DIFFERENTIABILITY OF PEANO DERIVATIVES

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Abstract. Peano differentiability is a notion of higher-order Fréchet differentiability. H. W. Oliver gave sufficient conditions for the \( m \)th Peano derivative to be a Fréchet derivative in the case of functions of a real variable. Here we generalize this theorem to functions of several variables.

1. Introduction

In 1891, G. Peano introduced the class of functions which can be best approximated by a polynomial of degree less than or equal to \( m \). These functions are called Peano differentiable. For a precise definition we use the following conventions. For a multi-index \( \alpha \in \mathbb{N}^n \) we set \(|\alpha| = \alpha_1 + \cdots + \alpha_n\), \( \alpha! = \alpha_1! \cdots \alpha_n! \), and, if \( x \in \mathbb{R}^n \), \( x^\alpha = x_1^{\alpha_1} \cdots x_n^{\alpha_n} \). The notion of Peano differentiability reads as follows.

Definition 1.1. Let \( U \subset \mathbb{R}^n \) be an open set and \( m \) a positive integer. The function \( f : U \to \mathbb{R} \) is called \( m \) times Peano differentiable at \( x_0 \in U \) if there are \( f_\alpha(x_0) \in \mathbb{R} \), \( |\alpha| \leq m \), such that

\[
(1.1) \quad \lim_{x \to x_0} \frac{f(x) - \sum_{|\alpha| \leq m} \frac{f_\alpha(x_0)}{\alpha!} (x - x_0)^\alpha}{\|x - x_0\|^m} = 0.
\]

A function with codomain \( \mathbb{R}^p \) is called \( m \) times Peano differentiable if each coordinate function possesses this property. The \( f_\alpha(x_0) \) are called Peano derivatives of order \( |\alpha| \). The set of all \( m \) times Peano differentiable functions with domain \( U \) forms a ring, and, moreover, the composition of \( m \) times Peano differentiable functions is again \( m \) times Peano differentiable.

The case \( m = 1 \) obviously represents the notion of Fréchet differentiability so that we can interpret the notion for \( m > 1 \) as differentiability of higher order.

Example 1.2. Let \( m > 1 \) be an integer and the function \( f : \mathbb{R} \to \mathbb{R} \) be defined by \( f(x) := x^{m+1} \sin(x^{-m}) \) for \( x \neq 0 \) and \( f(0) := 0 \). \( f \) is \( m \) times Peano differentiable in \( \mathbb{R} \), but the first derivative is not continuous.

The proof is straightforward and we omit it.

As a consequence we get that, in general, the \( m \)th Peano derivatives are not derivatives in the usual sense. Conversely, Taylor’s Theorem states that \( C^m \) functions are \( m \) times Peano differentiable.
H. W. Oliver found sufficient conditions which guarantee Fréchet differentiability. In [3] he proved the following statement.

**Theorem 1.3** (Oliver). Let $I$ be an interval in $\mathbb{R}$ and let $f : I \to \mathbb{R}$ be an $m$ times Peano differentiable function. If the $m^{th}$ Peano derivative is locally bounded from above or below at $x_0 \in I$, the Peano derivative of order $m - 1$ is Fréchet differentiable at $x_0$.

We will use Oliver’s theorem for the study of $m^{th}$ directional derivatives of $m$ times Peano differentiable functions, and we will prove that they are Fréchet derivatives if they are bounded from above or below. Knowing this fact we can deduce the following theorem.

**Theorem 1.4.** Let $U \subset \mathbb{R}^n$ be an open subset and let $f : U \to \mathbb{R}$ be an $m$ times Peano differentiable function. Let the $m^{th}$ Peano derivatives all be bounded from above or all be bounded from below. Then the Peano derivatives of order $m - 1$ are Fréchet differentiable at each $x_0 \in U$.

In [1] it is shown that if $f$ is $m$-convex and $m$ times Peano differentiable, it is $m$ times Fréchet differentiable. For Peano differentiable functions of several variables and $m \geq 3$, $m$-convexity is neither more general nor more special than the conditions given in Theorem 1.4. In the case $m = 2$ we need weaker conditions for obtaining Fréchet differentiability of the first derivatives; cf. Corollary 4.1, which is already known (personal communication with Clifford Weil).

2. Preliminary lemmas

The first lemma of this section deals with the number of zeros of the derivatives of a $p$ times differentiable map $f$ of one real variable. We compute an upper bound for the number of zeros of the derivatives under the assumption that the derivative of order $p$ is positive.

**Lemma 2.1.** Let $h : (a, b) \to \mathbb{R}$ be $p$ times differentiable and

\[ h^{(p)}(t) > 0, \quad t \in (a, b). \]

Then there is a subinterval $(c, d) \subset (a, b)$ of length

\[ |d - c| \geq |b - a|/p^2 \]

such that all derivatives of order $2, \ldots, p$ are either positive or negative on $(c, d)$.

**Proof.** We show by induction on $j$ that for $j = 0, \ldots, p - 1$ the number of zero-points of $h^{(p-j)}$ is bounded by $j$:

- $j = 0$: $h^{(p-0)}$ is a priori positive. Therefore it has no zero-point.
- $j \mapsto j + 1$:
  - By the assumption, $h^{(p-j)}$ has at most $j$ zero-points $x_1 < x_2 < \cdots < x_j$ in $(a, b)$.
  - We put $x_0 = a$ and $x_{j+1} = b$ so that on each of the $j + 1$ intervals $(x_l, x_{l+1})$ where $l = 0, \ldots, j$, the $(p - j)^{th}$ derivative of $h$ is strictly positive or negative.
  - Hence, $h^{(p-(j+1))}|_{(x_l, x_{l+1})}$ is a strictly monotone function and may therefore have at most one zero-point.
  - So there are at most $\sum_{l=0}^{p-1} l = \frac{p(p-1)}{2} \leq p^2$ points at which at least one of the $h^{(i)}$ equals 0 where $i = 2, \ldots, p$. Hence, we find an interval $(c, d) \subset (a, b)$ of length \[ |b - a|/p^2 \]
such that for \( i = 2, \ldots, p \), the \( i \)th derivative of \( h \) is either positive or negative. \( \square \)

Now we focus on \( p \) times differentiable functions of one variable with definite derivatives. We find the next two lemmas in [2]. Here we give alternative proofs.

**Lemma 2.2.** Let \( I \subset \mathbb{R} \) be an interval, \( \lambda : I \to \mathbb{R} \) be twice differentiable, and we have either \( \lambda^{(2)} \geq 0 \) on \( I \) or \( \lambda^{(2)} \leq 0 \) on \( I \). If \( t \in I \) and \( r > 0 \) such that \( [t-r,t+r] \subset I \), then

\[
\lambda^{(1)}(t) \leq 2 \sup_{s \in [t-r,t+r]} |\lambda(s)|.
\]

**Proof.** Without loss of generality we may assume that \( \lambda^{(2)} \leq 0 \) on \( I \) so that \( \lambda^{(1)} \) is monotonically decreasing on \( I \). According to the Mean-Value Theorem we obtain the following two inequalities:

\[
\lambda(t) - \lambda(t - r) \geq \inf_{s \in [t-r,t]} \lambda^{(1)}(s)r = \lambda^{(1)}(t)r,
\]

\[
\lambda(t + r) - \lambda(t) \leq \sup_{s \in [t,t+r]} \lambda^{(1)}(s)r = \lambda^{(1)}(t)r.
\]

Since \( |\lambda(t) - \lambda(t - r)| \) and \( |\lambda(t + r) - \lambda(t)| \) are both less than or equal to

\[
2 \sup_{s \in [t-r,t+r]} |\lambda(s)|,
\]

we get

\[
|\lambda^{(1)}(t)| \leq 2 \sup_{s \in [t-r,t+r]} |\lambda(s)|. \quad \square
\]

This lemma generalizes to the next one.

**Lemma 2.3.** Let \( \lambda : I \to \mathbb{R} \) be \( p \) times differentiable such that for \( i = 2, \ldots, p \), we have either \( \lambda^{(i)} \geq 0 \) on \( I \) or \( \lambda^{(i)} \leq 0 \) on \( I \). Let \( t \in I \) and \( r > 0 \) such that \( [t-r,t+r] \subset I \). Then, for \( j = 1, \ldots, p-1 \), there is a \( C_j > 0 \) such that

\[
|\lambda^{(j)}(t)| \leq r^{-j}C_j \sup_{s \in [t-r,t+r]} |\lambda(s)|.
\]

**Proof.** We proceed by induction on \( j \). The case \( j = 1 \) was proved in Lemma 2.2 with \( C_1 = 2 \). So we assume that the statement of the lemma holds true for \( j \). Let \( u \in [t - \frac{r}{2}, t + \frac{r}{2}] \subset [t-r,t+r] \subset I \). Then the interval \([u - \frac{r}{2}, u + \frac{r}{2}] \subset I \), and we obtain

\[
|\lambda^{(j)}(u)| \leq C_j \left(\frac{r}{2}\right)^{-j} \sup_{s \in [t-r,t+r]} |\lambda(s)|.
\]

Since \( \lambda^{(j+2)} : [t - \frac{r}{2}, t + \frac{r}{2}] \to \mathbb{R} \) is non-negative or non-positive we can apply Lemma 2.2 to \( \lambda^{(j)} \) and get with \( C_{j+1} := C_j 2^{j+2} \),

\[
|\lambda^{(j+1)}(t)| \leq \frac{1}{r} \sup_{u \in [t - \frac{r}{2}, t + \frac{r}{2}]} |\lambda^{(j)}(u)|
\]

\[
\leq \frac{1}{r} C_j \left(\frac{r}{2}\right)^{-j} \sup_{s \in [t-r,t+r]} |\lambda(s)|
\]

\[
= C_{j+1} r^{-(j+1)} \sup_{s \in [t-r,t+r]} |\lambda(s)|. \quad \square
\]

As a consequence of Lemma 2.1 and Lemma 2.3 we obtain
Corollary 2.4. Let \( h : (a, b) \to \mathbb{R} \) be \( p \) times differentiable and
\[
(2.12) \quad h^{(p)}(t) > 0, \quad t \in (a, b).
\]
Then, there is a constant \( \tilde{C}_j \) depending only on \( j \), and a subinterval \((c, d) \subset (a, b)\) of length
\[
(2.13) \quad |d - c| \geq \frac{|b - a|}{p^2}
\]
such that for \( j = 1, \ldots, p - 1 \):
\[
(2.14) \quad |h^{(j)}(s)| \leq \tilde{C}_j \sup_{t \in (c, d)} \frac{|h(t)|}{|d - c|^{j}}, \quad s \in \left( c + \frac{d - c}{4}, d - \frac{d - c}{4} \right).
\]

Proof. According to Lemma 2.1 we get a subinterval \((c, d) \subset (a, b)\) of the desired length where each of the derivatives of \( h \) is definite. We apply Lemma 2.3 to \( h \) restricted to \((c, d)\) with \( r = (d - c)/4 \) and \( t \in \left( c + \frac{d - c}{4}, d - \frac{d - c}{4} \right) \).

\[\square\]

3. Directional derivatives

In this section we consider \((m - 1)^{th}\) directional derivatives of \( m \) times Peano differentiable functions where \( m \geq 2 \). Here we interpret higher-order directional derivatives in the sense of Peano.

Lemma 3.1. Let \( U \subset \mathbb{R}^n \) be open, let \( f : U \to \mathbb{R} \) be an \( m \) times Peano differentiable function and \( \nu \in \mathbb{R}^n \) a unit vector. If the \( m^{th} \) directional derivative of \( f \) with respect to \( \nu \) is locally bounded from below (above) at \( x_0 \in U \), the corresponding \((m - 1)^{th}\) directional derivative is Fréchet differentiable at \( x_0 \).

Proof. We denote by \( B(r) \) the ball with centre 0 and radius \( r \). Without loss of generality we may assume that \( U = B(1) \), \( x_0 = 0 \), and all Peano derivatives up to order \( m \) vanish at 0. Furthermore, we may assume that the \( m^{th} \) directional derivative with respect to \( \nu \) is bounded from below on \( B(1) \) by \( K < 0 \).

We show that the mapping
\[
(3.1) \quad x \mapsto \frac{\partial^{m-1}}{\partial \nu^{m-1}} f(x)
\]
is \( o(\|x\|) \).

In order to show this we assume that the mapping above is not \( o(\|x\|) \); e.g., there is an \( L > 0 \) and a zero-sequence \((y_l)_{l \in \mathbb{N}}\) such that
\[
(3.2) \quad \forall l \quad \frac{\partial^{m-1}}{\partial \nu^{m-1}} f(y_l) \geq L\|y_l\| \quad \text{(or \leq -L\|y_l\|)}.
\]

Step 1. We consider the following family of functions \( g_l : [-\frac{1}{2}, \frac{1}{2}] \to \mathbb{R}, \quad l \in \mathbb{N} \) with
\[
(3.3) \quad g_l(t) := f(y_l + t\nu).
\]
Since \( g_l \) is an \( m \) times Peano differentiable function of the variable \( t \) such that the derivative of order \( m \) is bounded from below, we know by Oliver’s theorem that \( g_l \) is \( m \) times Fréchet differentiable in \([-\frac{1}{2}, \frac{1}{2}]\).

We may apply the Mean-Value Theorem to the \((m - 1)^{th}\) derivative of \( g_l \). If \( s \geq 0 \),
\[
(3.4) \quad g_l^{(m-1)}(s) - g_l^{(m-1)}(0) \geq K(s - 0)
\]
since $K$ is a lower bound for the $m^{th}$ derivative of $g_l$, which is in fact the function $t \mapsto \frac{\partial^m}{\partial t^m}(y_l + tv)$.

Hence, if $0 \leq s \leq \frac{L\|y_l\|}{2K}$,

$$g_l^{(m-1)}(s) = g_l^{m-1}(0) + Ks$$

$$\geq g_l^{(m-1)}(0) - \frac{L\|y_l\|}{2}$$

$$= \frac{\partial^{m-1}}{\partial t^{m-1}}f(y_l) - \frac{L\|y_l\|}{2}$$

$$\geq \frac{L\|y_l\|}{2}.$$

In particular, $g_l^{(m-1)}$ is positive on $(0, \frac{L\|y_l\|}{2K})$.

**Step 2.** We apply Corollary 2.4 to $g_l$ restricted to the interval $(0, \frac{L\|y_l\|}{2K})$ with $p = m - 1$.

Thus, there is a subinterval $(c_l, d_l) \subset (0, \frac{L\|y_l\|}{2K})$ such that

$$|g_l^{(m-2)}(s)| \leq \tilde{C}_{m-2} \sup_{s' \in (c_l, d_l)} \frac{|g_l(s')|}{|d_l - c_l|^{m-2}}$$

whenever $s \in (c_l + \frac{d_l - c_l}{4}, d_l - \frac{d_l - c_l}{4})$. Moreover, we may assume that for $c_l$ and $d_l$,

$$|d_l - c_l| = \frac{L\|y_l\|}{2K(m - 1)^2}.$$ 

So, for $s \in (c_l + \frac{d_l - c_l}{4}, d_l - \frac{d_l - c_l}{4})$ we obtain the inequality

$$|g_l^{(m-2)}(s)| \leq N_1 \sup_{s' \in (c_l, d_l)} \frac{|g_l(s')|}{2\|y_l\|^{m-2}}$$

$$\leq N \sup_{z \in B(\|y_l\|)} \frac{f(z)}{\|z\|^{m-2}}$$

where $N_1$ and $N$ are positive constants depending only on $m, L, K$.

**Step 3a.** Since $g_l^{(m-2)}$ is continuous on the closed interval $[\frac{-1}{2}, \frac{1}{2}]$ we conclude that for $s_l := c_l + \frac{d_l - c_l}{4}$ and $t_l := d_l - \frac{d_l - c_l}{4}$,

$$|g_l^{(m-2)}(t_l) - g_l^{(m-2)}(s_l)| \leq 2N \sup_{z \in B(\|y_l\|)} \frac{f(z)}{\|z\|^{m-2}}.$$ 

**Step 3b.** According to the Mean-Value Theorem in connection with equation (3.5) we can give an alternative estimate for $|g_l^{(m-2)}(t_l) - g_l^{(m-2)}(s_l)|$:

$$|g_l^{(m-2)}(t_l) - g_l^{(m-2)}(s_l)| \geq (t_l - s_l) \inf_{u \in (s_l, t_l)} g_l^{(m-1)}(u)$$

$$\geq (t_l - s_l) \frac{L\|y_l\|}{2}$$

$$= \frac{L\|y_l\|}{2K(m - 1)^2}$$

$$\geq \tilde{N}\|y_l\|^2$$

with a positive constant $\tilde{N}$ depending only on $m, L, K$. 

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Step 4. Since $f$ is $o(\|x\|^m)$ there is for each $l_0 \in \mathbb{N}$, an $l \geq l_0$ such that

$$\sup_{z \in B(2\|y_1\|)} \frac{f(z)}{\|z\|^{m-2}} \leq \frac{N}{4N} \|y_l\|^2.$$  

Steps 3a and 3b now imply for this $l$ that

$$\frac{N}{2} \|y_l\|^2 \geq 2N \sup_{z \in B(2\|y_1\|)} \frac{f(z)}{\|z\|^{m-2}} \geq |g_l^{(m-2)}(t_i) - g_l^{(m-2)}(s_i)| \geq N \|y_l\|^2,$$

which is a contradiction.

So, the assumption of (3.2) is false.

Step 5. We can show analogously that the assumption

$$\forall l \frac{\partial^{m-1}}{\partial \nu^{m-1}} f(y_l) \leq -L \|y_l\|$$

leads to a contradiction.

Hence, $\frac{\partial^{m-1}}{\partial \nu^{m-1}} f(x)$ is $o(\|x\|)$; i.e., it is Fréchet differentiable at 0.

We note that in the case of $\nu$ being one of the standard unit vectors $e_i \in \mathbb{R}^n$, the directional derivatives of $f$ with respect to $\nu$ are the corresponding partial derivatives.

4. RESULTS

Lemma 3.1 leads us directly to the following corollary.

**Corollary 4.1.** Let $U \subset \mathbb{R}^n$ be open and $f: U \to \mathbb{R}$ a 2 times Peano differentiable function. If every second partial derivative of $f$ is locally bounded from above or below at $x_0 \in U$, then $f$ is 2 times Fréchet differentiable at $x_0$.

This corollary cannot directly be generalized to higher-order Peano differentiable functions. This is due to the fact that for $m \geq 3$ the Peano derivatives of order $m-1$ can be mixed derivatives; i.e., they are no longer directional derivatives. But, by some stronger assumptions, as they are given in Theorem 1.4, we can obtain an analogous statement for $m \geq 3$.

**Proof of Theorem 1.4.** If $\nu \in (\mathbb{R}_+^n)$, $t \in \mathbb{R}$, then

$$f(x + tv) = \sum_{|\alpha| \leq m} \frac{f_\alpha(x)}{\alpha!} \nu^\alpha t^{|\alpha|} + o(t^m);$$

cf. equation (1.1). So the $k^{\text{th}}$ directional derivative with respect to $\nu$ at the point $x$ is given by

$$\sum_{|\alpha| = k} \frac{k!}{\alpha!} f_\alpha(x) \nu^\alpha.$$

Since for each $\alpha$ with $|\alpha| = m$, $\frac{\partial^{m-1}}{\partial \nu^{m-1}} f_\alpha$ is non-negative and the $f_\alpha$ are bounded from below (above), the $m^{\text{th}}$ directional derivative of $f$ with respect to $\nu$ is bounded from below (above) so that Lemma 3.1 implies Fréchet differentiability for $\frac{\partial^{m-1}}{\partial \nu^{m-1}} f(x)$.

We show that every $(m-1)^{\text{th}}$ Peano derivative of $f$ is a linear combination of the $(m-1)^{\text{th}}$ directional derivatives with direction $\nu \in (\mathbb{R}_+^n)$. Let $\beta(1), \ldots, \beta(s)$ denote the multi-indices $\alpha \in \mathbb{N}^n$ with $|\alpha| = m - 1$. It remains to show the existence of $s$
linearly independent vectors \( w_l = (\nu_l^{\beta(1)}, \ldots, \nu_l^{\beta(s)}) \) with unit vectors \( \nu_l \in (\mathbb{R}_0^+)^n \), \( l = 1, \ldots, s \). We select a sequence \( p_1 < \cdots < p_n \) of prime numbers, and we put \( p(l) := (p'_l, \ldots, p''_l) \). Then we set \( \nu_l := p(l)/\|p(l)\| \), so \( \nu_l \in (\mathbb{R}_0^+)^s \) is a unit vector.

**Claim:** The vectors \( w_l := (\nu_l^{\beta(1)}, \ldots, \nu_l^{\beta(s)}) \), \( l = 1, \ldots, s \), are linearly independent.

We consider the vectors \( v_l := \|p(l)\|^{m-1}w_l = ((p^{\beta(1)}l)_l, \ldots, (p^{\beta(s)}l)_l) \), so

\[
V := \begin{pmatrix}
v_1 \\
\vdots \\
v_s 
\end{pmatrix}
\]

is a matrix of Vandermonde type. The uniqueness of prime decomposition implies for \( i \neq j \), \( p^{\beta(i)}l \neq p^{\beta(j)}l \), so \( V \) is invertible. Therefore, the \( v_l \) are linearly independent, so the \( w_l \) are linearly independent, so the claim is proved.

Hence, each \( f_{\alpha} \), \( |\alpha| = m - 1 \), is a linear combination of Fréchet differentiable functions, so they are Fréchet differentiable, too.

We note a consequence of Theorem 1.4.

**Corollary 4.2.** Let \( U \subset \mathbb{R}^n \) be an open subset and \( f : U \to \mathbb{R} \) be an \( m \) times Peano differentiable function. Let the \( m^{th} \) Peano derivatives all be locally bounded at \( x_0 \in U \). Then the \((m-1)^{th}\) Peano derivatives are Fréchet differentiable at \( x_0 \).

**References**


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