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MONOTONICITY OF THE PRINCIPAL EIGENVALUE OF THE p-LAPLACIAN IN AN ANNULUS

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Abstract. In this note we prove a monotonicity result related to the principal eigenvalue of the p-Laplacian in an annulus in $\mathbb{R}^N$.

1. Introduction

This note is motivated by [3], where the authors consider the principal eigenvalue for $-\Delta$, with respect to the Dirichlet boundary conditions, in $D(h) := B \setminus B_h$. Here B stands for the unit ball in $\mathbb{R}^N$ centered at the origin, and B_h is the ball centered at $(h, 0) \in \mathbb{R} \times \mathbb{R}^{N-1}$ with radius $a < 1$. They show the principal eigenvalue as a function of h decreasing over the interval $[0, 1-a]$.

In this work we show a similar result, but for $-\Delta_p$. For $p \in (1, \infty)$, Δ_p denotes the usual p-Laplacian operator; that is $\Delta_p u = \nabla \cdot (|\nabla u|^{p-2} \nabla u)$. Let us describe the problem. We fix $-1 < a_0 < 0$, and consider $f \in L^\infty(B)$. We assume $f^+(x) := \max(f(x), 0)$ is supported in $S := \{x \in B : \ x_1 < a_0\}$ and does not vanish identically. Also, we assume $f^- := f^+ - f$ is non-increasing in the x_1-variable. Let us consider the following eigenvalue problem:

$$\text{(BVP)} \quad \begin{cases} -\Delta_p u = \lambda f(x)|u|^{p-2}u & \text{in } D(h), \\ u = 0 & \text{on } \partial D(h). \end{cases}$$

By $\lambda(h)$ we denote the principal eigenvalue of (BVP), and by $u(h)$ the corresponding positive (principal) eigenfunction normalized by

$$\int_{D(h)} f(x) u^p(h)\, dx = 1. \tag{1}$$

It is well known that $\lambda(h)$ is simple, $u(h) \in C^1(\overline{D(h)})$, and $\partial u(h)/\partial \nu < 0$ on $\partial D(h)$, where ν stands for the unit outward normal on $\partial D(h)$; see for example [2]. The main result of the paper is the following

Theorem 1. *The principal eigenvalue $\lambda(h)$ is decreasing on $[-a_0, 1-a]$.*

The idea of the proof of Theorem 1 is straightforward; we show the derivative of $\lambda(h)$ is non-positive on $[-a_0, 1-a]$. This is the same idea employed in [3], however

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in our case the technicalities are very different. In particular we use the concept of domain derivative to a great extent, and the reference we use for this is [4].

2. Domain derivative

Throughout this section Ω denotes a bounded smooth domain in $\mathbb{R}^N$. For a vector field $V \in C^2(\mathbb{R}^N, \mathbb{R}^N)$, and non-negative t, we denote by Ω_t the image of Ω under the map $I + tV$. It is well known that when t is small the map $I + tV$ is a diffeomorphism, hence Ω_t would be as "good" as Ω, roughly speaking.

Suppose $\mathcal{L}$ is a differential operator that maps $W^{m,p}(\Omega')$ into $\mathcal{D}'(\Omega')$, the space of distributions on Ω', where Ω' is any open set inside Ω. Suppose w is the solution of the following boundary value problem:

$$\begin{cases} \mathcal{L}u = 0 & \text{in } \Omega', \\ u = 0 & \text{on } \partial\Omega'. \end{cases} \tag{2}$$

Also suppose w^t is the solution of (2) with Ω' replaced by Ω'_t. Then the function w' defined on Ω' by

$$w'(x) = \lim_{t \to 0^+} \frac{w^t(x^t) - w(x)}{t}$$

where $x^t := x + tV(x)$ is called the domain derivative of w at Ω' in the direction of V. In practice w' usually satisfies

$$\begin{cases} \frac{\partial \mathcal{L}}{\partial u} w' = 0 & \text{in } \Omega', \\ w' = -\frac{\partial w}{\partial \nu}\, V \cdot \nu & \text{on } \partial\Omega', \end{cases}$$

where ν stands for the unit outward normal to $\partial\Omega'$. The reader is referred to [4] for a thorough discussion on domain derivatives. As an example we consider the (BVP) introduced in the previous section. For simplicity we write u and λ in place of $u(h)$ and $\lambda(h)$, respectively. Then we have the following result

Lemma 1. *With the notation introduced above, u satisfies*

$$\begin{cases} -\nabla \cdot \left((p-2)|\nabla u|^{p-1} \frac{\nabla u \cdot \nabla u'}{|\nabla u|^3}\, \nabla u + |\nabla u|^{p-2} \nabla u'\right) \\ \qquad = \lambda' f(x) u^{p-1} + (p-1)\lambda f(x) u^{p-2} u' & \textit{in } D(h), \\ u' = -\frac{\partial u}{\partial \nu}\, V \cdot \nu & \textit{on } \partial D(h). \end{cases} \tag{3}$$

Proof. The boundary equation in (3) follows directly from [4, Theorem 3.2, p. 664], and the right hand side of the differential equation follows from the product rule for domain derivatives, noting that $f(x)$ does not depend on the domain. To derive the left hand side of the differential equation we apply [4, Theorem 3.1, p. 663]. Indeed since the domain differential operator commutes with the divergence operator we only need to show

$$\left(|\nabla u|^{p-2} \nabla u\right)' = (p-2)|\nabla u|^{p-1} \frac{\nabla u \cdot \nabla u'}{|\nabla u|^3}\, \nabla u + |\nabla u|^{p-2} \nabla u',$$

where the "prime" denotes the domain derivative. The identity

$$|\nabla u|^{p-2} \nabla u = (\nabla u \cdot \nabla u)^{\frac{p-2}{2}} \nabla u$$

is clear, so when we differentiate both sides, applying the product rule to the right hand side, we obtain

$$\begin{aligned}\left(|\nabla u|^{p-2}\nabla u\right)' &= \frac{p-2}{2}(\nabla u\cdot\nabla u)^{\frac{p-2}{2}-1}(\nabla u'\cdot\nabla u+\nabla u\cdot\nabla u')\nabla u\\ &+ (\nabla u\cdot\nabla u)^{\frac{p-2}{2}}\nabla u'\\ &= (p-2)|\nabla u|^{p-1}\frac{\nabla u\cdot\nabla u'}{|\nabla u|^3}\,\nabla u+|\nabla u|^{p-2}\nabla u',\end{aligned}$$

as desired. □

3. Proof of Theorem 1

In this section we prove Theorem 1, for which we need the following "weak comparison principle".

Theorem 2. *Let $\Omega\subset\mathbb{R}^N$ be a bounded domain with piecewise smooth boundary $\partial\Omega$. Suppose ω is an open smooth subset of $\partial\Omega$. Suppose $g:\Omega\times[0,\infty)\to[0,\infty)$ is a Carathèodory function such that $g(x,\cdot)$ is non-decreasing for every $x\in\Omega$. Let $v,\ u\in C^1(\overline{\Omega})$ be non-negative weak solutions of the following differential inequalities:*

$$\left\{\begin{array}{l}-\Delta_p v+g(x,v)\le 0,\\ -\Delta_p u+g(x,u)\ge 0,\end{array}\right. \tag{4}$$

in Ω, respectively. Suppose $u=v$ on ω and $v\le u$ on $\partial\Omega\setminus\omega$. Also, suppose $\partial u/\partial\nu<0$ on ω. Then there is a neighborhood of ω, denoted $\mathcal{N}(\omega)\subset\Omega$, such that $u\ge v$ in $\mathcal{N}(\omega)$.

Proof. It is clear that if $\{x\in\Omega:\ v(x)>u(x)\}$ is empty, then the assertion of the theorem follows trivially; so we assume otherwise. It suffices to show that the boundary of the open set $\{x\in\Omega:\ v(x)>u(x)\}$ does not intersect ω. Note that since $u\ge v$ on $\partial\Omega$, $(v-u)^+\in W_0^{1,p}(\Omega)$; thus it can be used as a test function in both inequalities in (4). After multiplying each of the inequalities in (4) by $(v-u)^+$ and integrating over Ω we obtain

$$\int_\Omega|\nabla v|^{p-2}\nabla v\cdot\nabla(v-u)^+\,dx+\int_\Omega g(x,v)(v-u)^+\,dx\le 0$$

and

$$\int_\Omega|\nabla u|^{p-2}\nabla u\cdot\nabla(v-u)^+\,dx+\int_\Omega g(x,u)(v-u)^+\,dx\ge 0.$$

Thus

$$\begin{aligned}&\int_\Omega(|\nabla v|^{p-2}\nabla v-|\nabla u|^{p-2}\nabla u)\cdot\nabla(v-u)^+\,dx\\ &\le\int_\Omega(g(x,u)-g(x,v))(v-u)^+\,dx\\ &=\int_{\{v>u\}}(g(x,u)-g(x,v))(v-u)^+\,dx.\end{aligned}$$

The last integral is non-positive since $g(x,\cdot)$ is non-decreasing. This in conjunction with

$$\nabla(v-u)^+=\left\{\begin{array}{ll}\nabla(v-u) & \text{if } v>u,\\ 0 & \text{if } v\le u,\end{array}\right.$$

yields

$$\int_{\{v>u\}} (|\nabla v|^{p-2}\nabla v - |\nabla u|^{p-2}\nabla u)\cdot\nabla(v-u)\,dx \le 0. \tag{5}$$

At this stage we recall the following result, whose proof can be found in [1, Lemma 2.1]: There exists $c > 0$ such that for any $\eta,\ \eta' \in \mathbb{R}^N$ with $|\eta| + |\eta'| > 0$, the following holds:

$$(|\eta|^{p-2}\eta - |\eta'|^{p-2}\eta')\cdot(\eta-\eta') \ge c\,(|\eta| + |\eta'|)^{p-2}|\eta-\eta'|^2. \tag{6}$$

Applying (6) to (5) we find

$$c\int_{\{v>u\}} (|\nabla u| + |\nabla v|)^{p-2}|\nabla u - \nabla v|^2\,dx \le 0. \tag{7}$$

In order to derive a contradiction let us assume $\partial\{x\in\Omega:\ v(x) > u(x)\}$ intersects ω, and consider $x_0 \in \partial\{x\in\Omega:\ v(x) > u(x)\}\cap\omega$. From the hypotheses we infer $\frac{\partial u}{\partial\nu}(x_0) < 0$. Since $u \in C^1(\overline{\Omega})$, it follows that there exists a positive constant γ and an open set, relative to Ω, denoted $V(x_0)$, containing x_0 such that $|\nabla u(x)| \ge \gamma$ for every $x \in V(x_0)$. Note that $x_0 \in \partial W(x_0)$, where $W(x_0) := \{x\in\Omega:\ v(x) > u(x)\}\cap V(x_0)$. From (7) we deduce that

$$c\int_{W(x_0)} (|\nabla u| + |\nabla v|)^{p-2}|\nabla u - \nabla v|^2\,dx \le 0.$$

Hence

$$c\inf_{x\in W(x_0)}\{|\nabla u(x)| + |\nabla v(x)|\}^{p-2}\int_{W(x_0)} |\nabla(u-v)|^2\,dx \le 0.$$

Since $|\nabla u| > \gamma$ on $W(x_0)$, it follows from the above inequality that

$$c\gamma^{p-2}\int_{W(x_0)} |\nabla(u-v)|^2\,dx \le 0,$$

so $\int_{W(x_0)} |\nabla(u-v)|^2\,dx = 0$. This, in turn, implies that $u - v = const.$, in $W(x_0)$. Since $x_0 \in \partial W(x_0)$, and $u(x_0) = v(x_0)$, we infer $u \equiv v$, in $W(x_0)$. However, this is not possible since $W(x_0) \subseteq \{x\in\Omega:\ v(x) > u(x)\}$. So, as desired, $\partial\{x\in\Omega:\ v(x) > u(x)\}$ does not intersect ω. □

Remark. It is worth noting that if $p \ge 2$, the condition $\frac{\partial u}{\partial\nu} < 0$, on ω, in Theorem 2 can be dropped. Indeed, in order to show that $\partial\{x\in\Omega:\ v(x) > u(x)\}$ does not intersect ω, we proceed along the same lines as in the proof of the theorem until we reach (5). Then we use the following inequality whose proof like (6) can be found in [1, Lemma 2.1]: If $p \ge 2$, then there exists $\hat{c} > 0$ such that for every $\eta,\ \eta' \in \mathbb{R}$,

$$(|\eta|^{p-2}\eta - |\eta'|^{p-2}\eta')\cdot(\eta-\eta') \ge \hat{c}\,|\eta-\eta'|^p. \tag{8}$$

Applying (8) to (5) yields

$$\hat{c}\int_{\{v>u\}} |\nabla u - \nabla v|^p\,dx \le 0,$$

so $\int_{\{v>u\}} |\nabla u - \nabla v|^p\,dx = 0$. Whence $u - v = const.$, on $\{x\in\Omega:\ v(x) > u(x)\}$. So if $x_0 \in \partial\{x\in\Omega:\ v(x) > u(x)\}\cap\omega$, then $u \equiv v$ in $\{x\in\Omega:\ v(x) > u(x)\}$, which is false, and we have a contradiction.

We are now ready to prove Theorem 1. As in the previous section we replace $u(h)$ by u.

Proof of Theorem 2. We multiply the differential equation in (3) by u and integrate the result over $D(h)$. Thus we obtain

(9)
$$(p-1)\int_{D(h)} |\nabla u|^{p-2}\nabla u\cdot\nabla u'\,dx=\lambda'(h)\int_{D(h)} f\,u^p\,dx+(p-1)\lambda(h)\int_{D(h)} f\,u^{p-1}u'\,dx.$$

The right-hand side of (9) simplifies a great deal. Indeed from (1) we infer that the (domain) derivative of $\int_{D(h)} f\,u^p\,dx$ must be zero. Hence using [4, Theorem 3.3] we obtain

$$\left(\int_{D(h)} f\,u^p\,dx\right)' = p\int_{D(h)} f\,u^{p-1}u'\,dx + \int_{\partial D(h)} f\,u^p V\cdot\nu\,d\sigma = 0,$$

where $d\sigma$ stands for the surface measure, and the "prime" denotes the domain derivative with respect to some vector field V. Since u is zero on $\partial D(h)$ we deduce that $\int_{D(h)} fu^{p-1}u'\,dx = 0$, hence from (9) we find the following formula:

$$(p-1)^{-1}\lambda'(h) = \int_{D(h)} |\nabla u|^{p-2}\nabla u\cdot\nabla u'\,dx, \tag{10}$$

again thanks to (1). Let us recall the following characterization of the principal eigenvalue:

$$\lambda(h) = \int_{D(h)} |\nabla u|^p\,dx.$$

Hence, another application of [4, Theorem 3.3] yields

$$\lambda'(h) = p\int_{D(h)} |\nabla u|^{p-2}\nabla u\cdot\nabla u'\,dx + \int_{\partial D(h)} |\nabla u|^p V\cdot\nu\,d\sigma. \tag{11}$$

From (10) and (11) we find

$$\lambda'(h) = (1-p)\int_{\partial D(h)} |\nabla u|^p\,V\cdot\nu\,d\sigma. \tag{12}$$

Let us choose a particular vector field as follows. Fix $\tilde\phi \in C_0^\infty(B)$ such that $0 \le \tilde\phi \le 1$ and $\tilde\phi \equiv 1$ on a bounded neighborhood of B_h inside B. The zero extension of $\tilde\phi$ to the whole of $\mathbb{R}^N$ is denoted by ϕ. Finally we let $V := \phi\vec{i}$, where $\vec{i}$ is the standard unit vector in $\mathbb{R}^N$. Obviously $V \in C^2(\mathbb{R}^N;\mathbb{R}^N)$, so it can be inserted in (12). Thus we obtain

$$\lambda'(h) = (1-p)\int_{\partial B_h} |\nabla u|^p\nu_1\,d\sigma, \tag{13}$$

where ν_1 is the first component of ν. Note that since $p > 1$ the assertion of the theorem follows once we establish that the integral in (13) is non-negative. To this end we shift the coordinate axes so that the hyperplane $l = \{x \in \mathbb{R}^N :\ x_1 = 0\}$ is positioned in such a way that it passes through the center of B_h, hence for $(x_1, y) \in B_h$ we have $(-x_1, y) \in B_h$, as well; that is, B_h is symmetric with respect to l. Let us introduce some more notation. By A_l we denote the part of $D(h)$ for which $x_1 \ge 0$, and the image of A_l, with respect to l, is denoted A_l^*. Also B_h^+ will stand for the part of B_h for which $x_1 \ge 0$, and B_h^-, the part of B_h for which $x_1 \le 0$, in the new coordinate system. Next we introduce the function

$$v(x) = \begin{cases} u(x) & \text{if } x \in A_l, \\ u(x_l) & \text{if } x \in A_l^*, \end{cases}$$

where x_l stands for the reflection of x about l. Observe that since $h \in [a_0, 1-a]$, $f(x) = -f^-(x)$, for $x \in A_l^*$. Therefore

$$-\Delta_p u + \lambda f^-(x) u^{p-1} = 0, \quad x \in A_l^*. \tag{14}$$

On the other hand, using the definition of v, we have

$$\begin{aligned} -\Delta_p v(x) + \lambda f^-(x) v^{p-1}(x) &= -\Delta_p u(x_l) + \lambda f^-(x) u^{p-1}(x_l) \\ &\leq -\Delta_p u(x_l) + \lambda f^-(x_l) u^{p-1}(x_l) \\ &= -\Delta_p u(x_l) + \lambda f(x_l) u^{p-1}(x_l) \\ &= 0, \end{aligned}$$

for $x \in A_l^*$. In the inequality above we used the fact that f^- is decreasing in the x_1-variable. So we have

$$-\Delta_p v + \lambda f^-(x) v^{p-1} \leq 0, \quad x \in A_l^*. \tag{15}$$

Thus we can apply Theorem 2, with $g(x,z) := \lambda f^-(x) z^{p-1}$, $\omega := \partial B_h^-$, to (14) and (15). Hence we infer existence of a neighborhood of ∂B_h^-, denoted $\mathcal{N}(\partial B_h^-)$, such that $u \geq v$ in $\mathcal{N}(\partial B_h^-)$. So, since u and v both vanish on ∂B_h^-, we obtain

$$\frac{\partial}{\partial \nu}(u-v)(z) = \lim_{t \to 0^-} \frac{(u-v)(z+t\nu) - (u-v)(z)}{t} \leq 0, \quad z \in \partial B_h^-,$$

provided t is small enough to ensure $z + t\nu \in \mathcal{N}(\partial B_h^-)$. From the above inequality we infer $\frac{\partial u}{\partial \nu}(z) \leq \frac{\partial v}{\partial \nu}(z)$, for $z \in \partial B_h^-$. However, since $\frac{\partial v}{\partial \nu}(z) \leq 0$, for $z \in \partial B_h^-$, it follows that $|\frac{\partial u}{\partial \nu}(z)| \geq |\frac{\partial v}{\partial \nu}(z)|$, for $z \in \partial B_h^-$. Note that for any $z \in \partial B_h^-$, we have $\nabla u(z) = -|\nabla u(z)| \nu(z)$. This, in turn, implies $\frac{\partial u}{\partial \nu}(z) = -|\nabla u(z)|$, so $|\frac{\partial u}{\partial \nu}(z)| = |\nabla u(z)|$, for $z \in \partial B_h^-$. Similarly one can show $|\frac{\partial v}{\partial \nu}(z)| = |\nabla v(z)|$, for $z \in \partial B_h^-$. Whence we derive $|\nabla u(z)| \geq |\nabla v(z)|$, for $z \in \partial B_h^-$. Thus

$$\begin{aligned} \int_{\partial B_h^-} |\nabla u(z)|^p \nu_1(z)\, d\sigma(z) &\geq \int_{\partial B_h^-} |\nabla v(z)|^p \nu_1(z)\, d\sigma(z) \\ &= \int_{\partial B_h^-} |\nabla u(z_l)|^p \nu_1(z)\, d\sigma(z) \\ &= \int_{\partial B_h^+} |\nabla u(z)|^p \nu_1(z_l)\, d\sigma(z) \\ &= -\int_{\partial B_h^+} |\nabla u(z)|^p \nu_1(z)\, d\sigma(z). \end{aligned}$$

From this we get $\int_{\partial B_h^-} |\nabla u|^p \nu_1\, d\sigma \geq -\int_{\partial B_h^+} |\nabla u|^p \nu_1\, d\sigma$, hence $\int_{\partial B_h} |\nabla u|^p \nu_1\, d\sigma \geq 0$, as desired. □

Remark. From Theorem 1, we infer that if the support of f^+ is concentrated at $(-1,0) \in \mathbb{R} \times \mathbb{R}^{N-1}$, then $\lambda(h)$ is maximized when the two balls B and B_h are almost concentric.

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