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NORMING ALGEBRAS AND AUTOMATIC COMPLETE BOUNDEDNESS OF ISOMORPHISMS OF OPERATOR ALGEBRAS

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Abstract. We combine the notion of norming algebra introduced by Pop, Sinclair and Smith with a result of Pisier to show that if $\mathcal{A}_1$ and $\mathcal{A}_2$ are operator algebras, then any bounded epimorphism of $\mathcal{A}_1$ onto $\mathcal{A}_2$ is completely bounded provided that $\mathcal{A}_2$ contains a norming C^*-subalgebra. We use this result to give some insights into Kadison's Similarity Problem: we show that every faithful bounded homomorphism of a C^*-algebra on a Hilbert space has completely bounded inverse, and show that a bounded representation of a C^*-algebra is similar to a $*$-representation precisely when the image operator algebra λ-norms itself. We give two applications to isometric isomorphisms of certain operator algebras. The first is an extension of a result of Davidson and Power on isometric isomorphisms of CSL algebras. Secondly, we show that an isometric isomorphism between subalgebras $\mathcal{A}_i$ of C^*-diagonals $(\mathcal{C}_i, \mathcal{D}_i)$ $(i = 1, 2)$ satisfying $\mathcal{D}_i \subseteq \mathcal{A}_i \subseteq \mathcal{C}_i$ extends uniquely to a $*$-isomorphism of the $\mathcal{C}^*$-algebras generated by $\mathcal{A}_1$ and $\mathcal{A}_2$; this generalizes results of Muhly-Qiu-Solel and Donsig-Pitts.

1. Introduction and norming algebras

Central to this paper are the notions of operator spaces and their mappings. The theory surrounding these concepts has its antecedents in a result of Stinespring [30] characterizing those maps on a C^*-algebra obtained by compressing a $*$-representation to a subspace and began to flourish with the publication of the very influential papers of Arveson [1, 3]. Tremendous growth in the subject followed the abstract characterizations of operator spaces by Ruan [26] and operator algebras by Blecher, Ruan and Sinclair [5]. Details of these developments may be found in one of the several excellent texts on the subject now available; see [4, 9, 18, 23].

An interesting application of operator space theory is the following theorem of Ruan [27]: a locally compact group G is amenable if and only if its Fourier algebra $A(G)$ is operator amenable. Operator amenability is the notion of amenability appropriate for the category of operator algebras and completely bounded maps. Johnson [13] has shown that $A(SU(2))$ is not amenable as a Banach algebra. Since $SU(2)$ is compact, it is an amenable group; hence Ruan's characterization need not hold if operator amenability of $A(G)$ is replaced by Banach algebra amenability

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of $A(G)$. Other interesting and deep results of the theory concern similarity of operators and algebras. We shall say more about similarities later.

An *operator algebra* is a norm-closed subalgebra of $\mathcal{B}(\mathcal{H})$, where $\mathcal{B}(\mathcal{H})$ is the algebra of all bounded linear operators on the complex Hilbert space $\mathcal{H}$; similarly an *operator space* is a norm closed subspace of $\mathcal{B}(\mathcal{H})$. For positive integers m and n, the set $M_{mn}(\mathcal{B}(\mathcal{H}))$ of all $m \times n$ matrices over $\mathcal{B}(\mathcal{H})$ has a Banach space structure obtained by viewing $M_{mn}(\mathcal{B}(\mathcal{H}))$ as the set of all bounded linear maps from the direct sum of n copies of $\mathcal{H}$ to the direct sum of m copies of $\mathcal{H}$. Given an operator space $\mathcal{X}$, the inclusion $\mathcal{X} \subseteq \mathcal{B}(\mathcal{H})$ induces an inclusion $M_{mn}(\mathcal{X}) \subseteq M_{mn}(\mathcal{B}(\mathcal{H}))$, and the latter inclusion may be used to norm $M_{mn}(\mathcal{X})$. When $u : \mathcal{X} \to \mathcal{Y}$ is a linear map between operator spaces $\mathcal{X}$ and $\mathcal{Y}$, $u_{mn} : M_{mn}(\mathcal{X}) \to M_{mn}(\mathcal{Y})$ is the map that applies u to each entry of the matrix $X = (X_{ij}) \in M_{mn}(\mathcal{X})$, so $u_{mn}(X)_{ij} = u(X_{ij})$. If $m = n$, write $M_n(\mathcal{X})$ for $M_{nn}(\mathcal{X})$ and u_n for u_{nn}. Define $\|u\|_{\mathrm{cb}} := \sup_n \|u_n\|$. When $\|u\|_{\mathrm{cb}} < \infty$, u is called *completely bounded*, and is called *completely contractive* if $\|u\|_{\mathrm{cb}} \leq 1$. Finally, u is a *complete isometry* if u_n is an isometry for every n. The operator spaces $\mathcal{X}$ and $\mathcal{Y}$ are *completely equivalent* if there is a linear isomorphism $u : \mathcal{X} \to \mathcal{Y}$ such that both u and u^{-1} are completely bounded; if u can be taken to be a complete isometry, $\mathcal{X}$ and $\mathcal{Y}$ are called *completely isometric* operator spaces.

Let $\mathcal{A}$ be a unital operator algebra and $u : \mathcal{A} \to \mathcal{B}(\mathcal{H})$ be a homomorphism. If u is contractive (resp. isometric or bounded), examples show that it is not generally the case that u is completely contractive (resp. completely isometric or completely bounded). However, in some cases it is possible to conclude that if u is isometric or contractive, then u is completely isometric or completely contractive. For example, the contractive homomorphisms of a C^*-algebra are exactly the $*$-homomorphisms, and hence any contractive homomorphism of a C^*-algebra into $\mathcal{B}(\mathcal{H})$ is completely contractive. It is not known if every bounded representation of a C^*-algebra is completely bounded; a result of Haagerup (stated as Theorem 2.1 below) shows this question is equivalent to Kadison's similarity problem.

The purpose of this note is to give a sufficient condition on an operator algebra $\mathcal{B}$ which ensures that every bounded epimorphism $u : \mathcal{A} \to \mathcal{B}$ of the operator algebra $\mathcal{A}$ onto $\mathcal{B}$ is completely bounded. We show that this condition can be used to give simple proofs of several results in the literature, and also that every bounded faithful representation of a C^*-algebra is bounded below.

Throughout, we typically use $\mathcal{A}$ and $\mathcal{B}$ to denote operator algebras. An *operator $\mathcal{A}$-$\mathcal{B}$-bimodule* is an operator space $\mathcal{M}$ that is a left-$\mathcal{A}$, right-$\mathcal{B}$ bimodule where the bimodule action is completely contractive in the sense that for any $A \in M_{np}(\mathcal{A})$, $M \in M_p(\mathcal{M})$ and $B \in M_{pn}(\mathcal{B})$,

$$\|AMB\|_{M_n(\mathcal{M})} \leq \|A\|_{M_{np}(\mathcal{A})} \|M\|_{M_p(\mathcal{M})} \|B\|_{M_{pn}(\mathcal{B})} .$$

(We shall generally not write subscripts on the norms in the sequel, unless necessary for clarity.) We will sometimes write ${}_A\mathcal{M}_{\mathcal{B}}$ when $\mathcal{M}$ is an A-$\mathcal{B}$-bimodule.

For convenience, we will always assume that operator algebras are unital unless explicitly stated otherwise. However, many of the results in the sequel are valid for non-unital algebras.

Given such an $\mathcal{A}$-$\mathcal{B}$ operator bimodule $\mathcal{M}$, we may define a family of norms η_n on $M_n(\mathcal{M})$ as follows: for $X \in M_n(\mathcal{M})$,

$$\eta_n(X) := \sup\{\|RXC\| : R \in M_{1n}(\mathcal{A}), C \in M_{n1}(\mathcal{B}) \text{ and } \max\{\|C\|, \|R\|\} \leq 1\}.$$

Clearly $\eta_n(X) \leq \|X\|_{\mathcal{M}_n(\mathcal{M})}$.

Definition 1.1 ([25])**.** Let $\lambda > 0$ be a real number. We say that ${}_{\mathcal{A}}\mathcal{M}_{\mathcal{B}}$ is λ-*normed* by $\mathcal{A}$ and $\mathcal{B}$ if for every $n \in \mathbb{N}$ and $X \in M_n(\mathcal{M})$,

$$\lambda \|X\| \leq \eta_n(X).$$

When $\lambda = 1$, we say that ${}_{\mathcal{A}}\mathcal{M}_{\mathcal{B}}$ is *normed* by $\mathcal{A}$-$\mathcal{B}$. When $\mathcal{A} = \mathcal{B}$, we simply say that $\mathcal{M}$ is normed by $\mathcal{A}$.

When $\mathcal{A}$ and $\mathcal{B}$ are C^*-algebras, C. Pop [24] shows that $(\mathcal{M}, \{\eta_n\})$ is the smallest operator space structure on the Banach space $\mathcal{M}$ that is compatible with the module structure of $\mathcal{M}$.

We will make essential use of the following result.

Theorem 1.2 ([22, Lemma 7.7, p. 128])**.** *Let $\mathcal{A}$ be a C^*-algebra and suppose that $u : \mathcal{A} \to \mathcal{B}(\mathcal{H})$ is a bounded homomorphism. Then for any $R \in M_{1,n}(\mathcal{A})$ and $C \in M_{n,1}(\mathcal{A})$ we have*

$$\|u_{1,n}(R)(u_{1,n}(R))^*\| \leq \|u\|^4 \|RR^*\| \qquad \textit{and}$$
$$\|(u_{n,1}(C))^* u_{n,1}(C)\| \leq \|u\|^4 \|C^*C\|.$$

Remark 1.3. Actually, [22, Lemma 7.7, p. 128] gives the statement and proof for the column C. However to obtain the statement for R, one applies the statement for C to the opposite algebra $\mathcal{A}^{\mathrm{op}}$ and the homomorphism ψ of $\mathcal{A}^{\mathrm{op}}$ on the conjugate Hilbert space $\overline{\mathcal{H}}$ given by $d \in \mathcal{A} \mapsto \psi(d)$, where $\psi(d)\overline{h} = \overline{(u(d))^* h}$.

The following is our central observation. The key idea is to combine the techniques of [25, Theorem 2.10] with Theorem 1.2. (See also [29, Theorem 2.1].)

Theorem 1.4. *Let $\mathcal{B}$ be a norm-closed operator algebra containing a C^*-algebra $\mathcal{D}$ such that $\mathcal{B}$ is λ-normed by $\mathcal{D}$ for some $\lambda > 0$. If $\mathcal{A}$ is a norm-closed operator algebra, $u : \mathcal{A} \to \mathcal{B}$ is a bounded epimorphism and $\tilde{u} : \mathcal{A}/\ker u \to \mathcal{B}$ is the induced isomorphism, then u is completely bounded and*

$$\|u\|_{\mathrm{cb}} \leq \lambda^{-1} \|u\| \left\|\tilde{u}^{-1}\right\|^4.$$

Proof. We shall prove that when u is a bounded isomorphism, then u is completely bounded and $\|u\|_{\mathrm{cb}} \leq \lambda^{-1} \|u\| \left\|u^{-1}\right\|^4$. The full result is then obtained as follows. When u is a bounded epimorphism, equip $\mathcal{A}/\ker u$ with the quotient operator algebra structure; the quotient map $q : \mathcal{A} \to \mathcal{A}/\ker u$ then satisfies $\|q\|_{\mathrm{cb}} = 1$ (see [4, Proposition 2.3.4] for details). Thus, $\|u\|_{\mathrm{cb}} \leq \|\tilde{u}\|_{\mathrm{cb}} \leq \lambda^{-1} \|\tilde{u}\| \left\|\tilde{u}^{-1}\right\|^4 = \|u\| \left\|\tilde{u}^{-1}\right\|^4$, as desired.

So assume u is a bounded isomorphism. Let $T = (T_{ij}) \in M_n(\mathcal{A})$. Then for any $R \in M_{1n}(\mathcal{D})$ and $C \in M_{n1}(\mathcal{D})$ with $\|R\| \leq 1$ and $\|C\| \leq 1$ we have

$$\begin{aligned}
\|Ru_n(T)C\| &= \left\| \sum_{i,j=1}^{n} R_i u(T_{ij}) C_j \right\| \\
&\leq \|u\| \left\| \sum_{i,j=1}^{n} u^{-1}(R_i) T_{ij} u^{-1}(C_j) \right\| \\
&= \|u\| \left\| u_{1,n}^{-1}(R) T u_{n,1}^{-1}(C) \right\| \\
&\leq \|u\| \left\| u_{1,n}^{-1}(R) \right\| \|T\| \left\| u_{n,1}^{-1}(C) \right\|.
\end{aligned}$$

We may assume that $\mathcal{A}$ is represented completely isometrically as operators acting on a Hilbert space $\mathcal{H}$. Thus, $u^{-1}|_{\mathcal{D}}$ is a bounded homomorphism of $\mathcal{D}$ into $\mathcal{B}(\mathcal{H})$. By Theorem 1.2, $\left\|u_{n,1}^{-1}(C)\right\| \leq \left\|u^{-1}\right\|^2$ and $\left\|u_{1,n}^{-1}(R)\right\| \leq \left\|u^{-1}\right\|^2$. Taking suprema over R and C gives

$$\lambda \left\|u_n(T)\right\| \leq \|u\| \left\|u^{-1}\right\|^4 \|T\|,$$

as desired. □

When the isomorphism is isometric, more can be said. For any operator algebra $\mathcal{B}$, let $C^*_{\mathrm{env}}(B)$ be the C^*-envelope of $\mathcal{B}$.

Corollary 1.5. *For $i = 1, 2$, suppose that $\mathcal{A}_i$ are operator algebras and $\mathcal{D}_i \subseteq \mathcal{C}_i$ is a norming C^*-subalgebra of $\mathcal{A}_i$. If $u : \mathcal{A}_1 \to \mathcal{A}_2$ is an isometric isomorphism, then u extends uniquely to a $*$-isomorphism $\tilde{u} : C^*_{\mathrm{env}}(\mathcal{A}_1) \to C^*_{\mathrm{env}}(\mathcal{A}_2)$.*

Proof. Theorem 1.4 shows that u and u^{-1} are complete contractions, so that u is a complete isometry. The result follows from the universal property of C^*-envelopes. □

2. Applications

In this section we record some consequences of Theorem 1.4. We shall require the following closely related results of Haagerup and Paulsen.

Theorem 2.1 (Haagerup [11])**.** *Suppose $\mathcal{A}$ is a C^*-algebra and $u : \mathcal{A} \to \mathcal{B}(\mathcal{H})$ is a completely bounded homomorphism. Then there exists an invertible operator $S \in \mathcal{B}(\mathcal{H})$ with $\left\|S\right\| \left\|S^{-1}\right\| = \|u\|_{\mathrm{cb}}$ such that for every $a \in \mathcal{A}$,*

$$a \mapsto Su(a)S^{-1}$$

is completely contractive (and hence a $*$-representation).

Theorem 2.2 (Paulsen [19])**.** *Suppose $\mathcal{A}$ is a unital operator algebra and $u : \mathcal{A} \to \mathcal{B}(\mathcal{H})$ is a completely bounded unital homomorphism. Then there exists an invertible operator $S \in \mathcal{B}(\mathcal{H})$ with $\left\|S\right\| \left\|S^{-1}\right\| = \|u\|_{\mathrm{cb}}$ such that for every $a \in \mathcal{A}$,*

$$a \mapsto Su(a)S^{-1}$$

is a completely contractive homomorphism.

2.1. **Applications to C^*-algebras and Kadison's similarity problem.** We begin with a new proof of a result of Gardner. We use the following notation in the sequel: if $T \in \mathcal{B}(\mathcal{H})$ is an invertible operator, let $\operatorname{Ad} T$ denote the spatial isomorphism of $\mathcal{B}(\mathcal{H})$ given by $(\operatorname{Ad} T)(X) = TXT^{-1}$.

Theorem 2.3 (Gardner [10])**.** *Suppose $\mathcal{A}$ and $\mathcal{B}$ are C^*-algebras and $u : \mathcal{A} \to \mathcal{B}$ is an isomorphism. Then u is completely bounded and there exists a $*$-isomorphism $\alpha : \mathcal{A} \to \mathcal{B}$ and a bounded automorphism β of $\mathcal{B}$ such that $u = \beta \circ \alpha$. If $\mathcal{B} \subseteq \mathcal{B}(\mathcal{H})$, there exists a positive invertible operator $S \in \mathcal{B}(\mathcal{H})$ with $\left\|S\right\| \left\|S^{-1}\right\| \leq \|u\|$ so that $\beta = \operatorname{Ad} S$.*

Proof. By [25, Lemma 2.3(i)], $\mathcal{B}$ norms itself,[1] and as observed by Gardner, u is bounded. (One can also use a result of B. Johnson [12] (see also [28]) concerning automatic continuity of a homomorphism from a Banach algebra onto a semi-simple Banach algebra.) Theorem 1.4 implies that u is completely bounded. Assume that $\mathcal{B} \subseteq \mathcal{B}(\mathcal{H})$. By Theorem 2.1, there exists an invertible operator T with $\|T\| \, \|T^{-1}\| = \|u\|_{\mathrm{cb}}$ such that $\operatorname{Ad} T \circ u$ is a completely contractive homomorphism, so that $\operatorname{Ad} T \circ u$ is therefore a $*$-homomorphism. Let S be the positive square root of T^*T and let U be the polar part of T, so $T = US$. Since U is a unitary operator, $(\operatorname{Ad} S)(\mathcal{B})$ is a C^*-algebra. Thus $S\mathcal{B}S^{-1} = (S\mathcal{B}S^{-1})^* = S^{-1}\mathcal{B}S$, and it follows that $S^2\mathcal{B}S^{-2} = \mathcal{B}$. By Gardner's Invariance Theorem [10, Theorem 3.5], $\beta := \operatorname{Ad} S^{-1}$ is an automorphism of $\mathcal{B}$. Now let $\alpha = \operatorname{Ad}(U^*T) \circ u$. Since $\operatorname{Ad} T \circ u$ is a $*$-isomorphism of $\mathcal{A}$ onto its range, α is a $*$-isomorphism of $\mathcal{A}$ onto $\mathcal{B}$. Then $u = \beta \circ \alpha$. □

Remark 2.4. Gardner's original arguments give somewhat more. In particular, he shows that if $\mathcal{A}$ and $\mathcal{B}$ are faithfully represented using the universal atomic representations, then α can be taken to have the form $\operatorname{Ad} U$ for some unitary U.

The following is an immediate corollary of Theorem 1.4.

Theorem 2.5. *Suppose $\mathcal{A}$ is an operator algebra and $\mathcal{B}$ is a C^*-algebra. If $u : \mathcal{A} \to \mathcal{B}$ is an isomorphism, then u is automatically completely bounded and $\|u\|_{\mathrm{cb}} \leq \|u\| \, \|u^{-1}\|^4$.*

Proof. Continuity of u follows from Johnson's theorem [12]; then complete boundedness follows from Theorem 1.4 together with the fact that a C^*-algebra norms itself. □

We now wish to make some observations regarding Kadison's Similarity Problem. Recall that this problem asks whether every bounded representation of a C^*-algebra is similar to a $*$-representation, which as noted above, is equivalent to the question of whether bounded representations of C^*-algebras are automatically completely bounded. Theorem 2.5 can be used to prove that bounded representations of C^*-algebras are (modulo the kernel) "completely bounded below."

Theorem 2.6. *Suppose $\mathcal{A}$ is a C^*-algebra and $u : \mathcal{A} \to \mathcal{B}(\mathcal{H})$ is a bounded homomorphism. Put $\mathcal{J} = \ker u$. Then there exists a real number $k > 0$ such that for every $n \in \mathbb{N}$ and every $T \in M_n(\mathcal{A})$,*

$$k \operatorname{dist}(T, M_n(\mathcal{J})) \leq \|u_n(T)\| .$$

Proof. Since $\mathcal{A}/\mathcal{J}$ is a C^*-algebra isomorphic under the map induced by u to $u(\mathcal{A})$, without loss of generality we may assume that u is one-to-one. Thus our task is to prove that $u := u^{-1}$ is completely bounded. This will follow from Theorem 2.5 once we prove that the image $\mathcal{B} := u(\mathcal{A})$ is closed, and hence an operator algebra. We may also assume that $\mathcal{A}$ is unital and that $u(I) = I$.

Let x be a non-zero element of $\mathcal{A}$, and let $\mathcal{C}$ be the unital C^*-subalgebra of $\mathcal{A}$ generated by x^*x. Since $\mathcal{C}$ is abelian, its unitary group (regarded as a discrete

[1] Here is a sketch of an alternate argument, communicated by David Blecher, for showing that a unital C^*-algebra $\mathcal{B}$ norms itself. For $n \in \mathbb{N}$, let $\mathcal{E}_n$ be the direct sum of n copies of $\mathcal{B}$; then $\mathcal{E}_n$ is a right Hilbert $\mathcal{B}$-module with $\mathcal{B}$-valued inner product given by $\langle e, f \rangle_{\mathcal{E}_n} = \sum_{j=1}^n e_j^* f_j$. Then the algebra of adjointable operators on $\mathcal{E}_n$ is $M_n(\mathcal{B})$. For $T \in M_n(\mathcal{B})$, we have $\|T\| = \sup\{\left\|\langle e, Tf \rangle_{\mathcal{E}_n}\right\| : e, f \in \mathcal{E}_n, \|e\| < 1, \|f\| < 1\}$. Rewriting shows that $\mathcal{B}$ norms itself.

group) is amenable, and the Dixmier-Day Theorem on amenable groups implies that there exists an invertible operator S with $\|S\|\,\|S^{-1}\| \leq \|u|_{\mathcal{C}}\|^2$ such that $(\operatorname{Ad} S)\circ u|_{\mathcal{C}}$ is a $*$-homomorphism. Since u is one-to-one, we have

$$\|x\|^2 = \|x^*x\| = \|Su(x^*x)S^{-1}\| \leq \|u\|^2\,\|u(x^*)u(x)\| \leq \|u\|^3\,\|x\|\,\|u(x)\|\,,$$

so that $\|x\| \leq \|u\|^3\,\|u(x)\|$. Thus, u is bounded below, so that the range of u is closed. □

Combining Theorems 2.2 and 2.6 with the structure of completely contractive representations yields the following corollary.

Corollary 2.7. *Suppose $\mathcal{A} \subseteq \mathcal{B}(\mathcal{H})$ is a C^*-algebra and $u : \mathcal{A} \to \mathcal{B}(\mathcal{H}_u)$ is a faithful bounded representation. Then there exists a Hilbert space $\mathcal{K}$, a $*$-representation $\pi : \mathcal{B}(\mathcal{H}_u) \to \mathcal{B}(\mathcal{K})$, an isometry $W : \mathcal{H} \to \mathcal{K}$ and an invertible operator $S \in \mathcal{B}(\mathcal{H})$ with $\|S\|\,\|S^{-1}\| \leq \|u^{-1}\|\,\|u\|^4$ such that for every $x \in \mathcal{A}$,*

$$x = SW^*\pi(u(x))WS^{-1}. \tag{2.1}$$

Proof. Let $\mathcal{B} = u(\mathcal{A})$. We may assume that $\mathcal{A}$ is unital and $u(I) = I$. Theorem 2.6 shows that u^{-1} is a completely bounded map from $\mathcal{B}$ to $\mathcal{B}(\mathcal{H})$. Therefore, by Theorem 2.2, there exists an invertible operator $S \in \mathcal{B}(\mathcal{H})$ with $\|S\|\,\|S^{-1}\| = \|u^{-1}\|_{\mathrm{cb}}$ so that $\psi := \operatorname{Ad}(S^{-1})\circ u^{-1}$ is completely contractive. By Arveson's Structure Theorem for completely contractive representations of operator algebras, there exists a Hilbert space $\mathcal{K}$, a $*$-representation $\pi : \mathcal{B}(\mathcal{H}_u) \to \mathcal{B}(\mathcal{K})$ and an isometry W so that for all $b \in \mathcal{B}$, $\psi(b) = W^*\pi(b)W$. Letting $b = u(x)$ (for $x \in \mathcal{A}$) yields (2.1). The estimate for the condition number of S follows from Theorem 1.4. □

Remark 2.8. Unfortunately, we have been unable to solve for $u(x)$ in (2.1); doing so would of course lead to a solution of Kadison's problem.

The range of the isometry W appearing in Corollary 2.1 is a semi-invariant subspace for $\pi(\mathcal{B})$. Thus, $WW^* = PQ^\perp$ for some projections $P, Q \in \operatorname{Lat}(\pi(\mathcal{B}))$, with $Q \leq P$. The map $x \mapsto WS^{-1}xSW^*$ is a homomorphism into the "$2,2$-diagonal piece" of $\pi(u(x))$ relative to the block decomposition of $\pi(u(x))$ according to $I = Q + PQ^\perp + Q^\perp P^\perp$.

We now show that Kadison's problem is equivalent to the issue of whether the image of a C^*-algebra under a bounded homomorphism λ-norms itself.

Theorem 2.9. *Suppose $\mathcal{A}$ is a C^*-algebra and $u : \mathcal{A} \to \mathcal{B}(\mathcal{H})$ is a bounded representation with image $\mathcal{B} := u(\mathcal{A})$, and let $\tilde{u} : \mathcal{A}/\ker u \to \mathcal{B}$ be the induced map. If $\mathcal{B}$ λ-norms itself for some $\lambda > 0$, then u is completely bounded and $\|u\|_{\mathrm{cb}} \leq \lambda^{-1}\,\|u\|^9\,\|\tilde{u}^{-1}\|^2$. Conversely, if u is completely bounded, then $\mathcal{B}$ λ-norms itself for any λ with $0 < \lambda \leq \dfrac{1}{\|u\|_{\mathrm{cb}}\,\|\tilde{u}^{-1}\|\,\|u\|^4}$.*

Proof. We may assume that u is a monomorphism, so Theorem 2.5 gives $\|u^{-1}\|_{\mathrm{cb}} \leq \|u\|^4\,\|u^{-1}\|$. Suppose that $\mathcal{B}$ is λ-normed by itself for some $\lambda > 0$, and let $T \in M_n(\mathcal{A})$. The calculation in the proof of Theorem 1.4 shows that if $R \in M_{1,n}(\mathcal{B})$ and $C \in M_{n,1}(\mathcal{B})$ satisfy $\|R\| \leq 1$ and $\|C\| \leq 1$, then

$$\|Ru_n(T)C\| \leq \|u\|\,\|u_{1,n}^{-1}(R)\|\,\|T\|\,\|u_{n,1}^{-1}(C)\| \leq \|u\|^9\,\|u^{-1}\|^2\,\|T\|\,.$$

Taking suprema over all such R and C gives

$$\lambda \|u_n(T)\| \leq \|u\|^9 \left\|u^{-1}\right\|^2 \|T\|,$$

so u is completely bounded with $\|u\|_{\mathrm{cb}} \leq \lambda^{-1} \|u\|^9 \left\|u^{-1}\right\|^2$.

Conversely, suppose that u is completely bounded, and view $\mathcal{B}$ as a bimodule over itself. For any non-zero $R \in \mathcal{M}_{1,n}(\mathcal{A})$, $C \in \mathcal{M}_{n,1}(\mathcal{A}))$ with $\|R\|, \|C\| \leq 1$, we have

$$\begin{aligned}
\left\|Ru_n^{-1}(T)C\right\| &\leq \left\|u^{-1}\right\| \left\|u(Ru_n^{-1}(T)C)\right\| \\
&= \left\|u^{-1}\right\| \|u_{1,n}(R)\| \|u_{n,1}(C)\| \left\|\frac{u_{1,n}(R)}{\|u_{1,n}(R)\|} T \frac{u_{n,1}(C)}{\|u_{n,1}(C)\|}\right\| \\
&\leq \left\|u^{-1}\right\| \|u\|^4 \eta_n(T) \quad \text{(using Lemma 1.2)}.
\end{aligned}$$

Taking suprema yields $\left\|u_n^{-1}(T)\right\| \leq \left\|u^{-1}\right\| \|u\|^4 \eta_n(T)$, and hence

$$\|T\| \leq \|u_n\| \left\|u_n^{-1}(T)\right\| \leq \|u\|_{\mathrm{cb}} \left\|u^{-1}\right\| \|u\|^4 \eta_n(T).$$

Thus $\mathcal{B}$ λ-norms itself for any $\lambda \leq \dfrac{1}{\|u\|_{\mathrm{cb}} \|u^{-1}\| \|u\|^4}$. □

Corollary 2.10. *Suppose $\mathcal{B} \subseteq \mathcal{B}(\mathcal{H})$ is an operator algebra that is isomorphic to a C^*-algebra $\mathcal{A}$. Then there exists an invertible operator $S \in \mathcal{B}(\mathcal{H})$ such that $S\mathcal{B}S^{-1}$ is a C^*-algebra if and only if $\mathcal{B}$ λ-norms itself for some $\lambda > 0$.*

Proof. Let $u : \mathcal{A} \to \mathcal{B}$ be the isomorphism. Then Theorem 2.9 shows that if $\mathcal{B}$ λ-norms itself, then u is completely bounded; hence by Theorem 2.1, u is similar to a $*$-representation, and so $\mathcal{B}$ is similar to a C^*-algebra. Conversely, if $\mathcal{B}$ is similar to a C^*-algebra, then u is similar to a $*$-representation and an application of Theorem 2.9 completes the proof. □

The following question is thus a reformulation of Kadison's question.

Question 2.11. Suppose $\mathcal{B}$ is an operator algebra that is isomorphic to a C^*-algebra. Does $\mathcal{B}$ λ-norm itself for some $\lambda > 0$?

Remark 2.12. Haagerup [11] showed that every bounded, cyclic representation u of a C^*-algebra $\mathcal{A}$ is completely bounded with $\|u\|_{\mathrm{cb}} \leq \|u\|^3$. So given an arbitrary representation u of $\mathcal{A}$ on $\mathcal{H}$, let $\mathcal{B} = u(\mathcal{A})$. For each unit vector $\xi \in \mathcal{H}$, let P_ξ be the projection onto the cyclic subspace $[\mathcal{B}\xi]$ and let $\mathcal{B}_\xi$ be the restriction of $\mathcal{B}$ to $\mathcal{H}_\xi := P_\xi \mathcal{H}$. The representation u_ξ of $\mathcal{A}$ given by $u_\xi(x) = u(x)P_\xi$ is thus completely bounded and $\|u_\xi\|_{\mathrm{cb}} \leq \|u\|_{\mathrm{cb}}$. Theorem 2.2 shows that there exists an invertible operator $S_\xi \in \mathcal{B}(\mathcal{H}_\xi)$ such that $\|u_\xi\|_{\mathrm{cb}} = \left\|S_\xi\right\| \left\|{S_\xi}^{-1}\right\|$ and $(\operatorname{Ad} S_\xi)(\mathcal{B}_\xi)$ is a C^*-algebra, call it $\mathcal{A}_\xi$. Applying Theorem 2.9 to $\operatorname{Ad} S_\xi^{-1} : \mathcal{A}_\xi \to \mathcal{B}_\xi$, we find $\|\operatorname{Ad} S_\xi\| \leq \left\|S_\xi\right\| \left\|{S_\xi}^{-1}\right\|$, so that $\mathcal{B}_\xi$ λ-norms itself with $\lambda = \|u\|^{-18}$.

In the following example, we show that there exists an operator algebra that does not λ-norm itself. The idea for the proof is due to Ken Davidson.

Example 2.13. Let $\mathbb{D} \subseteq \mathbb{C}$ be the open unit disk and let $\mathbb{A}(\mathbb{D}) \subseteq C(\overline{\mathbb{D}})$ be the disk algebra, that is, the collection of all continuous functions on the closed unit disk that are analytic in $\mathbb{D}$. We use $\mathcal{P} \subseteq \mathbb{A}(\mathbb{D})$ to denote the collection of all polynomials.

Recall that an operator $T \in \mathcal{B}(\mathcal{H})$ is polynomially bounded if there exists $K > 0$ such that for every $p \in \mathcal{P}$, $\|p(T)\| \leq K \|p\|$. Polynomial boundedness of T is

equivalent to the existence of a bounded homomorphism $u : \mathbb{A}(\mathbb{D}) \to \mathcal{B}(\mathcal{H})$ such that for every $p \in \mathcal{P}$, $u(p) = p(T)$. A polynomially bounded operator T is *completely polynomially bounded* if u is completely bounded. Paulsen [20] showed that T is completely polynomially bounded if and only if T is similar to a contraction. Pisier [21] showed that there exists a polynomially bounded operator $T \in \mathcal{B}(\mathcal{H})$ that is not completely polynomially bounded, so T is not similar to a contraction.

Fix a polynomially bounded operator $T \in \mathcal{B}(\mathcal{H})$. Notice that the spectrum of T is contained in $\overline{\mathbb{D}}$. If U is a unitary operator with $\sigma(U) = \mathbb{T}$, then T is completely polynomially bounded if and only if $T \oplus U$ is completely polynomially bounded. We may therefore assume that

$$\mathbb{T} \subseteq \sigma(T).$$

Let $u : \mathbb{A}(\mathbb{D}) \to \mathcal{B}(\mathcal{H})$ be the extension of the map $p \in \mathcal{P} \mapsto p(T)$ and set $\mathcal{B} = u(\mathbb{A}(\mathbb{D}))$. Since $\mathbb{T} \subseteq \sigma(T)$, we have $\|u(f)\| \geq \|f\|$ for every $f \in \mathbb{A}(\mathbb{D})$, so that $\mathcal{B}$ is closed and u^{-1} is contractive. Since $\mathbb{A}(\mathbb{D}) \subseteq C(\overline{\mathbb{D}})$, the operator space structure on $\mathbb{A}(\mathbb{D})$ is minimal among all possible operator space structures on $\mathbb{A}(\mathbb{D})$ (see [4, Paragraph 1.2.21]); hence u^{-1} is completely contractive.

View $\mathcal{B}$ as a $\mathcal{B}$-bimodule and let η_n be the norm on $M_n(\mathcal{B})$ as in Definition 1.1. We shall show that for every n, the norm $\eta_n(u_n(\cdot))$ and the usual norm $\|\cdot\|_{M_n(\mathbb{A}(\mathbb{D}))}$ are equivalent norms on $M_n(\mathbb{A}(\mathbb{D}))$.

Choose $X \in M_n(\mathbb{A}(\mathbb{D}))$, $R \in M_{1n}(\mathbb{A}(\mathbb{D}))$ and $C \in M_{n1}(\mathbb{A}(\mathbb{D}))$. Since u is bounded, we have $\|u_{1n}(R)u_n(X)u_{n1}(C)\| = \|u(RXC)\| \leq \|u\| \, \|RXC\|$. Since $\|u_{1n}(R)\| \geq \|R\|$ and $\|u_{n1}(C)\| \geq \|C\|$, we obtain

$$\eta_n(u_n(X)) \leq \|u\| \, \|X\| \, . \tag{2.2}$$

On the other hand, a result of Bourgain [6] (see also [22, Theorem 9.9]) shows that there exists a constant $s > 0$ (independent of u) such that for every n, $\max\{\|u_{1n}\| \, , \|u_{n1}\|\} \leq s \, \|u\|$. Therefore,

$$\begin{aligned} \|RXC\| &\leq \|u_{1n}(R)u_n(X)u_{n1}(C)\| \\ &\leq s^2 \, \|u\|^2 \left\| \frac{u_{1n}(R)}{\|u_{1n}(R)\|} u_n(X) \frac{u_{n1}(C)}{\|u_{n1}(C)\|} \right\| \|R\| \, \|C\| \, . \end{aligned}$$

Since the operator space structure on $\mathbb{A}(\mathbb{D})$ is minimal, $\mathbb{A}(\mathbb{D})$ norms itself. Taking the supremum over R and C with norm one, we obtain

$$\|X\| \leq s^2 \, \|u\|^2 \, \eta_n(u_n(X)). \tag{2.3}$$

Combining (2.2) and (2.3) establishes the claim.

Thus, when T is chosen to be polynomially bounded but not completely polynomially bounded, we see that $\mathcal{B}$ cannot λ-norm itself.

2.2. **An application to CSL algebras.** Let $\mathcal{C}$ be a C^*-algebra. A subalgebra $\mathcal{S} \subseteq \mathcal{C}$ is *selfadjoint* if it is closed under the adjoint operation, i.e. $\mathcal{S} = \mathcal{S}^*$. Zorn's lemma shows that there are always maximal abelian selfadjoint subalgebras of $\mathcal{C}$; such an algebra is called a *MASA in* $\mathcal{C}$. Every MASA is a C^*-algebra.

A *commutative subspace lattice algebra*, usually abbreviated as *CSL algebra*, is an operator algebra $\mathcal{A} \subseteq \mathcal{B}(\mathcal{H})$ that is both reflexive and such that there exists a MASA $\mathcal{D} \subseteq \mathcal{B}(\mathcal{H})$ with $\mathcal{D} \subseteq \mathcal{A}$. The lattice of invariant projections of a CSL algebra is a commutative family of projections, and when this lattice is completely distributive, the CSL algebra is called a *completely distributive* CSL algebra. CSL

algebras were studied by Arveson in [2]; see [7] for additional information and references concerning CSL algebras.

Davidson and Power proved that isometric isomorphisms of completely distributive CSL algebras are unitarily implemented. Their techniques involved homological ideas and were somewhat intricate. We can give a simpler proof of their result, and which also extends theirs. For convenience, we assume irreducibility. If $\mathcal{A}$ is not irreducible, one can use direct integrals along the center of $\mathcal{A} \cap \mathcal{A}^*$, together with an appropriate hypothesis on the lattices of each term in the direct integral, to obtain a more general result.

Theorem 2.14. *For $i = 1, 2$, let $\mathcal{H}_i$ be Hilbert spaces, $\mathcal{K}(\mathcal{H}_i) \subseteq \mathcal{B}(\mathcal{H}_i)$ be the compact operators, and $\mathcal{A}_i \subseteq \mathcal{B}(\mathcal{H}_i)$ be CSL algebras such that $\mathcal{A}_i \cap \mathcal{K}(\mathcal{H}_i) \neq (0)$. Suppose further that $\mathcal{A}_1$ is irreducible. If $u : \mathcal{A}_1 \to \mathcal{A}_2$ is an isometric isomorphism, then there exists a unitary operator $U \in \mathcal{B}(\mathcal{H}_1, \mathcal{H}_2)$ such that $u = \operatorname{Ad} U$.*

Proof. Since the isomorphism u is isometric, its restriction to $\mathcal{A}_1 \cap (\mathcal{A}_1)^*$ is a $*$-isomorphism, so in particular, $\operatorname{Lat}(\mathcal{A}_2)$ is isomorphic to $\operatorname{Lat}(\mathcal{A}_1)$. Thus, since $\mathcal{A}_1$ is irreducible, so is $\mathcal{A}_2$. Let $\mathcal{C}_i$ be the C^*-subalgebra of $\mathcal{B}(\mathcal{H}_i)$ generated by $\mathcal{A}_i$. Then $\mathcal{C}_i$ is an irreducible C^*-algebra containing a compact operator; hence $\mathcal{K}(\mathcal{H}_i) \subseteq \mathcal{C}_i$. There exists a $*$-epimorphism $\pi_i : \mathcal{C}_i \to C^*_{\mathrm{env}}(\mathcal{A}_i)$ such that $\pi_i|_{\mathcal{A}_i} = \iota_i$, where ι is the canonical inclusion of $\mathcal{A}_i$ into $C^*_{\mathrm{env}}(\mathcal{A}_i)$. If $\ker \pi_i \neq (0)$, then since $\mathcal{C}_i$ contains the compact operators, $\ker \pi_i \cap \mathcal{K}(\mathcal{H}_i) \neq (0)$. Hence $\mathcal{K}(\mathcal{H}_i) \subseteq \ker \pi_i$, since $\ker \pi_i \cap \mathcal{K}(\mathcal{H}_i)$ is an ideal in an irreducible C^*-algebra. But this is impossible, since π_i is isometric on $\mathcal{A}_i \cap \mathcal{K}(\mathcal{H}_i)$. Therefore $\ker \pi_i = (0)$, so that $\mathcal{C}_i$ is the C^*-envelope of $\mathcal{A}_i$.

Theorem 2.7 of [25] shows that any MASA is norming for $\mathcal{B}(\mathcal{H})$; hence the $\mathcal{A}_i$ contain norming C^*-subalgebras. Theorem 1.4 shows that u and u^{-1} are completely contractive, so that u is a complete isometry. By the universal property of C^*-envelopes (applied to u and u^{-1}), u extends to a $*$-isomorphism $\tilde{u}$ of $\mathcal{C}_1$ onto $\mathcal{C}_2$. Now $\mathcal{K}(\mathcal{H}_i)$ is the smallest closed two-sided ideal contained in $\mathcal{C}_i$; hence $\tilde{u}|_{\mathcal{K}(\mathcal{H}_1)}$ is a $*$-isomorphism of $\mathcal{K}(\mathcal{H}_1)$ onto $\mathcal{K}(\mathcal{H}_2)$. Therefore, there exists a unitary operator U so that $(\operatorname{Ad} U)|_{\mathcal{K}(\mathcal{H}_1)} = \tilde{u}|_{\mathcal{K}(\mathcal{H}_1)}$. Finally, if $T \in \mathcal{C}_1$ and if $\eta \in \mathcal{H}_2$, we may find a finite rank projection P so that $2\|T\| \left\|P^\perp \eta\right\| < \varepsilon$. Then since $\tilde{u}^{-1}(P) = (\operatorname{Ad} U^*)(P)$, we have

$$\begin{aligned}\|(\tilde{u}(T) - (\operatorname{Ad} U)(T))\,\eta\| &\leq \left\|\left(\tilde{u}(T\tilde{u}^{-1}(P)) - (\operatorname{Ad} U)(T(\operatorname{Ad} U^*(P))\right)\eta\right\| \\ &\quad + \left\|(\tilde{u}(T) - (\operatorname{Ad} U)(T))\,P^\perp \eta\right\| \\ &= \left\|(\tilde{u}(T) - (\operatorname{Ad} U)(T))\,P^\perp \eta\right\| < \varepsilon,\end{aligned}$$

so $\tilde{u}(T) = (\operatorname{Ad} U)(T)$. Since $\mathcal{A}_1 \subseteq \mathcal{C}_1$, the proof is complete. □

2.3. **Applications to subalgebras of C^*-diagonals.** In this subsection, we provide applications to subalgebras of certain classes of C^*-algebras.

A C^*-diagonal is a pair $(\mathcal{C}, \mathcal{D})$ of C^*-algebras such that $\mathcal{D}$ is abelian and such that

i) every pure state of $\mathcal{D}$ extends uniquely to a pure state of $\mathcal{C}$;
ii) the conditional expectation $E : \mathcal{C} \to \mathcal{D}$ (whose existence is guaranteed by (i)) is faithful;
iii) the closed linear span of the set $\{v \in \mathcal{C} : v\mathcal{D} = \mathcal{D}v\}$ is $\mathcal{C}$.

We will assume that both $\mathcal{C}$ and $\mathcal{D}$ are unital. The extension property then implies that $\mathcal{D}$ is a MASA in $\mathcal{C}$.

Such pairs were introduced by Kumjian [14], who used slightly different, but essentially equivalent, axioms (see [8] for a discussion of the equivalence). Also, C^*-diagonals and their subalgebras were studied further in several papers; see for example [8, 16, 17].

Our first task is to show that $\mathcal{D}$ norms $\mathcal{C}$.

Lemma 2.15. *Suppose $(\mathcal{C}, \mathcal{D})$ is a C^*-diagonal. Then $\mathcal{C}$ is normed by $\mathcal{D}$.*

Proof. Theorem 5.9 of [8] shows that there exists a faithful $*$-representation $\pi : \mathcal{C} \to \mathcal{B}(\mathcal{H})$ such that $\pi(\mathcal{D})''$ is a MASA in $\mathcal{B}(\mathcal{H})$. It follows from [25, Lemma 2.2 and Theorem 2.7] that $\pi(\mathcal{D})$ norms $\mathcal{B}(\mathcal{H})$; hence $\pi(\mathcal{D})$ norms $\pi(\mathcal{C})$. As π is a faithful $*$-representation of a C^*-algebra, it is a complete isometry, so $\mathcal{D}$ norms $\mathcal{C}$. $\square$

The following notation will be useful. When $(\mathcal{C}, \mathcal{D})$ is a C^*-diagonal and $\mathcal{A}$ is a norm closed algebra with $\mathcal{D} \subseteq \mathcal{A} \subseteq \mathcal{C}$, we will write $\mathcal{A} \subseteq (\mathcal{C}, \mathcal{D})$. If $\mathcal{A} \subseteq (\mathcal{C}, \mathcal{D})$ and $\mathcal{A} \cap \mathcal{A}^* = \mathcal{D}$, $\mathcal{A}$ is called *triangular*.

For $i = 1, 2$, let $(\mathcal{C}_i, \mathcal{D}_i)$ be C^*-diagonals. Muhly, Qiu and Solel [17, Theorem 1.1] proved that when $\mathcal{A}_i \subseteq (\mathcal{C}_i, \mathcal{D}_i)$ are triangular algebras that generate $\mathcal{C}_i$ and $(\mathcal{C}_i, \mathcal{D}_i)$ are nuclear, then an isometric isomorphism $u : \mathcal{A}_1 \to \mathcal{A}_2$ extends to a $*$-isomorphism of $\mathcal{C}_1$ onto $\mathcal{C}_2$. Later Donsig and Pitts [8, Theorem 8.9] extended this result: they showed that the hypothesis of nuclearity can be removed. To prove their result, Donsig and Pitts showed that the isometric isomorphism between $\mathcal{A}_1$ and $\mathcal{A}_2$ induces an isomorphism of CSL algebras associated with $\mathcal{A}_1$ and $\mathcal{A}_2$, then used the structure theory for isomorphisms of CSL algebras.

Donsig and Pitts [8, Theorem 4.21] showed that the C^*-envelope of any subalgebra (triangular or not) $\mathcal{A} \subseteq (\mathcal{C}, \mathcal{D})$ is the C^*-subalgebra, $C^*(\mathcal{A})$, of $\mathcal{C}$ generated by $\mathcal{A}$. In general, it is not clear that the pair $(C^*(\mathcal{A}), \mathcal{D})$ is a C^*-diagonal—one needs to verify that condition (iii) of the definition of C^*-diagonal holds. In the context of both [17, Theorem 1.1] and [8, Theorem 8.9], $(C^*(\mathcal{A}_i), \mathcal{D}_i)$ is a C^*-diagonal due to the hypothesis that $(\mathcal{C}_i, \mathcal{D}_i) = (\mathcal{C}^*(\mathcal{A}_i), \mathcal{D}_i)$. Also, the techniques used to prove [17, Theorem 1.1] and [8, Theorem 8.9] do not apply for non-triangular subalgebras. The following consequence of [8, Theorem 4.21], Corollary 1.5 and Lemma 2.15 is therefore a significant extension of [17, Theorem 1.1] and [8, Theorem 8.9].

Theorem 2.16. *Let $\mathcal{A}_i \subseteq (\mathcal{C}_i, \mathcal{D}_i)$ be norm-closed subalgebras of C^*-diagonals. If $u : \mathcal{A}_1 \to \mathcal{A}_2$ is an isometric isomorphism, then u extends uniquely to a $*$-isomorphism of $C^*(\mathcal{A}_1)$ onto $C^*(\mathcal{A}_2)$.*

Remark 2.17. In [15], Mercer proves a result similar to Theorem 2.16, but where the algebras $\mathcal{A}_i$ are taken to be weak-$*$ closed subalgebras of von Neumann algebras $\mathfrak{M}_i$ and there are Cartan MASAs $\mathcal{D}_i \subseteq \mathfrak{M}_i$ such that $\mathcal{D}_i \subseteq \mathcal{A}_i \subset \mathcal{M}_i$. We expect that Cartan MASAs norm their containing von Neumann algebras, and thus expect that it should be possible to give a proof of Mercer's result based on Theorem 1.4 as well.

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