ON A FRAGMENT OF THE UNIVERSAL BAIRE PROPERTY
FOR $\Sigma^1_2$ SETS

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Abstract. There is a well-known global equivalence between $\Sigma^1_2$ sets having the universal Baire property, two-step $\Sigma^1_3$ generic absoluteness, and the closure of the universe under the sharp operation. In this note, we determine the exact consistency strength of $\Sigma^1_2$ sets being $(2^\omega)^{+\text{-cc}}$-universally Baire, which is below $0^{\#}$. In a model obtained, there is a $\Sigma^1_2$ set which is weakly $\omega^2$-universally Baire but not $\omega^2$-universally Baire.

1. Introduction

Consider the following two properties of a set of reals $A \subset \omega^\omega$ at an infinite cardinal $\kappa$:

1. For every continuous $f : \kappa^\omega \to \omega^\omega$, there is a dense set of $p \in \kappa^{<\omega}$ such that $f^{-1}(A)$ is either meager or comeager below $p$.

2. For every continuous $f : \kappa^\omega \to \omega^\omega$, there is a dense set of $p \in \kappa^{<\omega}$ such that $f^{-1}(A) \cap \sigma^\omega$ is either meager below $p$ in $\sigma^\omega$ for a club of $\sigma \in [\kappa]^{\omega\omega}$ or comeager below $p$ in $\sigma^\omega$ for a club of $\sigma \in [\kappa]^{\omega\omega}$.

The first property asserts that $A$ is $\kappa$-universally Baire or fully captured at $\kappa$, and the second asserts that $A$ is weakly $\kappa$-universally Baire (the author’s coinage) or weakly captured at $\kappa$ (see [9] and Lemma 4.1 below). The implication of (1) to (2) is immediate as a club of $\sigma \in [\kappa]^{\omega\omega}$ is closed under a Banach–Mazur strategy in the space $\kappa^\omega$. Regarding the reverse implication, any set of reals of size $\omega_1$ is a counterexample at $\kappa = \omega_2$ assuming Martin’s Maximum (see Theorem 2.6 of [9] and Theorem 3.1 of [7]). The place to look for a definable counterexample is the pointclass $\Sigma^2_2$ with $\kappa$ either $\omega_1$ or $\omega_2$. This is because (1) and (2) are equivalent for $\kappa = \omega$ and for $\Delta^1_2$ sets as a whole. Since the particular scenario for a counterexample suggested by [9] involves the question of the consistency strength of $\Sigma^1_2$ sets being $\omega_1$- or $\omega_2$-universally Baire, in particular whether this is possible without sharps, this paper was motivated by the following question: What fragment of the universal Baire property can $\Sigma^1_2$ sets have below $0^{\#}$, as measured by the weight or cellularity of the preimage space? In Theorem 3.4 of [1] a global equivalence is established between $\Sigma^1_2$ sets being universally Baire, two-step $\Sigma^1_3$ generic absoluteness, and the closure of the universe under the sharp operation. This equivalence, however, is not true level-by-level. In particular, the relevant part of their argument (originating...
in [8] would require that $\Sigma^1_2$ sets be $\omega_{\omega+1}$-universally Baire to prove the existence of $0^\#$. Using the full strength of covering for $L$, this can be reduced to $\omega_3$. On the other hand, Woodin has shown that $\Pi^1_2$ sets can be $\omega_2$-cc-universally Baire in a forcing extension of $L$.

**Definition 1.1.** $A \subset \mathbb{R}$ is $\kappa$-cc-universally Baire if $f^{-1}(A)$ has the Baire property in $X$ for every completely regular space $X$ of cellularity less than $\kappa$ and every continuous map $f : X \rightarrow \mathbb{R}$.

**Theorem 1.2** (Woodin). Assume $\lambda_0 < \lambda_1 < \lambda_2$ are cardinals of $L$ and that there is an elementary embedding $\pi : L_{\lambda_1} \rightarrow L_{\lambda_2}$ with a critical point $\lambda_0$ such that $\pi(\lambda_0) = \lambda_1$. Then $\Sigma^1_3$ sets are $\omega_2$-cc-universally Baire in a forcing extension of $L$ in which CH holds.

For equivalent versions of Definition 1.1, the reader is referred to Theorem 2.1 of [1]. In particular, we will use that a $\kappa$-cc-universally Baire set remains $\kappa$-cc-universally Baire after forcing with a $\kappa$-cc poset. When combined with the argument of Theorem 3.4 of [1], the above shows that two-step $\Sigma^1_3$ generic absoluteness for $(2^\omega)^+\text{-cc forcings } \mathbb{P} \ast \mathbb{Q}$ can hold in a forcing extension of $L$. In this note we reduce the hypothesis of Theorem 1.2 to obtain an exact equiconsistency.

**Definition 1.3.** An ordinal $\kappa$ is $L$-large to $\lambda$ if for every $\alpha < \lambda$ there is an elementary $j : L_\alpha \rightarrow L_\beta$ with critical point $\kappa$ such that $j(\kappa) \geq \alpha$.

Note that an ordinal $\kappa$ is $L$-large to $(\kappa^+)^L$ if and only if $\kappa$ is weakly compact in $L$. We obtain stronger notions by requiring that $\lambda$ be inaccessible or weakly compact in $L$. Neither notion implies that $0^\#$ exists (simply collapse $\lambda^+$ and use absoluteness of $L[g]$), though $L$ cannot see such an embedding with $\alpha \geq (\kappa^+)^L$.

**Theorem 1.4.** The following are equiconsistent:

1. $\Sigma^1_2$ sets are $(2^\omega)^+\text{-cc-universally Baire}$;
2. There is a $\kappa$ which is $L$-large to a weakly compact $L$.

In the last section of this paper, we argue that in the model of Theorem 1.4(1) there must be a $\Sigma^1_2$ set which has a weak capturing term at $\omega_2$ but no full capturing term at $\omega_2$.

**Theorem 1.5.** It is consistent that $\Sigma^1_2$ sets are weakly $\omega_2$-universally Baire but not $\omega_2$-universally Baire.

In what follows, any space $X^\omega$ will carry the product topology, with the set $X$ given the discrete topology. All pointclasses are boldface, and every statement below involving $\Sigma^1_2$ sets applies equally well to $\Pi^1_2$ sets. We would like to thank Hugh Woodin for sharing his proof of Theorem 1.2 and allowing us to include elements of it here and Stevo Todorcevic for several helpful comments regarding earlier drafts of this paper.

2. Preliminaries

Call a term $\dot{A}$ a $\text{Col}(\omega, \kappa)$ capturing term for a set of reals $A$ if there is a club of countable elementary submodels $X \prec H(\theta)$ with transitivization $H$ and collapse map $\pi$ such that

$$\pi(\dot{A})_g = A \cap H[g]$$
for every \(H\)-generic \(g \subset \text{Col}(\omega, \pi(\kappa))\). A set of reals \(A\) has a \(\text{Col}(\omega, \kappa)\) capturing term if and only if \(A\) is \(\kappa\)-universally Baire (see Lemma 1.6 of [9]). The proof below uses this observation and an argument from [1].

**Theorem 2.1.** The following are equivalent for a cardinal \(\kappa\):

1. \(\Sigma^1_2\) sets are \(\kappa\)-universally Baire;
2. For all sufficiently large \(\theta\), there is a club of countable \(X \prec H(\theta)\) such that \(X[g]\) is \(\Sigma^1_2\) elementary in \(V\) for every \(X\)-generic \(g \subset \text{Col}(\omega, \kappa \cap X)\) which belongs to \(V\).

**Proof.** Let \(A\) be \(\Sigma^1_2\) defined by a formula \(\phi\) (we suppress any parameter). Assuming (2), let \(\dot{A}\) be the set of pairs \((p, \tau)\) such that \(p \models \phi(\tau)\). Thus \(\dot{A}\) is a capturing term for \(A\) and it follows that \(A\) is \(\kappa\)-universally Baire. For the other direction assume \(S\) and \(T\) are trees which witness that a given \(\Sigma^1_2\) set is \(\kappa\)-universally Baire. Suppose this \(\Sigma^1_2\) set is defined by a formula \(\phi(x)\) (again suppressing parameters). The argument of Theorem 3.4 of [1] shows that

\[
p[S]^{V[G]} = \{x \mid \phi(x)\}^{V[G]},
\]

where \(G \subset \text{Col}(\omega, \kappa)\) is \(V\)-generic. This uses \(\Pi^1_1\)-uniformization. Let \(\phi\) be a \(\Sigma^1_1\) formula defining the universal \(\Sigma^1_2\) set \(A\) and let \(X \prec H(\theta)\) contain \(S\) and \(T\). Then \(p[S]^{X[g]} = A \cap X[g]\) and \(X[g]\) thinks \(p[S]^{X[g]}\) is the universal \(\Sigma^1_2\) set. Hence \(X[g]\) is \(\Sigma^1_2\) elementary in \(V\). \(\square\)

Using (2) and the full strength of covering for \(L\), we may now argue that \(\Sigma^1_2\) sets being \(\omega_3\)-universally Baire imply that \(0^\#\) exists. For a set of ordinals \(\sigma\) we let \(\text{otp}(\sigma)\) denote the order type of \(\sigma\).

**Theorem 2.2.** Assume \(\Sigma^1_2\) sets are \(\omega_3\)-universally Baire. Then \(0^\#\) exists.

**Proof.** Let \(\kappa = \omega_3\). We first argue that there are club many \(\sigma \in [\kappa]^{<\omega}\) such that \(\text{otp}(\sigma)\) is a regular cardinal of \(L\). Let \(\kappa \in X \prec H(\theta)\) be countable with transitive collapse \(\pi : X \rightarrow \dot{X}\). Note that \(\pi(\kappa) = \text{otp}(X \cap \kappa)\) and that there are club many such \(X \cap \kappa\). Thus \(\dot{X}\) thinks that \(\pi(\kappa)\) is a cardinal of \(L\). If \(\pi(\kappa)\) were not a regular cardinal of \(L\), then there would be a countable \(L_\gamma\) which sees this. Let \(g \subset \text{Col}(\omega, \pi(\kappa))\) be \(\dot{X}\)-generic, and let \(z \in \dot{X}[g]\) be a real coding a well-ordering of length \(\pi(\kappa)\). Then by Theorem 2.1, \(\dot{X}[g]\) thinks there is a level of \(L\) which sees that the ordinal coded by \(z\) is not regular. This is a contradiction as \(L^{\dot{X}} = L^{\dot{X}[g]}\). We now argue that the set of \(\alpha < \kappa\), such that \(\alpha\) is a regular cardinal of \(L\), contains a club in \(V\). Let \(f : \kappa^{<\omega} \rightarrow \kappa\) be such that any \(\sigma \in [\kappa]^{<\omega}\) which is closed under \(f\) has the property that \(\text{otp}(\sigma)\) is a regular cardinal of \(L\). Let \(\alpha < \kappa\) such that \(f[\alpha^{<\omega}] \subset \alpha\). Since there are club many such \(\alpha\), it suffices to show that \(\alpha\) is a regular cardinal of \(L\). Suppose not. There is a countable \(X \prec H(\kappa)\) with \(\alpha \in X\) such that \(X \cap \alpha = \sigma\) is closed under \(f\). Let \(X\) be the transitiveivization of \(X\) with a collapse map \(\pi\). Then \(\dot{X}\) thinks that \(\pi(\alpha)\) is not a regular cardinal of \(L\); hence, \(\pi(\alpha)\) is not a regular cardinal of \(L\) by absoluteness. This contradicts \(\pi(\alpha) = \text{otp}(\sigma)\). It follows that there is an \(\alpha\) with \(cf(\alpha) < \omega_3 < \alpha < \omega_3\) which is a regular cardinal of \(L\). Let \(\sigma \subset \alpha\) be unbounded in \(\alpha\) and have size \(cf(\alpha)\). Then \(\sigma\) cannot be covered by a set in \(L\) of size \(\omega_1\). \(\square\)

We conjecture that \(\omega_3\)-cc-universally Baire suffices for the argument above. Under this assumption \(\omega_3\) is weakly compact in \(L\) by Lemma 4 of [S] and a theorem in
We close this section with an equivalence between $\Sigma^1_2$ sets being $\omega_1$-universally Baire and the existence of a club of suitably closed submodels. We say that $\omega_2$ is inaccessible to $P(\omega_1)$ if $\omega_2$ is an inaccessible cardinal in $L[X]$ for every $X \subseteq \omega_1$.

**Lemma 2.3.** The following are equivalent:

1. $\omega_2$ is inaccessible to $P(\omega_1)$ and $\Sigma^1_2$ sets are $\omega_1$-universally Baire;
2. $\omega_2$ is inaccessible to $P(\omega_1)$ and for sufficiently large $\theta$ there is a club of $X < H(\theta)$ such that for every $\tau \in P(\omega_1) \cap X$ and every $L[\tau]$-cardinal $\gamma \in X \cap \omega_2$ the order type of $\gamma \cap X$ is itself an $L[\tau \cap X]$-cardinal;
3. For sufficiently large $\theta$ there is a club of $X < H(\theta)$ such that for every $\tau \in P(\omega_1) \cap X$ the order type of $X \cap \omega_2$ is an $L[\tau \cap X]$-cardinal.

**Proof.** By the argument of Theorem 2.2, condition (3) implies that $\omega_2$ is inaccessible to $P(\omega_1)$. Thus (2) and (3) are equivalent. Again by a boldface version of an argument from Theorem 2.2, (1) implies (2). Let $X < H(\theta)$ be as in (2). Let $\pi : X \to X$ be the collapse map and let $g \in \text{Col}(\omega, \omega_1 \cap X)$ be $X$-generic. Let $y = \pi(\tau)g$ be a real in $X[g]$. Since $g$ is also $L[\pi(\tau)]$-generic, we have

$$\pi(\omega_2) > (\pi(\omega_1)^+)_{L[\pi(\tau)]} \geq (\omega_1)^{L[g]}$$

so that $X[g]$ is correct about $\Sigma^1_2$ facts in the parameter $y$. \qed

3. **Equiconsistency results**

Fix a surjection $f_\gamma : \omega_1 \to \gamma$ for each $\gamma$ between $\omega_1$ and $\omega_2$. If $\gamma$ is a cardinal of $L$, let $S_\gamma$ denote the set of $\alpha < \omega_1$ such that the order type of $f_\gamma[\alpha]$ is an $L$-cardinal. Let $S$ be the set of $\sigma \in [\omega_2]^{\omega_1}$ such that

$$\sigma \cap \omega_1 \in \bigcap_{\gamma \in \sigma} S_\gamma.$$ 

If we assume that $\omega_2$ is inaccessible in $L$ and that there are stationary many $\sigma \in [\omega_2]^{\omega_1}$ such that $\text{otp}(\sigma)$ is an $L$-cardinal, then it follows that $S$ is stationary. Now let $Q$ be the countable support product of $Q_\gamma$, ranging over ordinals $\gamma < \omega_2$ which are $L$-cardinals, where $Q_\gamma$ is the poset for shooting a club through $S_\gamma$ with countable conditions. It follows that $Q$ is $(\omega, \infty)$-distributive. If $CH$ holds, then $Q$ satisfies the $\omega_2$-chain condition. The following key lemma is implicit in Woodin’s proof of Theorem 1.2. We thank the referee for pointing out that condition (2) below is considered in \[3\].

**Lemma 3.1.** Suppose that:

1. every subset of $\omega_1$ is $L$-generic for some poset $P \in L$ with $|P| < \omega_2$;
2. there are stationary many $\sigma \in [\omega_2]^{\omega_1}$ such that $\text{otp}(\sigma)$ is an $L$-cardinal;
3. $\omega_2$ is inaccessible in $L$ and $CH$ holds.

Then $\Sigma^1_2$ sets are $\omega_1$-universally Baire in $V[G]$ where $G \subseteq Q$ is $V$-generic.

**Proof.** We show that condition (3) of Lemma 2.3 is satisfied in $V[G]$. As discussed above, $Q$ preserves cardinals under these hypotheses and by design there is in $V[G]$ a club of $\sigma \in [\omega_2]^{\omega_1}$ such that $\text{otp}(\sigma)$ is a cardinal of $L$. Furthermore, condition (1) continues to hold in $V[G]$. Suppose $X < H(\theta)$ is such that $\text{otp}(X \cap \omega_2)$ is an $L$-cardinal. Let $\tau \in P(\omega_1) \cap X$. Then there are $P, H \in X$ such that $X$ thinks that $P \in L, |P| < \omega_2, H \in P$ is $L$-generic and $\tau \in L[H]$. Let $\pi : X \to \hat{X}$ be the transitivization map. As $\text{otp}(X \cap \omega_2) = (\omega_2)^X$ is a limit cardinal of $L$, it follows
that \( \pi(H) \subset \pi(\mathcal{P}) \) is \( L \)-generic and \( \tau \cap X \in L[\pi(H)] \). Thus \( \text{otp}(X \cap \omega_2) \) remains a cardinal in \( L[\tau \cap X] \) as desired. \( \square \)

Schindler pointed out to the author that condition (3) below is equiconsistent with the existence of a cardinal which is remarkable up to an inaccessible cardinal, a notion from his papers \([5]\) and \([6]\). This large cardinal concept has the advantage of not mentioning an inner model in its definition. We include this observation without proof.

**Theorem 3.2.** The following are equiconsistent:

1. \( \omega_2 \) is inaccessible in \( L \) and \( \Sigma^1_2 \) sets are \( \omega_1 \)-universally Baire;
2. There are club many \( \sigma \in [\omega_2]^\omega \) such that \( \text{otp}(\sigma) \) is an \( L \)-cardinal;
3. \( \omega_2 \) is inaccessible in \( L \) and there are stationary many \( \sigma \in [\omega_2]^\omega \) such that \( \text{otp}(\sigma) \) is a cardinal of \( L \);
4. There is a \( \kappa \) which is \( L \)-large to an \( L \)-inaccessible;
5. There is a cardinal \( \kappa \) which is remarkable up to an inaccessible cardinal.

**Proof.** (1) implies (2) outright by Lemma 2.3. The argument for (2) implies (3) is implicit in the proof of Theorem 2.2. Assume (3). Let \( g \subset \text{Col}(\omega, \omega_1) \) be \( V \)-generic. Let \( \kappa = \omega^Y \). Then in \( L[g] \) there is a stationary set of \( \sigma \in [\kappa]^\omega \) such that the order type of \( \sigma \) is an \( L \)-cardinal. Thus if \( h \subset \text{Col}(\omega_1, \omega^Y) \) is \( L[g] \)-generic in \( L[g][h] \), then the hypotheses of Lemma 3.1 are satisfied so that (1) holds in the forcing extension described there. Thus (1), (2) and (3) are equiconsistent. Assume (1). We will show that \( \omega^Y \) is \( L \)-large to \( \omega^Y \) in \( V[g] \) where \( g \subset \text{Col}(\omega, \omega_1) \) is \( V \)-generic. Let \( X \prec H(\theta) \) be countable. Let \( \pi: X \rightarrow H \) be the transitive collapse. Let \( Y \prec H(\theta) \) with \( \pi, H \in Y \) and let \( j: Y \rightarrow M \) be its trivialization. Note that \( j \circ \pi^{-1} = j(\pi) \). Call this map \( k \). We have that

\[
k \downarrow | L_{\omega^Y} : L_{\omega^Y} \rightarrow L_{\gamma}
\]

is fully elementary with critical point \( \omega^H \) and this map is an element of \( M \). Because \( Y \) sees that \( H \) is countable, we have

\[
k(\omega_1) = Y \cap \omega_1 > \omega^H.
\]

Let \( g \subset \text{Col}(\omega, \omega^H) \) be \( H \)-generic. Let \( \alpha < \omega^H \) be arbitrary and let \( x \in H[g] \) be a real coding a well-ordering of length \( \alpha \). The sentence asserting the existence of a transitive model of a sufficient fragment of set theory containing \( y \) which sees an embedding \( k: L_\alpha \rightarrow L_\beta \) with critical point \( \omega^H \) such that \( j(\omega^H) > \alpha \) is \( \Sigma^1_2 \) in the parameter \( x \). Hence \( H[g] \) sees such an embedding. As \( \alpha \) is arbitrary, we conclude that \( H[g] \) thinks that \( \omega^H \) is \( L \)-large to \( \omega^H \). Now apply \( \pi \). To connect (4) back to (1), assume that \( \kappa \) is \( L \)-large to some \( L \)-inaccessible \( \lambda \). Let \( g_\lambda \subset \text{Col}(\omega, \omega_1) \) be \( V \)-generic. Then \( g_\lambda \) is also \( L \)-generic for the same forcing. By folding the embeddings witnessing our hypothesis (4) into countable submodels and collapsing, we see that \( \kappa \) is \( L \)-large to \( \lambda = \omega^L_{\text{Col}} \) in \( L[g_\lambda] \). For ordinals \( \gamma < \lambda \) let \( g_\gamma \) denote \( g \cap \text{Col}(\omega, \omega_1) \). We claim that in \( L[g_\lambda] \), there are stationary many \( \sigma \in [\lambda]^\omega \) such that \( \text{otp}(\sigma) \) is an \( L \)-cardinal. It will then follow that (3) holds after forcing with \( \text{Col}(\omega_1, \omega_1) \). Let \( f: \lambda^{< \omega} \rightarrow \lambda \) belonging to \( L[g_\lambda] \) be arbitrary. Let \( \delta < \lambda \) be a cardinal of \( L[g_\lambda] \) such that \( f[\delta^{< \omega}] \subseteq \delta \). Let \( \alpha \) be a regular cardinal of \( L[g_\lambda] \) below \( \lambda \) which is greater than \( (\delta^+)^L \) so that \( f \upharpoonright \delta^{< \omega} \in L_\alpha[g_\lambda] \). In \( L[g_\lambda] \) there is an elementary embedding

\[
j: L_\alpha \rightarrow L_\beta
\]
with a critical point \( \kappa \) such that \( j(\kappa) > \alpha \). By standard arguments (using the fact that \( \text{Col}(\omega, \kappa) \) is \( \kappa\text{-cc} \) in \( L \)) this embedding extends to a fully elementary

\[
j : L_\alpha[g_\kappa] \to L_\beta[g_{j(\kappa)}].
\]

The embedding is defined by \( j(\text{val}(\tau, g_\kappa)) = \text{val}(j(\tau), g_{j(\kappa)}) \) and since it extends \( j \), we will also denote it by \( j \). Now, let \( \sigma \) denote the set \( j[\delta] \). Let \( z \) be a real in \( L_\beta[g_{j(\kappa)}] \) coding a well-ordering of length \( \delta \). The structure \( L_\beta[g] \) has a tree \( T \) consisting of pairs \( (s, t) \) with \( s \) a finite approximation to a set of ordinals \( \sigma \) closed under \( f \) and with \( t \) a finite approximation to an order isomorphism between \( \delta \) and \( \sigma \). Moreover, \( L_\beta[g_{j(\kappa)}] \) must see a branch through this tree and the result follows by reflection using \( j \).

\[ \square \]

**Lemma 3.3.** Suppose \( \kappa \) is a weakly compact cardinal and \( P \) is \( \kappa\text{-cc} \). Suppose \( \Pi^1_2 \) sets are \( \kappa\text{-cc-universally Baire} \) in \( V[G] \) where \( G \subset P \) is \( V \)-generic. Then \( \Pi^1_2 \) sets are \( \kappa\text{-cc-universally Baire} \) in \( V[G] \).

**Proof.** We will use the fact that a set \( A \subset \omega^\omega \) is \( \kappa\text{-cc-universally Baire} \) if and only if there are trees \( S, T \) on some \( \omega \times \lambda \) which project to \( A \) and its complement and continue to project to complements after forcing with any \( \kappa\text{-cc} \) poset. So let \( \mathcal{Q} \) be a \( \mathbb{P}\)-name for a poset which is forced by \( \mathbb{P} \) to have the \( \kappa\text{-cc} \). Thus \( \mathbb{P} * \mathcal{Q} \) has the \( \kappa\text{-cc} \) in \( V \). Now suppose \( \dot{x} \) is a \( \mathbb{P} * \mathcal{Q} \)-name for a real. Since \( \mathbb{P} * \mathcal{Q} \) is \( \kappa\text{-cc} \) and \( \kappa \) is weakly compact, there is an elementary suborder \( A \prec \mathbb{P} * \mathcal{Q} \) which has size strictly less than \( \kappa \), decides \( \dot{x} \), and has the property that maximal \( A \) antichains are maximal antichains in \( \mathbb{P} * \mathcal{Q} \). The upshot of this is that over \( V[G] \) where \( G \subset \mathbb{P} \) is \( V \)-generic, every real which is generic for a \( \kappa\text{-cc} \) forcing is generic for a forcing of size \( < \kappa \). Let \( A \) be a \( \Sigma^1_2 \) set and for each forcing \( \mathcal{Q} \) of size \( < \kappa \) (whose underlying set is some ordinal below \( \kappa \) say) let \( S_\mathcal{Q}, T_\mathcal{Q} \) be \( \mathcal{Q}\)-universally Baire representations of \( A \). These trees may be joined to produce the desired \( \kappa\text{-cc-universally Baire} \) representation of \( A \).

Schindler pointed out to the author that condition (4) could be added to the theorem below (see the remarks preceding Theorem 3.2).

**Theorem 3.4.** The following are equiconsistent:

1. \( \Sigma^1_2 \) sets are \( (2^\omega)^+\text{-cc-universally Baire} \);
2. \( \omega_2 \) is weakly compact in \( L \) and there are stationary many \( \sigma \in [\omega_2]^\omega \) such that \( \text{otp}(\sigma) \) is a cardinal of \( L \);
3. There is a \( \kappa \) which is \( L\text{-large} \) to a weakly compact cardinal of \( L \);
4. There is a cardinal which is remarkable up to a weakly compact cardinal.

**Proof.** This is identical to the proof of Theorem 3.2, using Theorem 3.4 in the argument from (2) to (1) to get the stronger conclusion. Of course we are using that \( CH \) holds in all models under consideration. We need to show that (1) implies that \( \omega_2 \) is weakly compact in \( L \). Assume (1) and let \( \mathcal{Q} \) be the poset for forcing Martin’s Axiom. Let \( \mathbb{P} = \text{Col}(\omega, \omega_1) * \mathcal{Q} \) and note that \( \mathbb{P} \) is \( \omega_2\text{-cc} \). In the extension \( V[G] \) by \( \mathbb{P} \) we will have \( \Sigma^1_2 \) sets ccc-universally Baire. This implies that \( \omega_1 \) is inaccessible to reals in this model by a result in [8]. Thus \( \omega_1 = \omega^L_1 \) is weakly compact in \( L \) in \( V[G] \) by a result of Harrington and Shelah (see [2] or Lemma 7 of [1]). \( \square \)
4. Weak capturing does not imply capturing

A weakening of the universal Baire property is presented in [3]. A set of reals $A$ is weakly captured at $\kappa$ if there is a $Col(\omega, \kappa)$-term $\dot{A}$ such that for sufficiently large $\theta$, for a club of countable $H \prec H(\theta)$, and for a comeager set of $g : \omega \to otp(H \cap \kappa)$,

$$\pi_H(\dot{A})_g = A \cap H[g],$$

where $otp(H \cap \kappa)$ is the order type of $H \cap \kappa$ and $\pi_H$ is the transitivization map. A less metamathematical characterization is the following.

**Lemma 4.1.** The following are equivalent:

1. $A$ is weakly captured at $\kappa$;
2. For every continuous $f : \kappa^\omega \to \omega^\omega$, there is a dense set of $p \in \kappa^{<\omega}$ such that $f^{-1}(A) \cap \sigma^\omega$ is either meager below $p$ in $\sigma^\omega$ for a club of $\sigma \in [\kappa]^{<\omega}$ or comeager below $p$ in $\sigma^\omega$ for a club of $\sigma \in [\kappa]^{<\omega}$.

**Proof.** (1) implies (2) is immediate as any condition $p \in \kappa^{<\omega}$ has a refinement $p \subseteq q$ such that $q \Vdash_{Col(\omega, \kappa)} f(\dot{g}) \in \dot{A}$ or $q \Vdash_{Col(\omega, \kappa)} f(\dot{g}) \notin \dot{A}$. For the other direction, if $\tau$ is a standard $Col(\omega, \kappa)$ term for a real, then $\tau$ gives rise to a function $\tau_{\dot{g}} : \kappa^\omega \to \omega^\omega$ defined by $\tau_{\dot{g}}(g) = \tau_g$ which is continuous on a comeager set. Define $\dot{A}$ to be the set of $(p, \tau)$ such that $f^{-1}(A) \cap \sigma^\omega$ is comeager below $p$ in $\sigma^\omega$ for a club of $\sigma \in [\kappa]^{<\omega}$. A straightforward argument shows that $\dot{A}$ is a weak capturing term for $A$. $\Box$

**Theorem 4.2.** It is consistent relative to the existence of a cardinal which is remarkable up to a weakly compact cardinal that $\Sigma^1_2$ is weakly captured at $\omega_2$ but not fully captured at $\omega_2$.

**Proof.** Suppose $\kappa$ is $L$-large to an $L$-weakly compact, and let $L[g][h]$ be the model of Theorem 3.2 in which $\kappa = \omega_1$ and $\lambda = \omega_2$. We have shown that there is a stationary set $S \subseteq [\omega_2]^{<\omega}$ such that whenever $X \prec H(\theta)$ is such that $X \cap \omega_2 \in S$, then $X[g]$ is $\Sigma^1_2$ element in $V$ for every $X$-generic $g \in Col(\omega, \omega_1 \cap X)$. The forcing $Q$ of Lemma 3.1 puts a club through $S$ and so in the extension all $\Sigma^1_2$ sets are $\omega_1$-universally Baire. We first argue that $WRP(\omega_2)$ holds in $L[g][h][G]$. For $a \subseteq \lambda$, let $Q_a$ denote the countable support product of $\mathbb{P}_a$ taken over $L$-cardinals $\gamma \in a$, with $S$ the underlying stationary set. Returning to $L[g]$ where $\lambda$ is still weakly compact, let $p$ be a condition and $\dot{T}$ a term such that

$$p \Vdash_{Col(\omega_1, \lambda^{<\omega})} \dot{T} \text{ is stationary in } [\omega_2]^{<\omega}.$$  

By the usual reflection argument we have an inaccessible $\delta < \omega_2$ such that

$$p \Vdash_{Col(\omega_1, \delta^{<\delta})} \dot{T}_\delta \text{ is stationary in } [\delta]^{\omega},$$  

where $\dot{T}_\delta$ denotes $\dot{T} \cap V_\delta$. Now let $h \in Col(\omega_1, \delta)$ be $L[g]$-generic and let $G \subseteq Q$ be $L[g][h]$-generic below the condition $p$. Let $h_\delta$ and $G_\delta$ be the restrictions to $Col(\omega_1, \delta)$ and $Q_\delta$, respectively. These are $L[g]$-generic as well and

$$\text{val}(\dot{T}_\delta, h_\delta \ast G_\delta) = \text{val}(\dot{T}, h \ast G) \cap [\delta]^{\omega}$$

in $L[g][h][G]$. We need to show that the stationarity of $T_\delta = \text{val}(\dot{T}_\delta, h_\delta \ast G_\delta)$ is preserved. It suffices to show that the stationarity of $T$ is preserved by $Q_{\lambda \setminus \delta}$ over $L[h][G_\delta]$. The key point is that $\{\sigma \cap \delta \mid \sigma \in S\}$ contains a club in $[\delta]^{\omega}$. Thus if $\dot{C}$ is a name for a club subset of $[\delta]^{\omega}$, we can find a dense set of conditions $t \in Q_{\lambda \setminus \delta}$
with a corresponding \( \sigma \in S \) such that \( \sigma \cap \delta \in T \) and \( t \) forces \( \sigma \in \check{C} \). Now let \( G \) be \( L[g][h] \)-generic for the forcing \( Q \). This forcing does not add countable sets of ordinals. Let \( A \) be a \( \Sigma_1^1 \) set in \( L[g][h][G] \). Then \( A = A^{L[g][h]} \). Fix such an \( A \). We claim that \( A \) is weakly captured in \( L[g][h][G] \). Otherwise, there is a condition \( t \in Q \), terms \( \check{T}_m \) and \( \check{T}_c \), and \( p \in \omega_1^\omega \) such that \( t \) forces the following to hold:

1. \( \check{T}_m \) and \( \check{T}_c \) are both stationary subset of \( S \);
2. \( \sigma \in \check{T}_m \) implies \( \check{f}^{-1}(A) \cap \sigma^\omega \) is meager below \( p \);
3. \( \sigma \in \check{T}_c \) implies \( \check{f}^{-1}(A) \cap \sigma^\omega \) is comeager below \( p \).

We may assume that there is a \( \delta < \omega_2 \) such that \( t \) forces both \( \check{T}_m \) and \( \check{T}_c \) to reflect to \( \delta \). Let \( \bar{t} \leq t \) and \( \bar{p} \leq p \) such that

\[
\bar{t} \Vdash_Q \check{f}^{-1}(A) \cap \delta^\omega \text{ is comeager below } \bar{p}.
\]

It follows that \( \bar{t} \) forces that \( \check{f}^{-1}(A) \cap \sigma^\omega \) is comeager below \( \bar{p} \) for a club of \( \sigma \in [\delta]^\omega \), a contradiction. To finish the proof of the theorem, we must show that \( \Sigma_1^2 \) sets are not \( \omega_2 \)-universally Baire in \( L[g][h][G] \). Let \( D \) be the set of \( \alpha < \lambda \) such that \( cf(\alpha) = \omega \) in \( L \). As \( L[g][h][G] \) is a \( \lambda \)-cc extension of \( L \), we know that \( D \) remains stationary in \( L[g][h][G] \). Thus the set of regular cardinals of \( L \) below \( \omega_2 \) cannot be club, as they would be if \( \Sigma_1^2 \) sets were \( \omega_2 \)-universally Baire by the argument of Theorem 2.2.

\[ \square \]

References


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