\tilde{B}$ are defined, respectively, by $\tilde{A} = (A_1, A_2, \ldots, A_n)$ and $\tilde{B} = (B_1, B_2, \ldots, B_n)$.

The elementary operator $\delta_{\tilde{A}, \tilde{B}}$ on $\mathcal{B}(\mathcal{H})$ induced by $\tilde{A}$ and $\tilde{B}$ is defined by

$$\delta_{\tilde{A}, \tilde{B}} X = \sum_{i=1}^{n} A_i X B_i, \text{ for } X \in \mathcal{B}(\mathcal{H}).$$

The norm $\| \delta_{\tilde{A}, \tilde{B}} \|$ of the elementary operator $\delta_{\tilde{A}, \tilde{B}}$ is defined by

$$\| \delta_{\tilde{A}, \tilde{B}} \| = \sup \{ \| \sum_{i=1}^{n} A_i X B_i \| : X \in \mathcal{B}(\mathcal{H})_1 \}. $$

The elementary operator as an operator on the Banach space $\mathcal{B}(\mathcal{H})$ has attracted much attention of many mathematicians. Some interesting results about the spectra, the ranges and the norms of elementary operators have been obtained (see [1]-[5]).

About the discussion of the norms of elementary operators, one can trace back to Stampfli’s theorem in 1970 (see [6]).
Moreover, the quantity in the above equality is the same as

\[ \sup \{ \| AX + XB \| : X \in \mathcal{B}(\mathcal{H})_1 \} = \min \{ \| A + \mu I \| + \| B - \mu I \| : \mu \in \mathbb{C} \}. \]

In the present paper, the inspiration originated from the main result obtained recently by Choi and Li in [1].

**Theorem C-L** (Theorem 2.1 in [1]). Let \( A, B \in \mathcal{B}(\mathcal{H}) \). Then

\[ \sup \{ \| U^* A U + V^* B V \| : U, V \in \mathcal{U}(\mathcal{H}) \} = \min \{ \| A + \mu I \| + \| B - \mu I \| : \mu \in \mathbb{C} \}. \]

Moreover, the quantity in the above equality is the same as

\[ \sup \{ \| AX + XB \| : X \in \mathcal{B}(\mathcal{H})_1 \}. \]

In this note, we shall concentrate on the norms of elementary operators. The main result in this paper is the following.

**Theorem 1.1.** Let \( A_1, \ldots, A_n, B_1, \ldots, B_n \in \mathcal{B}(\mathcal{H}) \) and let \( \delta_{\tilde{A}, \tilde{B}} : \mathcal{B}(\mathcal{H}) \to \mathcal{B}(\mathcal{H}) \) be defined by

\[ \| \delta_{\tilde{A}, \tilde{B}}(X) \| = \sup_{U \in \mathcal{U}(\mathcal{H})} \| \tilde{A}, \tilde{B}^* \| \| \tilde{A}, \tilde{B} \| . \]

Moreover, there is a contraction \( X \in \mathcal{B}(\mathcal{H})_1 \) such that \( \| \delta_{\tilde{A}, \tilde{B}}(X) \| = \| \delta_{\tilde{A}, \tilde{B}} \| \) if and only if there is a unitary \( U \in \mathcal{U}(\mathcal{H}) \) such that \( \| \delta_{\tilde{A}, \tilde{B}}(U) \| = \| \delta_{\tilde{A}, \tilde{B}} \| . \)

An elementary operator \( \delta_{\tilde{A}, \tilde{B}} \) is said to be norm-attainable if there is a contraction \( X \in \mathcal{B}(\mathcal{H})_1 \) such that \( \| \delta_{\tilde{A}, \tilde{B}}(X) \| = \| \delta_{\tilde{A}, \tilde{B}} \| . \)

By Theorem 1.1 and Theorem S, one can deduce Theorem C-L as follows. In fact, in Theorem 1.1, let \( n = 2, A_1 = A, A_2 = I, B_1 = I \) and \( B_2 = B \). Then

\[ \sup \{ \| AX + XB \| : X \in \mathcal{B}(\mathcal{H}), \| X \| \leq 1 \} = \sup \{ \| AU + UB \| : U \in \mathcal{U}(\mathcal{H}) \}. \]

It is clear that

\[ \sup \{ \| AU + UB \| : U \in \mathcal{U}(\mathcal{H}) \} = \sup \{ \| UV^* + UV^* B \| : U, V \in \mathcal{U}(\mathcal{H}) \} = \sup \{ \| U^* AU + V^* BV \| : U, V \in \mathcal{U}(\mathcal{H}) \}. \]

From Theorem S, we have

\[ \sup \{ \| AX + XB \| : X \in \mathcal{B}(\mathcal{H})_1 \} = \min \{ \| A + \mu I \| + \| B - \mu I \| : \mu \in \mathbb{C} \}. \]

Hence,

\[ \sup \{ \| U^* AU + V^* BV \| : U, V \in \mathcal{U}(\mathcal{H}) \} = \min \{ \| A + \mu I \| + \| B - \mu I \| : \mu \in \mathbb{C} \}. \]

2. **Proof of the Main Theorem and Auxiliary Results**

In this section, we begin with some notation and terminology.

An operator \( A \in \mathcal{B}(\mathcal{H}) \) is said to be positive if \( (Ax, x) \geq 0 \) for all \( x \in \mathcal{H} \). If \( A \) is positive, then the unique positive square root of \( A \) is denoted by \( A^\frac{1}{2} \). The spectrum, the null-space and the range of \( A \in \mathcal{B}(\mathcal{H}) \) are denoted by \( \sigma(A) \), \( \mathcal{N}(A) \) and \( \mathcal{R}(A) \), respectively. An operator \( V \in \mathcal{B}(\mathcal{H}) \) is said to be an isometry (or co-isometry) if \( V^* V = I \) (or \( VV^* = I \)). For a subspace \( M \subseteq \mathcal{H} \), if \( \dim M \) is infinite, then \( M \) is said to be infinite co-dimensional, where \( \dim K \) denotes the dimension of a subspace \( K \subseteq \mathcal{H} \) and \( K^\perp \) denotes the orthogonal complement of \( K \). The orthogonal projection on \( M \) is denoted by \( P_M \).

To complete the proof of Theorem 1.1, we need some auxiliary results.
Lemma 2.1. If $A \in \mathcal{B}(\mathcal{H})_1$, then there exist two isometries or co-isometries $V_1$ and $V_2$ in $\mathcal{B}(\mathcal{H})_1$ such that

$$A = \frac{1}{2}(V_1 + V_2).$$

Moreover, if $\dim N(A) = \dim N(A^*)$, then $V_1$ and $V_2$ can be taken to be unitaries.

Proof. Let $A \in \mathcal{B}(\mathcal{H})_1$ and $A = VP$ be the polar decomposition of $A$. Since $A \in \mathcal{B}(\mathcal{H})_1$, $P$ is a positive contraction in $\mathcal{B}(\mathcal{H})_1$, so $I - P^2$ is also a positive contraction in $\mathcal{B}(\mathcal{H})_1$. Define operators $U_1$ and $U_2$ by

$$U_1 = P + i(I - P^2)^{\frac{1}{2}} \quad \text{and} \quad U_2 = P - i(I - P^2)^{\frac{1}{2}},$$

respectively. Noting that $U_1^* = U_2$ and directly checking, $U_1U_1^* = U_1^*U_1 = I$ and $U_2U_2^* = U_2^*U_2 = I$, so $U_1$ and $U_2$ are unitaries and

$$P = \frac{1}{2}(U_1 + U_2).$$

If $\dim N(A) < \dim N(A^*)$, then $V$ can be taken to be an isometry. Take $V_1 = VU_1$ and $V_2 = VU_2$. Then $V_1$ and $V_2$ are isometries and

$$A = VP = V\left(\frac{1}{2}(U_1 + U_2)\right) = \frac{1}{2}(V_1 + V_2).$$

If $\dim N(A) > \dim N(A^*)$, then $V$ can taken to be a co-isometry. So $V_1$ and $V_2$ in (4) can be taken to be co-isometries.

If $\dim N(A) = \dim N(A^*)$, then $V$ can be taken to be a unitary. So $V_1$ and $V_2$ in (4) can be taken to be unitaries. \hfill \Box

Corollary 2.2. If the elementary operator $\delta_{A, B}$ is norm-attainable, then there exists an isometry or a co-isometry $V_0$ such that

$$\| \delta_{A, B} \| = \left\| \sum_{i=1}^{n} A_i V_0 B_i \right\|.$$ 

Proof. If the elementary operator $\delta_{A, B}$ is norm-attainable, then there exists an operator $X_0 \in \mathcal{B}(\mathcal{H})_1$ such that

$$\| \delta_{A, B} \| = \left\| \sum_{i=1}^{n} A_i X_0 B_i \right\|.$$ 

To complete the proof, it is sufficient to show that $X_0$ can be represented as an average of two isometries or two co-isometries. By (3) in Lemma 2.1, it is obvious. \hfill \Box

Corollary 2.3. If $A \in \mathcal{B}(\mathcal{H})_1$ is of finite rank, then there exist two unitaries $U_1$ and $U_2$ in $\mathcal{U}(\mathcal{H})$ such that

$$A = \frac{1}{2}(U_1 + U_2).$$

Proof. Since $A \in \mathcal{B}(\mathcal{H})_1$ is of finite rank, which implies that $\dim N(A) = \dim N(A^*)$, the corollary is evident by Lemma 2.1. \hfill \Box

An operator $A \in \mathcal{B}(\mathcal{H})$ is said to be norm-attainable if there exists a unit vector $x_0 \in \mathcal{H}$ such that $\|Ax_0\| = \|A\|$. 

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Lemma 2.4. If $A \in \mathcal{B}(\mathcal{H})$ is not norm-attainable, then there exists an infinite sequence $\{x_n\}_{n=1}^{\infty}$ of orthonormal vectors such that $\lim_{n \to \infty} \|Ax_n\| = \|A\|$, $Ax_i \neq 0$ for all $1 \leq i < \infty$ and $Ax_i \perp Ax_j$, $i \neq j$.

Proof. If $A = UP$ is the polar decomposition of $A$, where $U$ is a partial isometry from $\mathcal{R}(A^*)$ onto $\overline{\mathcal{R}(A)}$, then $\|Ax\| = \|Px\|$ for all $x \in \mathcal{H}$. This shows that $A$ is norm-attainable if and only if $P$ is norm-attainable. Denote the spectral decomposition of $P$ by

$$P = \int_0^\|P\| \lambda dE_\lambda$$

and suppose $A$ is not norm-attainable. Then $P$ is not norm-attainable either. Hence, $\|P\|$ is not an isolated point of $\sigma(P)$. So we can choose a sequence $\{\alpha_n\}$ of positive numbers such that $\{\alpha_n\}$ is strictly increasing, $\lim_{n \to \infty} \alpha_n = \|P\|$ and the spectral projection $E_n = E((\alpha_n, \alpha_{n+1})) \neq 0$, $1 \leq n < \infty$. For this situation, $[\alpha_n, \alpha_{n+1}) \cap \sigma(P) \neq \emptyset$ and $E_n \mathcal{H}$ is an invariant subspace of $P$ for $1 \leq n < \infty$.

Taking a unit vector $x_n \in E_n \mathcal{H}$, it is clear that $\{x_n\}$ is a sequence of orthonormal vectors with $A^*Ax_n = P^2x_n \in E_n \mathcal{H}$; thus $(A^*Ax_i, x_j) = 0$ since $E_iE_j = E_jE_i = 0$ for $i \neq j$, $1 \leq i, j < \infty$. So $(Ax_i, Ax_j) = 0$ for $i \neq j$, $1 \leq i, j < \infty$. In this case,

$$\|Ax_n\|^2 = (A^*Ax_n, x_n) = (P^2x_n, x_n) = \left(\int_0^\|P\| \lambda^2 dE_\lambda x_n, x_n\right)$$

$$= \left(\int_{\alpha_n}^{\alpha_{n+1}} \lambda^2 dE_\lambda x_n, x_n\right) \geq \alpha_n^2 \rightarrow \|P\|^2 = \|A\|^2.$$

Hence,

$$\lim_{n \to \infty} \|Ax_n\| = \|A\| \quad \Box$$

Lemma 2.5. For an operator $A \in \mathcal{B}(\mathcal{H})$, the operator $A$ is norm-attainable if and only if its adjoint $A^*$ is norm-attainable.

Proof. It is enough to show that $A \neq 0$ is norm-attainable implies that $A^*$ is norm-attainable. If $A$ is norm-attainable, then there exists a unit vector $x_0$ such that $\|Ax_0\| = \|A\|$. That is,

$$A^*Ax_0 = \|A\|^2 x_0.$$

Denote $y_0 = \frac{Ax_0}{\|A\|}$. Then $y_0$ is a unit vector and $\|A^*y_0\| = \|A\| = \|A^*\|$. Hence $A^*$ is norm-attainable. \hfill \Box

Lemma 2.6. Let $V$ be an isometry and let $\mathcal{M}$ be an infinite co-dimensional subspace of $\mathcal{H}$. If the restriction of $V$ on $\mathcal{M}$ is denoted by $V|_{\mathcal{M}}$, then $\dim \mathcal{N}(V|_{\mathcal{M}}) = \dim \mathcal{N}((V|_{\mathcal{M}})^*) = \infty$.

Proof. It is clear that $\dim \mathcal{N}(V|_{\mathcal{M}}) = \infty$.

By the assumption that $V$ is an isometry, if $x \in \mathcal{M}$ and $y \in \mathcal{M}^\perp$, then $(Vx, Vy) = (V^*Vx, y) = (x, y) = 0$. Hence $\mathcal{R}(V|_{\mathcal{M}}) \perp \mathcal{R}(V|_{\mathcal{M}^\perp})$. Observing that $\dim \mathcal{R}(V|_{\mathcal{M}^\perp}) = \dim \mathcal{M}^\perp = \infty$, so $\dim \mathcal{N}((V|_{\mathcal{M}})^*) \geq \dim \mathcal{R}(V|_{\mathcal{M}^\perp}) = \infty$. Therefore

$$\dim \mathcal{N}(V|_{\mathcal{M}}) = \dim \mathcal{N}((V|_{\mathcal{M}})^*) = \infty. \quad \Box$$
**Lemma 2.7.** Let \( \{y_i\}_{i=1}^{\infty} \) be a sequence of unit vectors with \( |(y_i, y_j)| < \frac{1}{2\max\{i, j\}+4} \) for \( i \neq j, 1 \leq i, j < \infty \). Then, for each \( k \in \mathbb{N} \), \( y_k \) is not contained in the closed subspace spanned by \( \{y_j, j \neq k\} \).

**Proof.** Let \( \{y_i\}_{i=1}^{\infty} \) be a sequence of unit vectors with \( |(y_i, y_j)| < \frac{1}{2\max\{i, j\}+4} \) for \( i \neq j \). Firstly, we shall show that if \( y_{i_0} = \sum_{j \neq i_0} \lambda_j y_j \), then the set \( \{ |\lambda_j| : j \neq i_0 \} \) is bounded. Moreover, \( |\lambda_j| \leq 4, 1 \leq j < \infty \).

On the contrary, assume that there exists a \( k_0 \) with \( |\lambda_{k_0}| > 4 \). If \( \epsilon < \frac{1}{2} \) is small enough, then there exists an \( n_0 \) with \( n_0 > k_0 \) such that

\[
\|y_{i_0} - \sum_{j \neq i_0, j = 1}^{n_0} \lambda_j y_j \| < \epsilon.
\]

Denote

\[
x_{i_0, n_0} = y_{i_0} - \sum_{j \neq i_0, j = 1}^{n_0} \lambda_j y_j
\]

and \( |\lambda_{k_0}| = \max\{|\lambda_j| : 1 \leq j \leq n_0\} \), where \( 1 \leq k_0 \leq n_0 \). Then \( y_{i_0} = \sum_{j \neq i_0, j = 1}^{n_0} \lambda_j y_j + x_{i_0, n_0} \) and \( |\lambda_{k_0}| > 4 \). So

\[
|(y_{i_0}, y_{k_0})| \geq \Big( \sum_{j \neq i_0, j = 1}^{n_0} \lambda_j y_j + x_{i_0, n_0}, y_{k_0} \Big) \geq |\lambda_{k_0}| - \left( \sum_{j = 1, j \neq i_0, j \neq k_0}^{n_0} \frac{1}{2\max\{k_0, j\}+4} |\lambda_j| + \epsilon \right)
\]

\[
\geq |\lambda_{k_0}| \left( 1 - \sum_{j = 1, j \neq i_0, j \neq k_0}^{n_0} \frac{1}{2\max\{k_0, j\}+4} \right) - \epsilon
\]

\[
\geq \frac{1}{2} |\lambda_{k_0}| - \epsilon
\]

\[
\geq \frac{1}{4} |\lambda_{k_0}|.
\]

Moreover,

\[
\frac{1}{2\max\{i_0, k_0\}+4} \geq |(y_{i_0}, y_{k_0})| \geq \frac{1}{4} |\lambda_{k_0}| \geq 1.
\]

This is a contradiction.

Secondly, we shall show that, for \( k \in \mathbb{N} \), \( y_k \) is not contained in the closed subspace spanned by \( \{y_j, j \neq k\} \).

On the contrary again, assume that there exists a vector \( y_{k_0} \) such that

\[
y_{k_0} = \sum_{j = 1, j \neq k_0} \lambda_j y_j.
\]
Observing that $|\lambda_j| \leq 4$ for $j \neq k_0$, then
\[
1 = (y_{k_0}, y_{k_0}) = |\left( \sum_{j=1, j \neq k_0} \lambda_j y_j, y_{k_0} \right)| = |\left( \sum_{j=1, j \neq k_0} (\lambda_j y_j, y_{k_0}) \right)| \leq \sum_{j=1, j \neq k_0} \frac{1}{2^{\max\{k_0, j\}+4}} |\lambda_j| \leq 4\left( \frac{1}{2^{k_0}} \left( k_0^2 + \frac{1}{2^{k_0}} \right) \right) \leq \frac{1}{2} (|\lambda_k| \leq 4).
\]
This is a contradiction. \qed

From Lemma 2.7, we have the following definition: A sequence $\{y_i\}_{i=1}^\infty$ of unit vectors is said to be almost-orthonormal if $|(y_i, y_j)| < \frac{1}{2^{\max\{i, j\}+1}}$ for all $1 \leq i, j < \infty$ and $i \neq j$.

By Lemma 2.7, all the vectors in an almost-orthonormal sequence are linearly independent.

**Lemma 2.8.** Let $\{x_i\}_{i=1}^\infty \subset \mathcal{H}$ be an infinite sequence of orthonormal vectors. If $\{y_j\}_{j=1}^n \subset \mathcal{H}$ is a finite sequence, then, for an arbitrary $\epsilon > 0$, there exists a positive integer $i_0$ such that $|(y_j, x_i)| < \epsilon$, for all $1 \leq j \leq n$ and $i > i_0$.

**Proof.** This is obvious. \qed

We shall devote the next section to a proof of Theorem 1.1.

**Proof of Theorem 1.1.** First, we shall show that
\[
\|\delta_{\tilde{A}, \tilde{B}}\| = \sup \{\|\sum_{i=1}^n A_i U B_i\| : U \in \mathcal{U}(\mathcal{H})\}.
\]

By the definition of the operator norm, there exists an operator sequence $\{X_m\}_{m=1}^\infty \subset \mathcal{B}(\mathcal{H})_1$ such that
\[
\|\delta_{\tilde{A}, \tilde{B}}\| = \lim_{m \to \infty} \|\delta_{\tilde{A}, \tilde{B}} X_m\| = \lim_{m \to \infty} \|\sum_{i=1}^n A_i X_m B_i\|.
\]

For each $m \in \mathbb{N}$, there exists a unit vector $x_m \in \mathcal{H}$ such that
\[
\|\sum_{i=1}^n A_i X_m B_i\| - \|\sum_{i=1}^n A_i X_m B_i x_m\| < \frac{1}{m}, 1 \leq i \leq n.
\]

Define an operator $X^0_m$ for $m \in \mathbb{N}$ by
\[
\begin{cases}
X^0_m B_i x_m = X_m B_i x_m, & 1 \leq i \leq n; \\
X^0_m y = 0, & \text{if } y \in (\mathcal{V}\{B_i x_m, 1 \leq i \leq n\})^\perp.
\end{cases}
\]

It is obvious that $\|X^0_m\| \leq \|X_m\| \leq 1$. That is, $X^0_m \in \mathcal{B}(\mathcal{H})_1$ and
\[
\|\sum_{i=1}^n A_i X_m B_i\| - \|\sum_{i=1}^n A_i X^0_m B_i x_m\| < \frac{1}{m}, m \in \mathbb{N}.
\]
From the definition of $X^0_m$, $X^0_m$ is of finite rank and $X^0_m \in \mathcal{B}(\mathcal{H})_1$, so by Corollary 2.3 there exist two unitaries $U^{(1)}_m$ and $U^{(2)}_m$ for each $m \in \mathbb{N}$ such that

$$X^0_m = \frac{1}{2}(U^{(1)}_m + U^{(2)}_m), \text{ for } m \in \mathbb{N}.$$ 

Observing that

$$\| \sum_{i=1}^{n} A_i X^0_m B_i x_m \| = \| \sum_{i=1}^{n} \frac{1}{2} A_i (U^{(1)}_m + U^{(2)}_m) B_i x_m \|$$

$$\leq \frac{1}{2} (\| \sum_{i=1}^{n} A_i U^{(1)}_m B_i x_m \| + \| \sum_{i=1}^{n} A_i U^{(2)}_m B_i x_m \|)$$

for $m \in \mathbb{N}$, this shows that at least one of the numbers $\| \sum_{i=1}^{n} A_i U^{(1)}_m B_i x_m \|$ and $\| \sum_{i=1}^{n} A_i U^{(2)}_m B_i x_m \|$ is greater than or equal to $\| \sum_{i=1}^{n} A_i X^0_m B_i x_m \|$. Without loss of generality, we can assume that

$$\| \sum_{i=1}^{n} A_i U^{(1)}_m B_i x_m \| \geq \| \sum_{i=1}^{n} A_i X^0_m B_i x_m \|, \text{ for } m \in \mathbb{N}.$$ 

Therefore,

$$\| \delta_{\tilde{A}, \tilde{B}} \| \geq \| \sum_{i=1}^{n} A_i U^{(1)}_m B_i \|$$

$$\geq \| \sum_{i=1}^{n} A_i U^{(1)}_m B_i x_m \|$$

$$\geq \| \sum_{i=1}^{n} A_i X^0_m B_i x_m \|$$

$$\geq \| \sum_{i=1}^{n} A_i X^0_m B_i \| - \frac{1}{m}.$$ 

So

$$\| \delta_{\tilde{A}, \tilde{B}} \| \geq \lim_{m \to \infty} \| \sum_{i=1}^{n} A_i U^{(1)}_m B_i \|$$

$$\geq \lim_{m \to \infty} (\| \sum_{i=1}^{n} A_i X^0_m B_i \| - \frac{1}{m})$$

$$= \lim_{m \to \infty} \| \sum_{i=1}^{n} A_i X^0_m B_i \|$$

$$= \| \delta_{\tilde{A}, \tilde{B}} \|.$$ 

That is,

$$\| \delta_{\tilde{A}, \tilde{B}} \| = \lim_{m \to \infty} \| \sum_{i=1}^{n} A_i U^{(1)}_m B_i \|.$$ 

Second, we shall prove that there is a contraction $X \in \mathcal{B}(\mathcal{H})_1$ such that $\| \delta_{\tilde{A}, \tilde{B}}(X) \| = \| \delta_{\tilde{A}, \tilde{B}} \|$ if and only if there is a unitary $U \in \mathcal{U}(\mathcal{H})$ such that $\| \delta_{\tilde{A}, \tilde{B}}(U) \| = \| \delta_{\tilde{A}, \tilde{B}} \|$. 

"$\Leftarrow$" is obvious.
It is enough to consider that \( \Rightarrow \).
For convenience, we divide the part of the proof into three cases.

**Case 1.** There exists an operator \( X_0 \in \mathcal{B}(\mathcal{H})_1 \) with \( \dim \mathcal{N}(X_0) = \dim \mathcal{N}(X_0^*) \) such that

\[
\| \delta_{A,B} \| = \| \delta_{A,B} X_0 \| = \left\| \sum_{i=1}^{n} A_i X_0 B_i \right\|.
\]

In this case, by Lemma 2.1, there exist two unitaries \( U_0^1 \) and \( U_0^2 \) such that

\[
X_0 = \frac{1}{2}(U_0^1 + U_0^2).
\]

Therefore,

\[
\| \delta_{A,B} \| = \| \delta_{A,B} X_0 \| = \| \delta_{A,B} X_0 U_0^1 + \delta_{A,B} X_0 U_0^2 \| \\
\leq \frac{1}{2} (\| \delta_{A,B} U_0^1 \| + \| \delta_{A,B} U_0^2 \|) \\
\leq \| \delta_{A,B} \|.
\]

So \( \frac{1}{2} (\| \delta_{A,B} U_0^1 \| + \| \delta_{A,B} U_0^2 \|) ) = \| \delta_{A,B} \| \). That is,

\[
\| \delta_{A,B} U_0^1 \| = \| \delta_{A,B} U_0^2 \| = \| \delta_{A,B} \|.
\]

**Case 2.** There exists an operator \( X_0 \in \mathcal{B}(\mathcal{H})_1 \) such that

\[
\| \delta_{A,B} \| = \| \delta_{A,B} X_0 \| = \left\| \sum_{i=1}^{n} A_i X_0 B_i \right\|
\]

and \( \sum_{i=1}^{n} A_i X_0 B_i \in \mathcal{B}(\mathcal{H}) \) is also norm-attainable.

In such a case, there exists a unit vector \( x_0 \in \mathcal{B}(\mathcal{H}) \) such that

\[
\left\| \sum_{i=1}^{n} A_i X_0 B_i x_0 \right\| = \left\| \sum_{i=1}^{n} A_i X_0 B_i \right\|.
\]

Define an operator \( X_0^0 \) by

\[
\begin{align*}
X_0^0 B_i x_0 & = X_0 B_i x_0, \quad 1 \leq i \leq n; \\
X_0^0 y & = 0, \quad y \in (\{ B_i x_0, 1 \leq i \leq n \})^\perp.
\end{align*}
\]

It is obvious that \( \| X_0^0 \| \leq \| X_0 \| \leq 1 \), so \( X_0^0 \in \mathcal{B}(\mathcal{H})_1 \). In this case,

\[
\| \delta_{A,B} \| = \left\| \sum_{i=1}^{n} A_i X_0 B_i \right\| \\
= \left\| \sum_{i=1}^{n} A_i X_0 B_i x_0 \right\| \\
= \left\| \sum_{i=1}^{n} A_i X_0^0 B_i x_0 \right\| \\
\leq \left\| \sum_{i=1}^{n} A_i X_0^0 B_i \right\| \\
\leq \| \delta_{A,B} \|.
\]
That is, \( \| \delta_{A, B} \| = \| \sum_{i=1}^{n} A_{i}X_{0}B_{i} \| \). By the definition of \( X_{0} \), \( X_{0} \) is of finite rank. Similar to the proof of Case 1 and by Corollary 2.3, there exists a unitary \( U_{00} \) such that \( \| \delta_{A, B} \| = \| \sum_{i=1}^{n} A_{i}U_{00}B_{i} \| \).

**Case 3.** There exists an operator \( X_{0} \in \mathcal{B}(\mathcal{H}) \) with \( \dim \mathcal{N}(X_{0}) \neq \dim \mathcal{N}(X_{0}') \) and \( \sum_{i=1}^{n} A_{i}X_{0}B_{i} \in \mathcal{B}(\mathcal{H}) \) is not norm-attainable such that

\[
\| \delta_{A, B} \| = \| \delta_{A, B}X_{0} \| = \| \sum_{i=1}^{n} A_{i}X_{0}B_{i} \| .
\]

In this case, if \( \dim \mathcal{N}(X_{0}) < \dim \mathcal{N}(X_{0}') \), by Lemma 2.1, we can assume that there exists an isometry \( V_{0} \) such that

\[
\| \delta_{A, B} \| = \| \delta_{A, B}V_{0} \| = \| \sum_{i=1}^{n} A_{i}V_{0}B_{i} \| .
\]

If \( \sum_{i=1}^{n} A_{i}V_{0}B_{i} \in \mathcal{B}(\mathcal{H}) \) is norm-attainable, by Case 2, there is nothing to do. So, in the next case, we assume that the operator \( \sum_{i=1}^{n} A_{i}V_{0}B_{i} \in \mathcal{B}(\mathcal{H}) \) is not norm-attainable.

In this case, by Lemma 2.4, there exists an orthonormal sequence \( \{x_{m}\}_{m=1}^{\infty} \subseteq \mathcal{R}(\sum_{i=1}^{n} B_{i}^{*}V_{0}A_{i}^{*}) \) of vectors such that

\[
\lim_{m \to \infty} \| \sum_{i=1}^{n} A_{i}V_{0}B_{i}x_{m} \| = \| \sum_{i=1}^{n} A_{i}V_{0}B_{i} \| ,
\]

and \( \sum_{i=1}^{n} A_{i}V_{0}B_{i}x_{j} \perp \sum_{i=1}^{n} A_{i}V_{0}B_{i}x_{k}, 1 \leq j, k < \infty, \) and \( j \neq k \).

Observing that if \( \| \delta_{A, B} \| = 0 \), the discussion is trivial, so we assume that \( \| \delta_{A, B} \| \neq 0 \) in the next case. If \( \| \delta_{A, B} \| \neq 0 \), by Lemma 2.4, there exist a positive number \( \alpha > 0 \) and \( m_{0} \in \mathbb{N} \) such that \( \| \sum_{i=1}^{n} A_{i}V_{0}B_{i}x_{m} \| > \alpha \) for each \( m > m_{0} \) and \( \sum_{i=1}^{n} A_{i}V_{0}B_{i}x_{m}, 1 \leq m < \infty \) is not contained in a finite-dimensional subspace of \( \mathcal{H} \). This shows that \( \{B_{i}x_{m}\}_{1 \leq i \leq n, 1 \leq m < \infty} \) is not contained in a finite-dimensional subspace of \( \mathcal{H} \). Furthermore, if \( \| \delta_{A, B} \| \neq 0 \), then there does not exist a subsequence \( \{z_{k}\}_{k=1}^{\infty} \subseteq \{x_{m}\}_{m=1}^{\infty} \) such that \( \lim_{k \to \infty} B_{i}z_{k} = 0 \) for all \( 1 \leq i \leq n \). This implies that there exist a subsequence \( \{y_{j}\}_{j=1}^{\infty} \subseteq \{x_{m}\}_{m=1}^{\infty} \) and \( 1 \leq i_{0} \leq n \) such that \( \{B_{i_{0}}y_{j}\}_{j=1}^{\infty} \) is bounded below. Without loss of generality, we can assume that \( \{x_{m}\} = \{y_{j}\} \) and \( i_{0} = 1 \). Hence, \( \{B_{1}x_{m}\}_{m=1}^{\infty} \) is bounded below. That is, there exists a \( \delta > 0 \) such that \( \|B_{1}x_{m}\| > \delta \) for all \( 1 \leq m < \infty \).

Since \( \|B_{1}x_{m}\| > \delta \), we have

\[
\left| \frac{B_{1}x_{i}}{\|B_{1}x_{i}\|} \cdot \frac{B_{1}x_{j}}{\|B_{1}x_{j}\|} \right| \leq \delta^{-2} \left| \frac{B_{1}^{*}B_{1}x_{i}, x_{j}}{\|B_{1}^{*}B_{1}x_{i}\|\|B_{1}x_{j}\|} \right| .
\]

Moreover, by Lemma 2.8, there exists a subsequence \( \{x_{m_{j}}\}_{j=1}^{\infty} \subseteq \{x_{m}\}_{m=1}^{\infty} \) such that

\[
\left| \frac{B_{1}x_{m_{j}}}{\|B_{1}x_{m_{j}}\|} \cdot \frac{B_{1}x_{m_{k}}}{\|B_{1}x_{m_{k}}\|} \right| < \frac{1}{2^{\text{max}(j,k)+4}},
\]

since \( \{x_{m}\}_{m=1}^{\infty} \) is an infinite orthonormal sequence. This means that \( \{ \frac{B_{1}x_{m_{j}}}{\|B_{1}x_{m_{j}}\|} \}_{j=1}^{\infty} \) is almost orthonormal. For the sake of convenience, we can think of the subsequence \( \{x_{m_{j}}\}_{j=1}^{\infty} \) as being the same as the sequence \( \{x_{m}\}_{m=1}^{\infty} \). By Lemma 2.7, \( B_{1}x_{m}, 1 \leq m < \infty \), are not contained in the closed subspace spanned by \( \{B_{1}x_{2m-1}\}_{m=1}^{\infty} \). This shows that the closed subspace spanned by \( \{B_{1}x_{2m-1}\}_{m=1}^{\infty} \) is infinite co-dimen-


Denote by $P$ the orthogonal projection on the closed subspace $\bigvee \{ B_1 x_{2m-1} \}_{m=1}^{\infty}$. Then $P^\perp \mathcal{H}$ is an infinite-dimensional subspace.

Next, we shall divide the remainder of the proof into two subcases.

**Subcase 1.** Suppose that there exists a subsequence $\{ z_k \}_{k=1}^{\infty} \subseteq \{ x_{2m-1} \}_{m=1}^{\infty}$ such that $P^\perp B_1 z_k \to 0$ as $k \to \infty$ for every $2 \leq i \leq n$. Denote by $\mathcal{M}$ the closed subspace $\bigvee \{ B_1 z_k : 1 \leq k \leq \infty \}$. Since $\bigvee \{ B_1 z_k : 1 \leq k \leq \infty \} \subseteq \bigvee \{ B_1 x_{2m-1} : 1 \leq m \leq \infty \}$, $\mathcal{M}$ is infinite co-dimensional. Define $D_0 := V_0 P$. It is clear that $\| D_0 \| \leq \| V_0 \| = 1$.

Observing that

$$\lim_{k \to \infty} \| \sum_{i=1}^{n} A_i D_0 B_i z_k \|$$

$$= \lim_{k \to \infty} \| \sum_{i=1}^{n} A_i V_0 P B_i z_k \|$$

$$= \lim_{k \to \infty} \| \sum_{i=1}^{n} A_i V_0 P B_i z_k + \sum_{i=2}^{n} A_i V_0 P^\perp B_i z_k \|$$

$$= \lim_{k \to \infty} \| \sum_{i=1}^{n} A_i V_0 P B_i z_k + \sum_{i=1}^{n} A_i V_0 P^\perp B_i z_k \|$$

$$= \lim_{k \to \infty} \| \sum_{i=1}^{n} A_i V_0 B_i z_k \|$$

$$= \lim_{m \to \infty} \| \sum_{i=1}^{n} A_i V_0 B_i x_{2m-1} \|$$

$$= \left\| \sum_{i=1}^{n} A_i V_0 B_i \right\| = \| \delta_{A,B} \| ,$$

then

$$\| \sum_{i=1}^{n} A_i D_0 B_i \| = \| \delta_{A,B} \| .$$

From the definition of $D_0$, $\dim \mathcal{N}(D_0) = \dim \mathcal{N}(D_0^\ast) = \infty$ by Lemma 2.6. Furthermore, by Case 1, there exists a unitary $U_0$ such that

$$\| \sum_{i=1}^{n} A_i U_0 B_i \| = \| \delta_{A,B} \| .$$

**Subcase 2.** Assume that there does not exist a subsequence $\{ z_k \}_{k=1}^{\infty} \subseteq \{ x_{2m-1} \}_{m=1}^{\infty}$ such that $P^\perp B_1 z_k \to 0$ as $k \to \infty$ for every $2 \leq i \leq n$. Then it is obvious that $\bigvee \{ P^\perp B_i z_k : 2 \leq i \leq n, 1 \leq k < \infty \}$ is infinite dimensional. Denote $C_i = P^\perp B_i Q$, $2 \leq i \leq n$. In such a case, instead of $(B_1, B_2, \ldots, B_n)$, we use $(C_2, C_3, \ldots, C_n)$ and repeat the programme as Subcase 1. Then in at most by $n - 1$ steps we can get a subsequence $\{ y_l \}_{l=1}^{\infty} \subseteq \{ x_{2m-1} \}_{m=1}^{\infty}$ such that the closed subspace $\bigvee \{ B_1 y_l : 1 \leq i \leq n_0, 1 \leq l < \infty \}$ is infinite co-dimensional and $P_{\lambda A_1} B_j y_l \to 0$, as $l \to \infty$ and $n_0 < j \leq n$, where it is necessary that we can change the order of the $n$-tuple
(B_1, B_2, \ldots, B_n) and let \( M_1 \) denote the closed subspace
\[
\vee \{ B_i y_l : 1 \leq i \leq n_0, 1 \leq l < \infty \}.
\]

Similar to Subcase 1, define
\[
D^2_0 := V_0 P_{M_1}.
\]
Then
\[
\lim_{l \to \infty} \| \sum_{i=1}^{n} A_i D^2_0 B_i y_l \| = \lim_{l \to \infty} \| \sum_{i=1}^{n} A_i V_0 P_{M_1} B_i y_l \|
= \lim_{l \to \infty} \| \sum_{i=1}^{n} A_i V_0 P_{M_1} B_i y_l + \sum_{i=n_0+1}^{n} A_i V_0 P_{M_1} B_i y_l \|
= \lim_{l \to \infty} \| \sum_{i=1}^{n} A_i V_0 B_i y_l \|
= \lim_{m \to \infty} \| \sum_{i=1}^{m} A_i V_0 B_i x_{2m-1} \| = \| \sum_{i=1}^{n} A_i V_0 B_i \| = \| \delta_{\tilde{A}, \tilde{B}} \|.
\]

So there exists a unitary \( U_0 \) such that
\[
\| \sum_{i=1}^{n} A_i U_0 B_i \| = \| \delta_{\tilde{A}, \tilde{B}} \|.
\]

If \( \dim \mathcal{N}(X_0) > \dim \mathcal{N}(X_0^*) \), by Lemma 2.5, we shall consider the operator
\[
\sum_{i=1}^{n} B_i^* X_0 A_i^*
\]
by an argument similar to the above. Then there exists a unitary \( W \) such that
\[
\| \delta_{\tilde{A}, \tilde{B}} \| = \| \sum_{i=1}^{n} B_i^* X_0 A_i^* \| = \| \sum_{i=1}^{n} B_i^* W A_i^* \| = \| \sum_{i=1}^{n} A_i W^* B_i \|.
\]

The proof is completed. \qed

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