PARTITIONING TRIPLES AND PARTIALLY ORDERED SETS

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Abstract. We prove that if \( P \) is a partial order and \( P \rightarrow (\omega_1)^1_\omega \), then
(a) \( P \rightarrow (\omega + \omega + 1, 4)^3 \), and
(b) \( P \rightarrow (\omega + m, n)^3 \) for each \( m, n < \omega \).
Together these results represent the best progress known to us on the following question of P. Erdős and others. If \( P \rightarrow (\omega_1)^1_\omega \), then does \( P \rightarrow (\alpha, n)^3 \) for each \( \alpha < \omega_1 \) and each \( n < \omega \)?

1. Introduction

The results presented here represent only the most recent leg of the journey begun in 1956 by P. Erdős and R. Rado when they proved the following.

**Theorem 1** (P. Erdős and R. Rado, [2]). If \( L \) is a real order, then \( L \rightarrow (\omega + m, 4)^3 \) for each \( m < \omega \).

(A real order is an uncountable linear order all of whose wellordered and anti-wellordered subsets are countable.) This result for triples stood alone for over thirty years, until 1987, when E. C. Milner and K. Prikry extended it from real orders to non-special linear orders.

**Theorem 2** (E. C. Milner and K. Prikry, [7]). If \( L \) is a non-special linear order, then \( L \rightarrow (\omega + m, 4)^3 \) for each \( m < \omega \).

(A partial order \( P \) is non-special if it cannot be decomposed into countably many anti-wellfounded subsets.) In particular, this implies that \( \omega_1 \rightarrow (\omega + m, 4)^3 \) for all \( m < \omega \), a previously unknown fact. Six years later, Milner and Prikry improved on this more specific result.

**Theorem 3** (E. C. Milner and K. Prikry, [8]). \( \omega_1 \rightarrow (\omega + \omega + 1, 4)^3 \).

In 1999, we were able to increase the four to an arbitrary finite integer, but at the cost of dropping back below \( \omega + \omega \) in the other color.

**Theorem 4** (A. L. Jones, [4]). \( \omega_1 \rightarrow (\omega + m, n)^3 \) for all \( m, n < \omega \).
The results presented below extend the domain of the results referenced above from that of the real orders, the non-special linear orders, or just \( \omega_1 \) to that of all non-special partial orders. Even so, they fall almost unbelievably short of resolving the following.

**Conjecture** (P. Erdős, E. C. Milner, K. Prikry, A. L. Jones, et al.). If \( P \) is a partial order and \( P \to (\omega)_\omega \), then \( P \to (\alpha, n)^3 \) for all \( \alpha < \omega_1 \) and all \( n < \omega \).

We remark that this conjecture is sharp in the sense that if \( P \to (\omega + 1, 4)^3 \), then necessarily \( P \to (\omega)_\omega \). The interested reader is kindly referred to [5] for more details.

## 2. Notation

Our set theoretic notation is essentially that of [3].

If \( P \) and \( Q \) are partial orders, then let \([P]^Q = \{ X \subseteq P \mid X \cong Q \}\) be the set of all subsets of \( P \) which are order-isomorphic to \( Q \). Thus, for any ordinal \( \alpha \), \([P]^\alpha\) is the set of wellordered chains of \( P \) of length \( \alpha \). For any ordinal \( \alpha \), let \([P]^{<\alpha} = \bigcup\{[P]^\beta \mid \beta < \alpha\}\), etc. Also, let \([P]^{n^{<\omega}} = \{ X \subseteq P \mid X \to (\omega)_\omega \}\) be the set of non-special subsets of \( P \). A partial order \( P \) is non-special if \( P \to (\omega)_\omega \).

If \( A \) and \( B \) are subsets of \( P \), then by \( A < B \) we mean that each element of \( A \) is less than each element of \( B \). If \( A \) is a subset of \( P \) and \( b \) is an element of \( P \), then by \( A < b \) we mean that each element of \( A \) is less than \( b \). If \( a \) is an element of \( P \) and \( B \) is a subset of \( P \), then by \( a < B \) we mean that \( a \) is less than each element of \( B \).

If \( P_1, \ldots, P_n \) and \( Q_1, \ldots, Q_n \) are partial orders, then

\[
[P_1, \ldots, P_n | Q_1^{Q_1}, \ldots, Q_n]_{Q_n} = \{ X_1 \cup \cdots \cup X_n \mid X_1 \in [P_1]^{Q_1}, \ldots, X_n \in [P_n]^{Q_n} \}.
\]

This notation will be most useful to us when \( n \) is small and each \( Q_i \) is a small ordinal. For example, if \( A \) and \( B \) are subsets of \( P \) with \( A < B \), then \([A, B]^{2,1}\) is the set of three-element chains of \( P \) with two elements in \( A \) and one element in \( B \). Similarly, \([A, B, C]^{1,1,1}\) might represent triples with one element in each of the sets \( A, B, \) and \( C \), etc.

If \( x, y \in [P]^{<\omega} \), then put \( x \subseteq y \) if \( x \neq y \) and there are \( u, v \in [P]^{<\omega} \) with \( u < v \), \( x = u \), and \( y = u \cup v \). Put \( x \ll y \) if \( x \neq y \) and there are \( u, v_0, v_1 \in [P]^{<\omega} \) with \( u < v_0 < v_1 \), \( x = u \cup v_0 \), and \( y = u \cup v_1 \).

If \( P, Q_0, \) and \( Q_1 \) are partial orders and \( m < \omega \), then the partition relation \( P \to (Q_0, Q_1)^m \) holds if for every partition \([P]^m = K_0 \cup K_1\) either there is \( X_0 \in [P]^{Q_0} \) with \([X_0]^m \subseteq K_0 \) or there is \( X_1 \in [P]^{Q_1} \) with \([X_1]^m \subseteq K_1 \).

Similarly, if \( P \) and \( Q \) are partial orders, \( \kappa \) is a cardinal, and \( m < \omega \), then the partition relation \( P \to (Q)^m_\kappa \) holds if for every partition \( f : [P]^m \to \kappa \) there is \( X \in [P]^Q \) on which \( f \) is constant (i.e., there is \( i < \kappa \) with \( f^{\sim}[X]^m \subseteq \{ i \} \)).

The study of these relations (and much of the notation defined above) was introduced by P. Erdős and R. Rado in [2].

## 3. Results

This section is devoted to our proof of the following.

**Proposition.** If \( P \) is a partial order and \( P \to (\omega)_\omega \), then

(a) \( P \to (\omega + \omega + 1, 4)^3 \), and
(b) \( P \rightarrow (\omega + m, n)^3 \) for all \( m, n < \omega \).

By a result of S. Todorčević it is enough to prove this proposition for non-special trees of cardinality less than that of the continuum, under the additional assumption that \( p = c \). We do so below, and refer the reader to either [10], [1], or [6] for a detailed explanation of this (now somewhat standard) metamathematical reduction.

Our kind reader should be familiar with several fundamental facts.

**Theorem 5** (F. P. Ramsey, [9]). \( \omega \rightarrow (\omega)^{m}_n \) for all \( m, n < \omega \).

A family \( \mathcal{F} \subseteq [\omega]^{\omega} \) is a filter base if the intersection of any finitely many of its elements is infinite. In particular, any subfamily of a non-principal ultrafilter over \( \omega \) is a filter base. A set \( X \in [\omega]^{\omega} \) is a pseudo-intersection of \( \mathcal{F} \subseteq [\omega]^{\omega} \) if \( X \subseteq^* F \) for every \( F \in \mathcal{F} \). Here, \( X \subseteq^* F \) if there is \( N < \omega \) with \( X \cap N \subseteq F \).

The cardinal \( p \) is the pseudo-intersection number, the minimal cardinality of a filter base for which there is no pseudo-intersection. Most importantly, every filter base of cardinality less than \( p \) must have a pseudo-intersection. As usual, the cardinal \( c \) is the cardinality of the continuum. Note that \( \omega_1 \leq p \leq c \).

An ultrafilter \( \mathcal{U} \) over \( \omega \) is a Ramsey ultrafilter if \( \omega \rightarrow (\mathcal{U})^{m}_n \) for all \( m, n < \omega \). If we assume either CH or Martin's Axiom (or more generally that \( p = c \)), then such filters are easily constructed.

**Theorem 6** (S. Todorčević, [10]). Non-special trees are much like \( \omega_1 \) in that

(i) if \( T \) is a non-special tree and \( f : T \rightarrow T \) is regressive (that is, \( f(t) < t \) for all \( t \in T \)), then there must be \( S \in \mathcal{T}^{n.s.} \) on which \( f \) is constant,

(ii) \( n.s. \) tree \( \rightarrow (n.s. \) tree, \( \omega + 1)^2 \), and

(iii) \( n.s. \) tree \( \rightarrow (\alpha)^n_\beta \) for all \( \alpha < \omega_1 \) and \( n < \omega \).

Each part of the above theorem is a generalization to non-special trees of an important result about \( \omega_1 \):

(i) Fodor’s (Pressing Down) Lemma,

(ii) The Erdős–Dushnik–Miller Theorem, and

(iii) The Baumgartner–Hajnal Theorem.

We refer the interested reader to [3], [2], [1], and [10] for more information on these and other related results.

If \( T \) is a non-special tree and \( P \) is a partial order, then a function \( f : T \rightarrow P \) is said to be almost always non-increasing if for every \( s \in T \) the set

\[ I_f(s) = \{ t \in T \mid s < t \land f(s) < f(t) \} \]

is special.

**Corollary 7.** If \( T \) is a non-special tree, \( W \) is a wellordering, and \( f : T \rightarrow W \) is almost always non-increasing, then there is \( S \in \mathcal{T}^{n.s.} \) on which \( f \) is constant.

**Proof.** Let \( I = \{ t \in T \mid \exists s < t \mid t \in I_f(s) \} \) be the diagonal union of \( \{ I_f(s) \mid s \in T \} \). Let \( \bar{T} = T \setminus I \). By (i), \( \bar{T} \) is non-special. Define \( [\bar{T}]^2 = K_0 \cup K_1 \) as follows. For \( s, t \in \bar{T} \) with \( s < t \), put \( \{ s, t \} \in K_0 \) if \( f(s) = f(t) \) and put \( \{ s, t \} \in K_1 \) if \( f(s) > f(t) \). Because \( W \) is wellfounded there can be no \( X_1 \in [\bar{T}]^{\omega} \) with \( [X_1]^2 \subseteq K_1 \). By (ii), there must be \( X_0 \in [\bar{T}]^{n.s.} \) with \( [X_0]^2 \subseteq K_0 \). Choose \( S \in [X_0]^{n.s.} \) with a unique minimal element. Evidently \( f \) is constant on \( S \). \( \square \)

\(^1\)Part (b) of the proposition first appeared in [4], though the proof given there was much more difficult to follow and was marred by a few typographical errors.
By (i) we can (and do henceforth) assume that each non-special tree \( T \) is well-pruned in that \( T(b) = \{ t \in T \mid b < t \} \) is non-special for each \( t \in T \). In particular, if \( A < B \) for \( A \subseteq T \) and \( b \in T \), then there must be \( B \in [T]^{n.s.} \) (namely \( B = T(b) \)) with \( A < B \).

We now proceed to our proof of the proposition. Part (a) of the proposition will follow from Lemma A and Lemma C directly, while part (b) of the proposition will follow from Lemma B and Lemma C via a straightforward inductive argument.

**Lemma A.** Let \( T \) be a non-special tree with \( |T|^3 = K_0 \cup K_1 \). If there are \( A \in [T]^\omega \) and \( B \in [T]^{n.s.} \) with \( A < B \) and \( [A,B]^{2,1} \subseteq K_0 \), then either

(i) there is \( X \in [T]^{\omega+\omega+1} \) with \( |X|^3 \subseteq K_0 \), or

(ii) there is \( Y \in [T]^4 \) with \( |Y|^3 \subseteq K_1 \).

**Proof.** Let \( \mathcal{U} \) be a non-principal ultrafilter over \( A \). Thus, for each pair \( \{r,s\} \in [B]^2 \), there are \( i_{r,s} \in \{0,1\} \) and \( A_{r,s} \subseteq \mathcal{U} \) with \( \{a,r,s\} \in K_{i_{r,s}} \) for each \( a \in A_{r,s} \). For each \( t \in B \), call the pair \( \langle x,y \rangle \) good for \( t \) if

1. \( x \in [A]^{<\omega} \), \( y \in [B]^{<\omega} \), and \( y < t \),
2. \( [x,y]^{1,2} \cup [x,y,\{t\}]^{1,1,1} \cup [y,\{t\}]^{2,1} \subseteq K_0 \),
3. \( i_{r,s} = 0 \) for each pair \( \{r,s\} \in [y \cup \{t\}]^{2,1} \).

If \( \langle x_0,y_0 \rangle \) and \( \langle x_1,y_1 \rangle \) are both good pairs for \( t \), then put \( \langle x_0,y_0 \rangle \sqcup \langle x_1,y_1 \rangle \) if both \( x_0 \sqcup x_1 \) and \( y_0 \sqcup y_1 \). Note that \( (\emptyset,\emptyset) \) is a good pair for each \( t \in B \).

**Claim.** If for some \( t \in B \) there is an infinite increasing sequence \( \langle x_0,y_0 \rangle \sqcup \langle x_1,y_1 \rangle \sqcup \langle x_2,y_2 \rangle \sqcup \cdots \) of pairs that are good for \( t \), then either (i) or (ii) holds.

**Proof.** Let \( X = \bigcup \{x_n \mid n < \omega\} \) and \( Y = \bigcup \{y_n \mid n < \omega\} \). Then \( X,Y \in [T]^{\omega} \) because \( \langle x_n,y_n \rangle \subseteq \langle x_{n+1},y_{n+1} \rangle \) for each \( n < \omega \). Note that \( X,Y \in [T]^{\omega} \) and

\[ [X,Y \cup \{t\}]^{2,1} \subseteq K_0 \]

because each \( \langle x_n,y_n \rangle \) is a good pair for \( t \). This is almost enough to make \( [X \cup Y \cup \{t\}]^3 \subseteq K_0 \); all that is lacking is that \( [X]^3 \subseteq K_0 \) and \( [Y]^3 \subseteq K_0 \). But because \( X \rightarrow (\omega,4)^3 \), either there are \( X_0 \in [X]^{\omega} \) and \( Y_0 \in [Y]^{\omega} \) with \( [X_0]^3 \subseteq K_0 \) and \( [Y_0]^3 \subseteq K_0 \), in which case \( X_0 \cup Y_0 \cup \{t\} \in [T]^{\omega+\omega+1} \) and \( [X_0 \cup Y_0 \cup \{t\}]^3 \subseteq K_0 \) (and hence (i) holds), or there is either \( Z \in [X]^4 \) or \( Z \in [Y]^4 \) with \( [Z]^3 \subseteq K_1 \) (and hence (ii) holds).

Without loss of generality, we may therefore assume that for each \( t \in B \) there is a \( \sqcup \)-maximal good pair \( \langle x_t,y_t \rangle \). Moreover, we may assume (by pressing down in \( B \), if necessary) that \( \langle x_t,y_t \rangle \) is the same pair \( \langle x,y \rangle \) for all \( t \in B \). Note that this implies that for each pair \( \{s,t\} \in [B]^2 \), either

1. there is \( r \in x \cup y \) with \( \{r,s,t\} \in K_1 \), or
2. \( i_{s,t} = 1 \).

Otherwise, any \( a \in \bigcap \{A_{p,q} \mid \{p,q\} \in [y \cup \{s\} \cup \{t\}]^2 \} \) would make \( \langle x \cup \{a\}, y \cup \{s\} \rangle \) a good pair for \( t \), contradicting the supposed maximality of \( \langle x,y \rangle \) for \( t \).

Because \( B \rightarrow (\omega+\omega+1)^{2,1}_{|x|+|y|+1} \), either there is \( r \in x \cup y \) and \( X \in [B]^{\omega+\omega+1} \) with \( \{r,s,t\} \in K_1 \) for each pair \( \{s,t\} \in [X]^2 \), or there is \( X \in [B]^{\omega+\omega+1} \) with \( i_{s,t} = 1 \) for all \( \{s,t\} \in [X]^2 \). In either case, either \( [X]^3 \subseteq K_0 \) (and hence (i) holds) or there
are $Y \in [X]^3$ and $r \in \bigcap \{A_{s,t} | \{s,t\} \in [Y]^2\}$ with $\{r\} \cup Y^3 \subseteq K_1$ (and hence (ii) holds).

**Lemma B.** Suppose that $m, n < \omega$ and that n.s. tree $\rightarrow (\omega + m, n)^3$. Let $T$ be a non-special tree with $[T]^3 = K_0 \cup K_1$. If there are $A \in [T]^{\omega}$ and $B \in [T]^{n.s.}$ with $A < B$ and $[A, B]^{2,1} \subseteq K_0$, then either

(i) there is $X \in [T]^{\omega+m}$ with $[X]^3 \subseteq K_0$, or

(ii) there is $Y \in [T]^{n+1}$ with $[Y]^3 \subseteq K_1$.

**Proof.** Let $\mathcal{U}$ be a non-principal ultrafilter over $A$. Thus, for each pair $\{r, s\} \in [B]^2$, there are $i_{r,s} \in \{0, 1\}$ and $A_{r,s} \in \mathcal{U}$ with $\{a, r, s\} \in K_{i_{r,s}}$ for each $a \in A_{r,s}$. Since $B \rightarrow (n.s. \text{ tree}, \omega)^2$, either

(1) there is $C \in [B]^{n.s.}$ with $i_{r,s} = 1$ for each pair $\{r, s\} \in [C]^2$, or

(2) there is $D \in [B]^{\omega}$ with $i_{r,s} = 0$ for each pair $\{r, s\} \in [D]^2$.

If (1) holds, then because $C \rightarrow (\omega + m, n)^3$, either there is $E \in [C]^{\omega+m}$ with $[E]^3 \subseteq K_0$ (and hence (i) holds), or there is $F \in [C]^n$ with $[F]^3 \subseteq K_1$. In the latter case, let $a$ be any element of $\bigcap \{A_{r,s} | \{r, s\} \in [F]^2\}$, and let $E = \{a\} \cup F$. Clearly, $F \in [T]^{n+1}$ and $[F]^3 \subseteq K_1$ (and hence (ii) holds).

But if (2) holds, then since $D \rightarrow (\omega, n + 1)^3$, either there is $\bar{D} \in [D]^{\omega}$ with $[\bar{D}]^3 \subseteq K_1$, or there is $F \in [D]^{n+1}$ with $[F]^3 \subseteq K_1$ (and hence (ii) holds). In the former case, choose any $E \in [\bar{D}]^m$ and let $A = \bigcap \{A_{r,s} | \{r, s\} \in [E]^2\}$. Since $A \rightarrow (\omega, n + 1)^3$, either there is $\bar{A} \in [A]^{\omega}$ with $[\bar{A}]^3 \subseteq K_0$, or there is $F \in [A]^{n+1}$ with $[F]^3 \subseteq K_1$ (and hence (ii) holds). In the former case, let $G = \bar{A} \cup E$. Clearly, $G \in [T]^{\omega+m}$ and $[G]^3 \subseteq K_0$ (and hence (i) holds). □

**Lemma C.** Let $T$ be a non-special tree with $[T] < p = c$ and $[T]^3 = K_0 \cup K_1$. Then either

(i) there are $A \in [T]^{\omega}$ and $B \in [T]^{n.s.}$ with $A < B$ and $[A, B]^{2,1} \subseteq K_0$, or

(ii) for each $n < \omega$ there is $C \in [T]^n$ with $[C]^3 \subseteq K_1$.

Informally, n.s. tree $\rightarrow ((\omega : \text{n.s. tree})^{2,1}, n)^3$ for each $n < \omega$.

**Proof.** Let $<$ be a fixed wellordering of $H(c^+)$, the collection of all sets of hereditary cardinality not greater than the continuum.

For each $n < \omega$ and $x, y \in [\omega]^n$, note that $x \ll y$ if and only if there are $m < n$, $u \in [\omega]^m$, and $v_0, v_1 \in [\omega]^{n-m}$ with $u < v_0 < v_1$, $x = u \cup v_0$, and $y = u \cup v_1$. For each $n < \omega$ and $A \in [T]^{\omega}$, there is a unique order isomorphism between $([\omega]^n, <_{\text{lex}})$ and $(A, <_T)$. For each $x \in [\omega]^n$, let $A(x)$ be the element of $A$ identified with $x$ via this isomorphism.

Fix a Ramsey ultrafilter $\mathcal{U}$ over $\omega$. For $n < \omega$, $m < n$, $A \in [T]^{\omega}$, and $b \in T$ with $A < b$, define the partition $f^b_{A,m} : [\omega]^{2n-m} \rightarrow \{0, 1\}$ as follows. Decompose each $x \in [\omega]^{2n-m}$ as $x = u \cup v_0 \cup v_1$ with $u < v_0 < v_1$, $|u| = m$, and $|v_0| = |v_1| = n - m$. Let

$$f^b_{A,m}(x) = i \quad \text{only if} \quad \{A(u \cup v_0), A(u \cup v_1), b\} \in K_i.$$

Because $\mathcal{U}$ is a Ramsey ultrafilter, there must be $i_{A,m} \in \{0, 1\}$ and $X^b_{A,m} \in \mathcal{U}$ with $f^b_{A,m}[X^b_{A,m}]^{2n-m} = \{i_{A,m}\}$. Let $X_A$ be the $<_\prec$-least $X \in [\omega]^\omega$ with $X \subseteq^* X^b_{A,m}$ for all $m < n$ and all $b \in T$ with $A < b$. Let $N^b_{A,m}$ be the least $N < \omega$ with $X_A \setminus N \subseteq X^b_{A,m}$. 

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For \( n < \omega, m < n, \) and \( A \in [T]^{\omega^n} \) let
\[
B_{A,m} = \{ b \in T \mid A < b \land i^b_{A,m} = 0 \}.
\]
For each \( n < \omega \) let \( \mathcal{B}_n \) be the collection of \( B \in [T]^{\omega \cdot n} \) for which there exist \( A \in [T]^{\omega^n} \) and \( m < n \) with \( B \subseteq B_{A,m} \).

Claim. If (i) fails, then \( \mathcal{B}_n \) is empty for all \( n < \omega \). (Or contrapositively, if \( \mathcal{B}_n \) is non-empty for some \( n < \omega \), then (i) holds.)

Proof. Fix \( n < \omega \) and suppose \( B \in \mathcal{B}_n \). Choose \( A \in [T]^{\omega^n} \) and \( m < n \) with \( B \subseteq B_{A,m} \). Because \( B \rightarrow (n.s. \text{ tree})^{\omega^1} \) there must be \( \tilde{B} \in [B]^{\omega \cdot n} \) and \( \tilde{N} < \omega \) with \( \tilde{N} = N^b_{A,m} \) for each \( b \in \tilde{B} \). Choose \( u \in [X_A]^m \) and \( \{ v_i \mid i < \omega \} \subseteq [X_A]^{n-m} \) with
\[
\tilde{N} < u < v_0 < v_1 < v_2 < \ldots.
\]
Let \( \tilde{A} = \{ A(u \cup v_i) \mid i < \omega \} \). Note that \( \tilde{A} \in [T]^{\omega^n}, \tilde{B} \in [T]^{\omega \cdot n}, \tilde{A} \prec \tilde{B}, \) and \( [\tilde{A}, \tilde{B}]^{2,1} \subseteq K_0 \). Thus (i) holds.

Suppose \( A, B \in T \). Then \( A \) is bounded in \( B \) if there is \( b \in B \) with \( A < b \). For each \( n < \omega \) let \( \mathcal{A}_n(B) \) be the collection of all \( A \in [B]^{\omega^n} \) with \( A \) bounded in \( B \) and \( \{ A(x), A(y), A(z) \} \in K_1 \) for all \( x, y, z \in [\omega]^n \) with \( x \ll y \ll z \) and \( |x \cap z| < |x \cap y| \).

Let \( \mathcal{A}_n = \mathcal{A}_n(T) \).

Claim. If \( \mathcal{B}_n \) is empty for all \( n < \omega \), then \( \mathcal{A}_n \) is non-empty for all \( n < \omega \).

Proof. Suppose that \( \mathcal{B}_n \) is empty for all \( n < \omega \). Then for each \( A \in [T]^{\omega^n} \) the set
\[
B_A = \bigcap \{ B_{A,m} \mid m < n \} = \{ b \in T \mid A < b \land \exists m < n [i^b_{A,m} = 0] \}
\]
is special. Consequently, for every \( B \in [T]^{\omega \cdot n} \), each \( A \in \mathcal{A}_n(B) \) is nicely bounded in \( B \), in that there is \( b \in B \) with \( A < b \) and \( i^b_{A,m} = 1 \) for all \( m < n \).

We will prove (by induction on \( n \)) that \( \mathcal{A}_n(B) \) is non-empty for every \( B \in [T]^{\omega \cdot n} \). Note that \( \mathcal{A}_0(B) \) and \( \mathcal{A}_1(B) \) are certainly non-empty for every \( B \in [T]^{\omega \cdot n} \), being vacuously equal to \( [B]^{\omega^1} \) and \( \{ A \mid [B]^{\omega^1} \cap B \} \in [B]^{\omega^1} \), respectively.

Fix a non-zero \( n < \omega \). Suppose that \( \mathcal{A}_n(B) \) is non-empty for every \( B \in [T]^{\omega \cdot n} \). Fix \( B \in [T]^{\omega \cdot n} \). We will prove that \( \mathcal{A}^{n+1}_1(B) \) is non-empty. For each \( b \in B \), we first try to construct a sequence \( \langle A^b_k \mid k < \omega \rangle \subseteq \mathcal{A}_n(B) \). Fix \( k < \omega \). Suppose that \( A^b_j \) has been defined for each \( j < k \). If there is \( A \in \mathcal{A}_n(B) \) with

1. \( A^b_j < A < b \) for all \( j < k \),
2. \( i^b_{A^b_j,m} = i^b_{A,m} = 1 \) for all \( j < k, m < n, \) and \( a \in A, \) and
3. \( N^b_{A^b_j,m} = N^b_{A,m} \) for all \( j < k, m < n, \) and \( a \in A, \)

then let \( A^b_k \) be the \( \prec \)-least such \( A \in \mathcal{A}_n(B) \). Otherwise, stop the construction and let \( k^b = k \).

We claim that there must be \( b \in B \) for which the construction of \( \langle A^b_k \mid k < \omega \rangle \) succeeds. If not, then by repeated application of Theorem 6 and its corollary, there must be \( \tilde{B} \in [B]^{\omega \cdot n}, \tilde{k} < \omega, \langle A_j \mid j < k \rangle \subseteq \mathcal{A}_n(B), \) and \( \langle N_{j,m} \mid j < k, m < n \rangle \subseteq \omega \) with

1. \( \tilde{k} = k^b \) for all \( b \in \tilde{B}, \)
2. \( A_j = A^b_j \) for all \( j < k \) and \( b \in \tilde{B}, \)
3. \( N_{j,m} = N^b_{A^b_j,m} \) for all \( j < k, m < n, \) and \( b \in \tilde{B} \).
Problem 1. Does $n.s.\text{ tree}$? 

The following asymmetry leaves us with the following unbalanced pair of open problems.

Problem 1. Does $\omega_1 \rightarrow (\omega + \omega + 2, 4)^3$?

Problem 2. Does n.s. tree $\rightarrow (\omega + \omega, 5)^3$?
References


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