ON THE COMPACTNESS OF THE PRODUCT OF HANKEL OPERATORS ON THE SPHERE

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Abstract. Consider Hankel operators $H_\phi$ and $H_\psi$ on the unit sphere in $\mathbb{C}^n$. If $n = 1$, then a necessary condition for $H_\phi^*H_\psi$ to be compact is $\lim_{|z| \uparrow 1} \|H_\phi k_z\|\|H_\psi k_z\| = 0$. We show that when $n \geq 2$, this condition is no longer necessary for $H_\phi^*H_\psi$ to be compact.

1. Introduction

Let $S$ denote the unit sphere $\{z \in \mathbb{C}^n : |z| = 1\}$ in $\mathbb{C}^n$. Let $\sigma$ be the positive, regular Borel measure on $S$ which is invariant under the orthogonal group $O(2n)$, i.e., the group of isometries on $\mathbb{C}^n \cong \mathbb{R}^{2n}$ which fix 0. Furthermore we normalize $\sigma$ such that $\sigma(S) = 1$. The Hardy space $H^2(S)$ is the norm closure in $L^2(S, d\sigma)$ of the collection of polynomials in the complex variables $z_1, ..., z_n$ [3, Section 5.6]. Let $P : L^2(S, d\sigma) \to H^2(S)$ be the orthogonal projection. For each $\psi \in L^\infty(S, d\sigma)$, the Toeplitz operator $T_\psi : H^2(S) \to H^2(S)$ and the Hankel operator $H_\psi : H^2(S) \to L^2(S, d\sigma) \ominus H^2(S)$ are respectively defined by the formulas

$$T_\psi h = P\psi h \quad \text{and} \quad H_\psi h = (1 - P)\psi h,$$

$h \in H^2(S)$. As usual, let $k_z$ denote the normalized reproducing kernel function for $H^2(S)$. That is, for each $z \in \mathbb{C}^n$ with $|z| < 1$, we write

$$k_z(w) = \frac{(1 - |z|^2)^n/2}{(1 - \langle w, z \rangle)^n}, \quad |w| \leq 1.$$

The main motivation for this investigation comes from the following sufficient condition for the compactness of $H_\psi^*H_\psi$ due to D. Zheng.

Theorem 1.1 ([6, Theorem 3]). Let $\varphi$ and $\psi$ be in BMO. Then the operator $H_\psi^*H_\psi$ is compact if

$$\lim_{|z| \uparrow 1} \|H_\psi k_z\|\|H_\psi^* k_z\| = 0.$$

(1.1)

Also see the comments on page 22 of [6]. This raises the obvious

Question 1.2. Is (1.1) a necessary condition for the compactness of $H_\psi^*H_\psi$?

In the case $n = 1$, the answer to this question is affirmative. Indeed Zheng proved

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1375
Theorem 1.3 ([5, Theorem 2]). Suppose that \( n = 1 \). If the operator \( H^*_\varphi H_\psi \) is compact, then
\[
\lim_{|z| \uparrow 1} \|H_\varphi k_z\|\|H_\psi k_z\| = 0.
\]

Moreover, for any complex dimension \( n \), if \( H^*_\varphi H_\psi \) is compact, then one trivially has
\[
\lim_{|z| \uparrow 1} \|H_\varphi k_z\|^2 = 0.
\]

Given these two facts, and the fact that (1.1) is such a natural-looking condition, one might be tempted to “extrapolate” that the answer to Question 1.2 is affirmative for all \( n \in \mathbb{N} \). The purpose of this paper is to report that that is not the case. In other words, Theorem 1.3 is actually something of an anomaly; in the case \( n \geq 2 \), (1.1) is not a necessary condition for the compactness of \( H^*_\varphi H_\psi \). More precisely, we will prove

Theorem 1.4. For each complex dimension \( n \geq 2 \), there exists a pair of functions \( \varphi \) and \( \psi \) in \( L^\infty(S, d\sigma) \) such that
\[
\limsup_{|z| \uparrow 1} \|H_\varphi k_z\|\|H_\psi k_z\| > 0
\]
and such that the operator \( H^*_\varphi H_\psi \) is compact.

This result tells us something that is somewhat anti-intuitive: while Theorem 1.1 cannot be improved in the case \( n = 1 \), for \( n \geq 2 \) one should try to prove the compactness of \( H^*_\varphi H_\psi \) under a condition that is weaker than (1.1)! This leads to the following question for future investigations.

Question 1.5. For \( n \geq 2 \), what is a necessary and sufficient condition for the compactness of \( H^*_\varphi H_\psi \)?

The basic idea behind Theorem 1.4 is the following. First of all, the problem can be converted to a problem for the product of Toeplitz operators. That is, if \( f \) and \( g \) are real valued and have disjoint supports, then \( H^*_f H_g = -T_f T_g \). If there is a positive distance between the supports of \( f \) and \( g \), then \( T_f T_g \) is compact. Furthermore, if \( f \) and \( g \) depend only on \( |z_1|, \ldots, |z_{n-1}| \), then \( T_f \) and \( T_g \) are diagonal operators with respect to the standard orthonormal basis \( \{e_i : i \in \mathbb{Z}_+^n\} \) in \( H^2(S) \). Therefore \( T_f T_g \) is a diagonal operator with eigenvalues \( \langle fe_i, e_i \rangle \langle ge_i, e_i \rangle, i \in \mathbb{Z}_+^n \). Thus in order for \( \|T_f T_g\| \) to be small, it suffices if one of the two factors \( \|fe_i, e_i\| \), \( \|ge_i, e_i\| \) is small for each \( i \in \mathbb{Z}_+^n \). The fact that we have two factors to manipulate allows us to construct \( f \) and \( g \) such that \( \|T_f T_g\| \) is arbitrarily small while \( \|H_f k_0\|\|H_g k_0\| \) has a predetermined lower bound. The desired functions \( \varphi \) and \( \psi \) are then obtained from a sequence of such \( f \)'s, a sequence of such \( g \)'s, and Möbius transforms. The lower bound for \( \|H_f k_0\|\|H_g k_0\| \) translates into (1.2), and the smallness of \( \|T_f T_g\| \) results in the compactness of \( H^*_\varphi H_\psi \).

The rest of the paper consists of the details of what we described above. Specifically, Section 2 contains the key step, Lemma 2.2. Section 3 recalls Möbius transforms and associated unitary operators. The proof of Theorem 1.4 is completed in Section 4.

For the rest of the paper we assume \( n \geq 2 \).
2. TWO TOEPLITZ OPERATORS

Let $Q$ denote the “first quadrant” of the closed unit ball in $\mathbb{R}^{n-1}$. In other words,

$$Q = \{(x_1, ..., x_{n-1}) \in \mathbb{R}^{n-1} : x_1^2 + ... + x_{n-1}^2 \leq 1 \text{ and } x_1 \geq 0, ..., x_{n-1} \geq 0\}.$$ 

On the compact set $Q$ we define the measure $d\mu$ by the formula

$$d\mu(x_1, ..., x_{n-1}) = (n-1)!2^{n-1}x_1...x_{n-1}dx_1...dx_{n-1}.$$ 

Using the technique described on page 17 of [3], it is straightforward to verify that

$$\int_Q x_1^{2i_1}...x_{n-1}^{2i_{n-1}}d\mu(x_1, ..., x_{n-1}) = \frac{(n-1)!i_1!...i_{n-1}!}{(n-1 + i_1 + ... + i_{n-1})!}$$

for all integers $i_1, ..., i_{n-1}$ in $\mathbb{Z}_+$. But for such $i_1, ..., i_{n-1}$ we also have

$$\int_S |z_1|^{2i_1}...|z_{n-1}|^{2i_{n-1}}d\sigma(z_1, ..., z_{n-1}, z_n) = \frac{(n-1)!i_1!...i_{n-1}!}{(n-1 + i_1 + ... + i_{n-1})!}$$

[3, Proposition 1.4.9]. Hence, by the Stone-Weierstrass approximation theorem, we have

$$(2.1) \quad \int_S \xi(|z_1|, ..., |z_{n-1}|)d\sigma(z_1, ..., z_{n-1}, z_n) = \int_Q \xi(x_1, ..., x_{n-1})d\mu(x_1, ..., x_{n-1})$$

for every $\xi \in C(S)$.

We will use the usual multi-index notation [3, page 3]. For each $i \in \mathbb{Z}_+^n$, define

$$c_i = \left\{ \frac{(n-1 + |i|)!}{(n-1)!i!} \right\}^{1/2}$$

and the function

$$e_i(z) = c_iz^i, \quad z \in S.$$ 

Then $\{e_i : i \in \mathbb{Z}_+^n\}$ is the standard orthonormal basis for $H^2(S)$ [3, Proposition 1.4.9].

For the rest of the paper, we set

$$\delta = \frac{1}{200(n-1)^{1/2}}.$$ 

With this fixed $\delta$, we define the subsets $A$ and $B$ of $S$ as follows:

$$A = \{(z_1, ..., z_{n-1}, z_n) \in S : \delta < |z_j| < 2\delta \text{ for } 1 \leq j \leq n-1\},$$

$$B = \{(z_1, ..., z_{n-1}, z_n) \in S : (n-1)^{-1/2} - \delta < |z_j| < (n-1)^{-1/2} - \delta \text{ for } 1 \leq j \leq n-1\}.$$ 

**Lemma 2.1.** (i) Let $f \in C(S)$ be such that $\|f\|_\infty \leq 1$. Furthermore, suppose that the support of $f$ is contained in $A$. Then for every $i = (i_1, ..., i_{n-1}, i_n)$ in $\mathbb{Z}_+^n$ satisfying the condition

$$i_1 + ... + i_{n-1} \geq i_n$$

we have $|\langle fe_i, e_i \rangle| \leq 2^{-|i|/2}$.

(ii) Let $g \in C(S)$ be such that $\|g\|_\infty \leq 1$. Furthermore, suppose that the support of $g$ is contained in $B$. Then for every $i = (i_1, ..., i_{n-1}, i_n)$ in $\mathbb{Z}_+^n$ satisfying the condition

$$i_1 + ... + i_{n-1} \leq i_n$$

we have $|\langle ge_i, e_i \rangle| \leq 2^{n-1}(10/3)^{-|i|/4}$. 


Proof. (i) Since \( \|f\|_\infty \leq 1 \) and the support of \( f \) is contained in \( A \), it is an easy consequence of (2.1) that for every \( i = (i_1, \ldots, i_{n-1}, i_n) \in \mathbb{Z}_+^n \) we have

\[
|\langle fe_i, e_i \rangle| \leq \int_A |e_i|^2 d\sigma = c_i^2 \int_A y_{i_1}^{2i_1} \cdots y_{i_{n-1}}^{2i_{n-1}} (1 - y_1^2 - \cdots - y_{n-1}^2)^i \, d\mu(x_1, \ldots, x_{n-1}),
\]

where

\[
\tilde{A} = \{(x_1, \ldots, x_{n-1}) : \delta < x_j < 2\delta \text{ for } 1 \leq j \leq n-1\},
\]

which is contained in \( Q \). By the definition of \( \delta \), \( Q \) also contains the set

\[
\tilde{C} = \{(y_1, \ldots, y_{n-1}) : 4\delta < y_j < 5\delta \text{ for } 1 \leq j \leq n-1\}.
\]

Also by the definition of \( \delta \), if \( (y_1, \ldots, y_{n-1}) \in \tilde{C} \), then \( y_1^2 + \cdots + y_{n-1}^2 < 1/5 \). On the other hand, if \( (x_1, \ldots, x_{n-1}) \in \tilde{A} \), then \( x_j + 3\delta > x_j + x_j = 2x_j \) for every \( 1 \leq j \leq n-1 \). Hence

\[
1 = \int_S |e_i|^2 d\sigma = c_i^2 \int_Q y_{i_1}^{2i_1} \cdots y_{i_{n-1}}^{2i_{n-1}} (1 - y_1^2 - \cdots - y_{n-1}^2)^i \, d\mu(y_1, \ldots, y_{n-1})
\]

\[
\geq c_i^2 \int_{\tilde{C}} y_{i_1}^{2i_1} \cdots y_{i_{n-1}}^{2i_{n-1}} (4/5)^i \, d\mu(y_1, \ldots, y_{n-1})
\]

\[
= (4/5)^i c_i^2 (n-1)!^2 (n-1)!^2 \int_A y_{i_1}^{2i_1} \cdots y_{i_{n-1}}^{2i_{n-1}} d\mu(x_1, \ldots, x_{n-1})
\]

\[
\geq (4/5)^i 2^i 2^{i_1+\cdots+i_{n-1}} \int_A x_{i_1}^{2i_1} \cdots x_{i_{n-1}}^{2i_{n-1}} \, d\mu(x_1, \ldots, x_{n-1})
\]

\[
\geq (4/5)^i 2^i 2^{i_1+\cdots+i_{n-1}} |\langle fe_i, e_i \rangle|.
\]

If \( i_1 + \cdots + i_{n-1} \geq i_n \), then \( (4/5)^i 2^i 2^{i_1+\cdots+i_{n-1}} \geq 2^i 2^{i_1+\cdots+i_{n-1}} \geq 2^{|i|}/2 \). This proves (i).

(ii) Since \( \|g\|_\infty \leq 1 \) and the support of \( g \) is contained in \( B \), it is an easy consequence of (2.1) that for every \( i = (i_1, \ldots, i_{n-1}, i_n) \in \mathbb{Z}_+^n \) we have

\[
|\langle ge_i, e_i \rangle| \leq \int_B |e_i|^2 d\sigma = c_i^2 \int_B x_{i_1}^{2i_1} \cdots x_{i_{n-1}}^{2i_{n-1}} (1 - x_1^2 - \cdots - x_{n-1}^2)^i \, d\mu(x_1, \ldots, x_{n-1}),
\]

where

\[
\tilde{B} = \{(x_1, \ldots, x_{n-1}) : (n-1)^{-1/2} - \delta < x_j < (n-1)^{-1/2} \text{ for } 1 \leq j \leq n-1\},
\]

which is contained in \( Q \). By the definition of \( \delta \), \( Q \) also contains the set

\[
\tilde{D} = \{(y_1, \ldots, y_{n-1}) : (n-1)^{-1/2} - 6\delta < y_j < (n-1)^{-1/2} - 5\delta \text{ for } 1 \leq j \leq n-1\}.
\]

The choice of \( \delta \) ensures that if \( (x_1, \ldots, x_{n-1}) \in \tilde{B} \), then

\[
1 - x_1^2 - \cdots - x_{n-1}^2 \leq 1/100,
\]

\[
1 - (x_1 - 5\delta)^2 - \cdots - (x_{n-1} - 5\delta)^2 \geq 1/30,
\]

and

\[
x_j - 5\delta \geq (9/10)x_j \text{ for } 1 \leq j \leq n-1.
\]
Therefore
\[
1 = \int_S |e_i|^2 d\sigma \geq c_i^2 \int_D y_1^{2i_1} \ldots y_{n-1}^{2i_{n-1}} (1 - y_1^2 - \ldots - y_{n-1}^2)^i_n d\mu(y_1, \ldots, y_{n-1})
\]
\[
= c_i^2 (n-1)!^2 2^{n-1} \int_D y_1^{2i_1+1} \ldots y_{n-1}^{2i_{n-1}+1} (1 - y_1^2 - \ldots - y_{n-1}^2)^i_n dy_1 \ldots dy_{n-1}
\]
\[
= c_i^2 (n-1)!^2 2^{n-1} \int_B \prod_{j=1}^{n-1} (x_j - 5\delta)^{2i_j+1} \left( 1 - \sum_{j=1}^{n-1} (x_j - 5\delta)^2 \right)^{i_n} dx_1 \ldots dx_{n-1}
\]
\[
\geq (9/10)^{2(i_1+\ldots+i_{n-1})+n-1}(10/3)^{i_n}
\]
\[
\times c_i^2 (n-1)!^2 2^{n-1} \int_B x_1^{2i_1+1} \ldots x_{n-1}^{2i_{n-1}+1} (1 - x_1^2 - \ldots - x_{n-1}^2)^i_n dx_1 \ldots dx_{n-1}
\]
\[
= (9/10)^{2(i_1+\ldots+i_{n-1})+n-1}(10/3)^{i_n}
\]
\[
\times c_i^2 \int_B x_1^{2i_1} \ldots x_{n-1}^{2i_{n-1}} \left( 1 - \sum_{j=1}^{n-1} x_j^2 \right)^{i_n} d\mu(x_1, \ldots, x_{n-1})
\]
\[
\geq 2^{-(n-1)} (9/10)^{2(i_1+\ldots+i_{n-1})}(10/3)^{i_n} |\langle ge_i, e_i \rangle|.
\]
Since \((9/10)^{2(10/3)^{1/2}} > 1\), if \(i_n \geq i_1 + \ldots + i_{n-1}\), then \((9/10)^{2(i_1+\ldots+i_{n-1})}(10/3)^{i_n} \geq (10/3)^{i_n/2} \geq (10/3)^{|i|/4}\). This completes the proof. \(\square\)

For each \(f \in L^2(S, d\sigma)\), denote
\[
\text{Var}(f) = \int \left| f - \int f d\sigma \right|^2 d\sigma.
\]

Lemma 2.2. For any given \(\epsilon > 0\), there exist real-valued \(\tilde{f}, \tilde{g} \in C(Q)\) with \(\|	ilde{f}\|_\infty \leq 1\) and \(\|	ilde{g}\|_\infty \leq 1\) such that the functions \(f\) and \(g\) defined by the formulas
\[
(2.4)
\]
\[
f(z_1, \ldots, z_{n-1}, z_n) = \tilde{f}(|z_1|, \ldots, |z_{n-1}|) \quad \text{and} \quad g(z_1, \ldots, z_{n-1}, z_n) = \tilde{g}(|z_1|, \ldots, |z_{n-1}|),
\]
\((z_1, \ldots, z_{n-1}, z_n) \in S\), have the following properties:
\[
(\alpha) \text{ The support of } f \text{ is contained in } A \text{ and the support of } g \text{ is contained in } B.
\]
\[
(\beta) \text{ Var}(f) \geq (1/3)\delta^{2(n-1)} \text{ and } \text{Var}(g) \geq (1/3)\delta^{2(n-1)}.
\]
\[
(\gamma) \|T_f T_g\| \leq \epsilon.
\]

Proof. Given \(\epsilon > 0\), let \(N \in \mathbb{N}\) be such that \(2n-1(10/3)^{-N/4} \leq \epsilon\). We first show that there is a real-valued \(\tilde{f} \in C(Q)\) with \(\|	ilde{f}\|_\infty \leq 1\) such that the function \(f\) defined by (2.4) has the following properties:
\[
(\text{i}) \text{ The support of } f \text{ is contained in } A.
\]
\[
(\text{ii}) |\langle f e_i, e_i \rangle| \leq \epsilon \text{ if } |i| \leq N.
\]
\[
(\text{iii}) \text{Var}(f) \geq (1/3)\delta^{2(n-1)}.
\]

To construct such an \(f\), let \(F = \{i \in \mathbb{Z}^n : |i| \leq N\}\) and let \(dm_{n-1}\) denote the standard Lebesgue measure on \(\mathbb{R}^{n-1}\). For each \(i = (i_1, \ldots, i_{n-1}, i_n) \in F\), define the function
\[
(2.5) u_i(x_1, \ldots, x_{n-1}) = c_i^2 (n-1)!^2 2^{n-1} x_1^{2i_1+1} \ldots x_{n-1}^{2i_{n-1}+1} (1 - x_1^2 - \ldots - x_{n-1}^2)^i_n
\]
on \(Q\). Since each \(u_i\) is continuous and since \(\text{card}(F) < \infty\), for the given \(\epsilon\) we can decompose the cube \(A\) defined by (2.2) as the union of a finite family of pairwise
disjoint subcubes \( \{ \tilde{A}_j : j \in J \} \) such that for each \( j \in J \) and each \( i \in F \), we have
\[
|u_i(x) - u_i(y)| \leq \epsilon \quad \text{for all } x, y \in \tilde{A}_j.
\]
Now, for each \( j \in J \), it is elementary to construct a real-valued function \( \tilde{f}_j \in C(Q) \) which has the following properties:
(a) The support of \( \tilde{f}_j \) is contained in the interior of \( \tilde{A}_j \).
(b) \(-1 \leq \tilde{f}_j \leq 1\).
(c) \(m_{n-1}(\{x : \tilde{f}_j(x) = 1\}) \geq (1/3)m_{n-1}(\tilde{A}_j)\).
(d) \(m_{n-1}(\{x : \tilde{f}_j(x) = -1\}) \geq (1/3)m_{n-1}(\tilde{A}_j)\).
(e) \(\int_{\tilde{A}_j} \tilde{f}_j dm_{n-1} = 0\).
Define \( \tilde{f} = \sum_{j \in J} \tilde{f}_j \). Then \( \tilde{f} \in C(Q) \), \(-1 \leq f \leq 1\), and the support of \( \tilde{f} \) is contained in \( \tilde{A} \). Hence the support of \( f \) is contained in \( A \), verifying (i).
To verify (ii), apply (2.1), (2.5) and (e). For each \( i \in F \) we have
\[
\langle f e_i, e_i \rangle = \int_Q u_i \tilde{f} dm_{n-1} = \sum_{j \in J} \int_{\tilde{A}_j} u_i \tilde{f}_j dm_{n-1} = \sum_{j \in J} \int_{\tilde{A}_j} (u_i - u_i(a_j)) \tilde{f}_j dm_{n-1},
\]
where \( a_j \) is any chosen point in \( \tilde{A}_j \). Combining this with (2.6), we conclude that
\[
|\langle f e_i, e_i \rangle| \leq \sum_{j \in J} m_{n-1}(\tilde{A}_j) \epsilon = m_{n-1}(\tilde{A}) \epsilon \leq \epsilon,
\]
i \( i \in F \). To prove (iii), observe that (c) and (d) together give us the estimate
\[
\int_{\tilde{A}} |\tilde{f} - c|^2 dm_{n-1} \geq \sum_{j \in J} \frac{1}{3} m_{n-1}(\tilde{A}_j) = \frac{1}{3} m_{n-1}(\tilde{A}) = \frac{1}{3} \delta^{n-1}
\]
for every \( c \in C \). By (2.1) and the fact that \( x_1 ... x_{n-1} \geq \delta^{n-1} \) if \( (x_1, ..., x_{n-1}) \in \tilde{A} \), we have
\[
\int_{Q} |f - c|^2 d\mu = \int_{Q} |\tilde{f} - c|^2 d\mu \geq \delta^{n-1} \int_{\tilde{A}} |\tilde{f} - c|^2 dm_{n-1} \geq \frac{1}{3} \delta^{2(n-1)}.
\]
This proves (iii) and completes the construction of \( f \).
To construct \( \tilde{g} \), consider the cube \( \tilde{B} \) defined by (2.3). It is elementary that there is a real-valued \( \tilde{g} \in C(Q) \) which has the following properties:
(1) The support of \( \tilde{g} \) is contained in the interior of \( \tilde{B} \).
(2) \(-1 \leq \tilde{g} \leq 1\).
(3) \(m_{n-1}(\{x : \tilde{g}(x) = 1\}) \geq (1/3)m_{n-1}(\tilde{B})\).
(4) \(m_{n-1}(\{x : \tilde{g}(x) = -1\}) \geq (1/3)m_{n-1}(\tilde{B})\).
Then (3) and (4) together imply that
\[
\int_{\tilde{B}} |\tilde{g} - c|^2 dm_{n-1} \geq \frac{1}{3} m_{n-1}(\tilde{B}) = \frac{1}{3} \delta^{n-1}
\]
for every \( c \in C \). By (2.3) and the definition of \( \delta \), if \( (x_1, ..., x_{n-1}) \in \tilde{B} \), then \( x_1 ... x_{n-1} \geq ((n-1)^{-1/2} - \delta)n^{-1} \geq \delta^{n-1} \). Thus it follows from (2.4) and (2.1) that
\[
\int_{Q} |\tilde{g} - c|^2 d\mu = \int_{Q} |\tilde{g} - c|^2 d\mu \geq \delta^{n-1} \int_{\tilde{B}} |\tilde{g} - c|^2 dm_{n-1} \geq \frac{1}{3} \delta^{2(n-1)}.
\]
This establishes (\( \alpha \)) and (\( \beta \)).
To prove (γ), note that (2.4) implies \( \langle fe_i, e_{i'} \rangle = 0 = \langle ge_i, e_{i'} \rangle \) for all \( i \neq i' \) in \( \mathbb{Z}_+^n \). Thus the Toeplitz operators \( T_f \) and \( T_g \) are diagonal operators with respect to the orthonormal basis \( \{ e_i : i \in \mathbb{Z}_+^n \} \). Consequently

\[
T_f T_g = \sum_{i \in \mathbb{Z}_+^n} \langle fe_i, e_i \rangle \langle ge_i, e_i \rangle e_i \otimes e_i.
\]

By Lemma 2.1, \( \langle fe_i, e_i \rangle \langle ge_i, e_i \rangle \leq 2^{n-1}(10/3)^{-|i|/4} \) for every \( i \in \mathbb{Z}_+^n \). By the choice of \( N \), this gives us \( \langle fe_i, e_i \rangle \langle ge_i, e_i \rangle \leq \epsilon \) in the case \( |i| \geq N \). But when \( |i| \leq N \), it follows from property (ii) for \( f \) that \( \langle fe_i, e_i \rangle \leq \epsilon \). Hence \( \|T_f T_g\| \leq \epsilon \).

### 3. Möbius transform

For each \( z \in \mathbb{C}^n \) with \( 0 < |z| < 1 \), define the Möbius transform

\[
\varphi_z(w) = \frac{1}{1 - \langle w, z \rangle} \left\{ z - \frac{\langle w, z \rangle}{|z|^2} z - (1 - |z|^2)^{1/2} \left( w - \frac{\langle w, z \rangle}{|z|^2} z \right) \right\}, \quad |w| \leq 1.
\]

Then \( \varphi_z \) is an involution, i.e., \( \varphi_z \circ \varphi_z = id \) [3, Theorem 2.2.2]. Recall that the formula

\[
(U_z f)(w) = f(\varphi_z(w))k_z(w)
\]

defines a unitary operator on \( L^2(S, d\sigma) \) with the property \( [U_z, P] = 0 \) [4, Section 6]. Therefore for any \( f \in L^\infty(S, d\sigma) \), \( \|H_f \varphi_z k_z\| = \|(1 - P)U_z f\| = \|(1 - P)f\| \). If \( f \) is a real-valued function, then \( 2\|f(1 - P)f\|^2 \geq \text{Var}(f) \) [4, (6.3)]. Thus we conclude that

\[
\|H_f \varphi_z k_z\|^2 \geq \frac{1}{2} \text{Var}(f)
\]

for every real-valued \( f \in L^\infty(S, d\sigma) \).

Recall that the formula \( d(u, v) = |1 - \langle u, v \rangle|^{1/2} \), \( u, v \in S \), defines a metric on \( S \) [3, page 66]. For \( u \in S \) and \( a > 0 \), let \( B(u, a) \) denote the open ball with respect to the metric \( d \). That is, we write

\[
B(u, a) = \{ v \in S : |1 - \langle u, v \rangle|^{1/2} < a \}.
\]

**Lemma 3.1.** Let \( 0 < a < 1 \). If \( 1 - (a^4/4) < r < 1 \), then for every \( u \in S \) we have

\[
\varphi_{ru}(S\backslash B(u, a)) \subset B(u, a).
\]

**Proof.** It is easy to see that \( \varphi_{ru}(-u) = u \) if \( 0 < r < 1 \) and \( u \in S \). For such \( r \) and \( u \), it follows from [3, Theorem 2.2.2] that

\[
1 - \langle \varphi_{ru}(w), u \rangle = 1 - \langle \varphi_{ru}(w), \varphi_{ru}(-u) \rangle = \frac{(1 - r)(1 + \langle w, u \rangle)}{1 - r \langle w, u \rangle}.
\]

It is elementary that if \( |c| \leq 1 \) and \( 0 < r < 1 \), then \( 2|1 - rc| \geq |1 - c| \). Hence

\[
|1 - \langle \varphi_{ru}(w), u \rangle| \leq \frac{4(1 - r)}{|1 - \langle w, u \rangle|}
\]

for \( 0 < r < 1 \) and \( w, u \in S \). Therefore if \( 1 - r < a^4/4 \) and \( |1 - \langle w, u \rangle| \geq a^2 \), then \( |1 - \langle \varphi_{ru}(w), u \rangle| \leq a^2 \). That is, if \( 1 - (a^4/4) < r < 1 \) and \( w \in S \backslash B(u, a) \), then \( \varphi_{ru}(w) \in B(u, a) \).
4. Proof of Theorem 1.4

Let $A$, $B$ be the same as in Section 2. Then the open set $V = S \setminus (\bar{A} \cup \bar{B})$ is obviously not empty. Thus there exist a sequence of points $\{u_j\}_{j=1}^\infty$ in $V$ and a sequence of positive numbers $\{a_j\}_{j=1}^\infty$ with $\lim_{j \to \infty} a_j = 0$ such that $B(u_j, a_j) \subset V$ for every $j$ and

$$B(u_j, 2a_j) \cap B(u_{j'}, 2a_{j'}) = \emptyset \quad \text{if} \quad j \neq j'. \quad (4.1)$$

For each $j \in \mathbb{N}$, pick an $r_j \in (1 - (a_j^4/4), 1)$. Then $\lim_{j \to \infty} r_j = 1$. Define $z(j) = r_j u_j$, $j \in \mathbb{N}$. Then Lemma 3.1 tells us that

$$\varphi_{z(j)}(S \setminus B(u_j, a_j)) \subset B(u_j, a_j) \quad (4.2)$$

for every $j$.

By Lemma 2.2, for each $j \in \mathbb{N}$ there exist real-valued $f_j, g_j \in C(S)$ such that

1. $\|f_j\|_\infty \leq 1$ and $\|g_j\|_\infty \leq 1$;
2. the support of $f_j$ is contained in $A$ and the support of $g_j$ is contained in $B$;
3. $\Var(f_j) \geq (1/3)\delta^{2(n-1)}$ and $\Var(g_j) \geq (1/3)\delta^{2(n-1)}$;
4. $\|T_{f_j}T_{g_j}\| \leq 2^{-j}$.

By (4.2) and the fact that $B(u_j, a_j) \subset V = S \setminus (\bar{A} \cup \bar{B})$, the supports of $f_j \circ \varphi_{z(j)}$ and $g_j \circ \varphi_{z(j)}$ are contained in $B(u_j, a_j)$. Combining this with (4.1), we have

$$f_j \circ \varphi_{z(j)} \cdot f_{j'} \circ \varphi_{z(j')} = 0 = g_j \circ \varphi_{z(j)} \cdot g_{j'} \circ \varphi_{z(j')} \quad \text{if} \quad j \neq j'. \quad (4.3)$$

Since $f_j g_j = 0$, we also have

$$f_j \circ \varphi_{z(j)} \cdot g_{j'} \circ \varphi_{z(j')} = 0 \quad \text{for all} \quad j, j' \in \mathbb{N}. \quad (4.4)$$

Denote $c = (1/6)\delta^{2(n-1)}$. Since $f_j, g_j$ are real-valued, (3.2) tells us that

$$\|H_{f_j \circ \varphi_{z(j)}} k_{z(j)}\| \|H_{g_j \circ \varphi_{z(j)}} k_{z(j)}\| \geq \frac{1}{2}(\Var(f_j)\Var(g_j))^{1/2} \geq c, \quad (4.5)$$

$j \in \mathbb{N}$.

It is well known that $\sigma(B(u, a)) \leq A_0 a^{2n}$ [3, Proposition 5.1.4]. Since $\|f_j\|_\infty \leq 1$ and $\|g_j\|_\infty \leq 1$ and the supports of $f_j \circ \varphi_{z(j)}$ and $g_j \circ \varphi_{z(j)}$ are contained in $B(u_j, a_j)$, we have

$$\lim_{j \to \infty} \|M_{f_j \circ \varphi_{z(j)}} h\| = 0 \quad \text{and} \quad \lim_{j \to \infty} \|M_{g_j \circ \varphi_{z(j)}} h\| = 0 \quad (4.6)$$

for every $h \in L^2(S, d\sigma)$. By (4.1) and a trivial estimate using the Cauchy integral formula for $P$ [3, Section 3.2],

$$\lim_{j \to \infty} \|M_{f_j \circ \varphi_{z(j)} P} M_{g_\nu \circ \varphi_{z(\nu)}}\| = 0 \quad \text{and} \quad \lim_{j \to \infty} \|M_{f_\nu \circ \varphi_{z(\nu)} P} M_{g_j \circ \varphi_{z(j)}}\| = 0 \quad (4.7)$$

for every $\nu \in \mathbb{N}$. Since $f_\nu \circ \varphi_{z(\nu)}, g_\nu \circ \varphi_{z(\nu)} \in C(S)$, the Hankel operators $H_{f_\nu \circ \varphi_{z(\nu)}}$ and $H_{g_\nu \circ \varphi_{z(\nu)}}$ are compact. Therefore for every $\nu \in \mathbb{N}$ we also have

$$\lim_{j \to \infty} \|H_{f_\nu \circ \varphi_{z(\nu)}} k_{z(j)}\| = 0 \quad \text{and} \quad \lim_{j \to \infty} \|H_{g_\nu \circ \varphi_{z(\nu)}} k_{z(j)}\| = 0. \quad (4.8)$$
Using (4.6), (4.7), (4.8) and a standard induction argument, we can select a strictly increasing sequence of natural numbers \( j_1 < \ldots < j_m < \ldots \) such that the inequalities

\[
\sum_{i=1}^{m-1} (\| M_{f_{j_m} \circ \varphi_{z(j_m)}} k_{z(j_i)} \| + \| M_{g_{j_m} \circ \varphi_{z(j_m)}} k_{z(j_i)} \|) \leq 2^{-m},
\]

\[
\sum_{i=1}^{m-1} (\| M_{f_{j_m} \circ \varphi_{z(j_m)}} PM g_{j_i} \circ \varphi_{z(j_i)} \| + \| M_{f_{j_i} \circ \varphi_{z(j_i)}} PM g_{j_m} \circ \varphi_{z(j_m)} \|) \leq 2^{-m},
\]

\[
\sum_{i=1}^{m-1} (\| H_{f_{j_i} \circ \varphi_{z(j_i)}} k_{z(j_m)} \| + \| H_{g_{j_i} \circ \varphi_{z(j_i)}} k_{z(j_m)} \|) \leq 2^{-m}
\]

hold for every \( m \geq 2 \).

To prove Theorem 1.4, we define

\[
\varphi = \sum_{m=1}^{\infty} f_{j_m} \circ \varphi_{z(j_m)} \quad \text{and} \quad \psi = \sum_{m=1}^{\infty} g_{j_m} \circ \varphi_{z(j_m)}.
\]

By (4.3) and the fact that \( \| f_j \|_\infty \leq 1 \) and \( \| g_j \|_\infty \leq 1 \), we have \( \| \varphi \|_\infty \leq 1 \) and \( \| \psi \|_\infty \leq 1 \). For each \( m \geq 2 \), it follows from (4.11) and (4.9) that

\[
\| H_\varphi k_{z(j_m)} \| \geq \| H_{f_{j_m} \circ \varphi_{z(j_m)}} k_{z(j_m)} \| - \sum_{i=1}^{m-1} \| H_{f_{j_i} \circ \varphi_{z(j_i)}} k_{z(j_m)} \|
\]

\[
\geq \| H_{f_{j_m} \circ \varphi_{z(j_m)}} k_{z(j_m)} \| - \sum_{i=m+1}^{\infty} \| H_{f_{j_i} \circ \varphi_{z(j_i)}} k_{z(j_m)} \|
\]

\[
\geq \| H_{g_{j_m} \circ \varphi_{z(j_m)}} k_{z(j_m)} \| - 2^{-m} - \sum_{i=m+1}^{\infty} 2^{-i}
\]

\[
= \| H_{g_{j_m} \circ \varphi_{z(j_m)}} k_{z(j_m)} \| - 2^{-m+1}.
\]

Similarly, \( \| H_\psi k_{z(j_m)} \| \geq \| H_{g_{j_m} \circ \varphi_{z(j_m)}} k_{z(j_m)} \| - 2^{-m+1} \). Combining this with (4.5), we have

\[
\| H_\varphi k_{z(j_m)} \| \| H_\psi k_{z(j_m)} \| \geq \| H_{f_{j_m} \circ \varphi_{z(j_m)}} k_{z(j_m)} \| \| H_{g_{j_m} \circ \varphi_{z(j_m)}} k_{z(j_m)} \| - 2^{-m+2}
\]

\[
\geq c - 2^{-m+2}
\]

for \( m \geq 2 \). Since \( |z(j_m)| = r_{j_m} \) and \( \lim_{m \to \infty} r_{j_m} = 1 \), this proves (1.2).

To prove that \( H_\varphi^* H_\psi \) is compact, observe that (4.4) gives us \( \varphi \psi = 0 \). Thus \( H_\varphi^* H_\psi = -T_\varphi T_\psi = -T_\varphi T_\psi \). Hence it suffices to show that \( T_\varphi T_\psi \) is compact. By (4.4) and the fact that \( f_j, g_j \) are continuous, the operator \( T_{f_j \circ \varphi_{z(j)}} T_{g_j \circ \varphi_{z(j')}} \) is compact for all \( j, j' \in \mathbb{N} \). Thus, by (4.12), to prove that \( T_\varphi T_\psi \) is compact, it suffices to show that

\[
\sum_{\ell=1}^{\infty} \sum_{m=1}^{\infty} \| T_{f_{j_\ell} \circ \varphi_{z(j_\ell)}} T_{g_{j_m} \circ \varphi_{z(j_m)}} \| < \infty.
\]
We write the above sum as $X + Y$, where

$$
X = \sum_{m=1}^{\infty} \|T_{j_m} \circ \varphi_{z(j_m)} T_{g_{j_m}} \circ \varphi_{z(j_m)}\|
$$

$$
Y = \sum_{i=1}^{\infty} \sum_{m=1}^{\infty} \left( \|T_{j_{m+i}} \circ \varphi_{z(j_{m+i})} T_{g_{j_m}} \circ \varphi_{z(j_m)}\| + \|T_{j_m} \circ \varphi_{z(j_m)} T_{g_{j_{m+i}}} \circ \varphi_{z(j_{m+i})}\| \right).
$$

By (4.10), we have

$$
Y \leq \sum_{i=1}^{\infty} \sum_{m=1}^{\infty} 2^{-(m+i)} < \infty.
$$

Recalling (3.1), we have

$$
U_{z(j)} T_{f_j} T_{g_j} U_{z(j)}^* = T_{f_j} \circ \varphi_{z(j)} T_{g_j} \circ \varphi_{z(j)}.
$$

Hence $\|T_{f_j} \circ \varphi_{z(j)} T_{g_j} \circ \varphi_{z(j)}\| = \|T_{f_j} T_{g_j}\| \leq 2^{-j}$, which leads to the conclusion $X < \infty$. This proves (4.13) and completes the proof of Theorem 1.4.

References


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