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# ANTIHOLOMORPHIC INVOLUTIONS OF SPHERICAL COMPLEX SPACES

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Abstract. Let $X$ be a holomorphically separable irreducible reduced complex space, $K$ a connected compact Lie group acting on $X$ by holomorphic transformations, $\theta : K \to K$ a Weyl involution, and $\mu : X \to X$ an antiholomorphic map satisfying $\mu^2 = \mathrm{Id}$ and $\mu(kx) = \theta(k)\mu(x)$ for $x \in X$, $k \in K$. We show that if $\mathcal{O}(X)$ is a multiplicity free $K$-module, then $\mu$ maps every $K$-orbit onto itself. For a spherical affine homogeneous space $X = G/H$ of the reductive group $G = K^{\mathbb{C}}$ we construct an antiholomorphic map $\mu$ with these properties.

## 1. Introduction

Let $X = (X, \mathcal{O})$ be a complex space on which a compact Lie group $K$ acts continuously by holomorphic transformations. Then the Fréchet space $\mathcal{O}(X)$ has the natural structure of a $K$-module. Recall that a $K$-module $W$ is called *multiplicity free* if any irreducible $K$-module occurs in $W$ with multiplicity 1 or does not occur at all. A self-map $\mu$ of a complex space $X$ is called an *antiholomorphic involution* if $\mu$ is antiholomorphic and $\mu^2 = \mathrm{Id}$. For complex manifolds, J. Faraut and E. G. F. Thomas gave an interesting and simple geometric condition which implies that $\mathcal{O}(X)$ is a multiplicity free $K$-module; see [FT]. Namely, for a complex manifold $X$ the $K$-action in $\mathcal{O}(X)$ is multiplicity free if

(FT) *there exists an antiholomorphic involution $\mu : X \to X$ with the property that, for every $x \in X$, there is an element $k \in K$ such that $\mu(x) = k \cdot x$.*

The proof of Theorem 3 in [FT] goes without changes for irreducible reduced complex spaces. It should be noted that the setting in [FT] is more general. Namely, the authors consider any, not necessarily compact, group of holomorphic transformations of $X$ and study invariant Hilbert subspaces of $\mathcal{O}(X)$. We will give a simplified proof of their result in our context; see Proposition 3.3. Our main purpose, however, is to prove the converse theorem for a special class of manifolds, namely, for Stein (or, equivalently, affine algebraic) homogeneous spaces of complex reductive groups.

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Let $G$ be a connected reductive complex algebraic group and $K \subset G$ a maximal compact subgroup. We prove that an affine homogeneous space $X = G/H$ is spherical (or, equivalently, $\mathcal{O}(X)$ is a multiplicity free $K$-module) if and only if the $K$-action on $X$ satisfies (FT); see Theorem 5.5. Recall that a diffeomorphism $\mu$ of a manifold $X$ with a $K$-action is said to be *$\theta$-equivariant* if $\theta$ is an automorphism of $K$ and $\mu(kx) = \theta(k)\mu(x)$ for all $x \in X$, $k \in K$. For $X = G/H$ spherical we can say more about $\mu$ in (FT). Namely, again by Theorem 5.5, $\mu$ can be chosen $\theta$-equivariant, where $\theta$ is a Weyl involution of $K$.

In order to prove Theorem 5.5, we consider $\theta$-equivariant antiholomorphic involutions in a more general context. Namely, let $X$ be a holomorphically separable irreducible reduced complex space, $K$ a connected compact Lie group of holomorphic transformations of $X$, and $\mu$ an antiholomorphic $\theta$-equivariant involution of $X$. Then our Theorem 4.1 asserts that $\mathcal{O}(X)$ is multiplicity free if and only if $\mu(x) \in Kx$ for all $x \in X$, i.e., if $X$ has property (FT) with respect to $\mu$.

Another important ingredient in the proof of Theorem 5.5 is the construction of two commuting involutions of $G$, a Weyl involution and a Cartan involution, which both preserve a given reductive spherical subgroup $H \subset G$; see Theorem 5.4. The proof is based on the results of [AV] and, therefore, on the classification of spherical subgroups.

At the end of our paper we give an example of an affine homogeneous space without $\theta$-equivariant antiholomorphic involutions; see Proposition 6.3.

## 2. Fourier series of Harish-Chandra

Harish-Chandra carried over the classical Fourier series to the representation theory of compact Lie groups in Fréchet spaces; see [H-C]. In this paper, we will need only the representations in $\mathcal{O}(X)$, where $X$ is a complex space. We recall the result of Harish-Chandra in this setting. The details can be found in [H-C]; see also [A], Ch. 5.

Let $K$ be a compact Lie group, $\hat{K}$ its unitary dual, and $dk$ the normalized Haar measure on $K$. For $\delta \in \hat{K}$ let $\chi_\delta$ denote the character of $\delta$ multiplied by the dimension of $\delta$. Suppose that $K$ acts by holomorphic transformations on a complex space $X$. Then we have a continuous representation of $K$ in $\mathcal{C}(X)$ and in $\mathcal{O}(X)$. We will assume that $X$ is reduced and irreducible, so the representation is given by $k \cdot f(x) = f(k^{-1}x)$, where $k \in K, x \in X$.

Define an operator family $\{E_\delta\}_{\delta \in \hat{K}}$ in $\mathcal{O}(X)$ by

$$E_\delta f(x) = \int_K \overline{\chi_\delta(k)} \cdot f(k^{-1}x) \cdot dk.$$

From orthogonality relations for characters it follows that all $E_\delta$ commute with the representation of $K$. Furthermore, $\{E_\delta\}_{\delta \in \hat{K}}$ is a family of projection operators, i.e., $E_\delta^2 = E_\delta$ and $E_\delta E_\epsilon = 0$ if $\delta \neq \epsilon$. Let $\mathcal{O}_\delta(X) = E_\delta \mathcal{O}(X)$. Then $\mathcal{O}_\delta(X) = \mathrm{Ker}(E_\delta - \mathrm{Id})$, so $\mathcal{O}_\delta(X)$ is a closed subspace. Again from orthogonality relations it follows that $\mathcal{O}_\delta(X)$ is the *isotypic component of type* $\delta$, i.e., $\mathcal{O}_\delta(X)$ consists of all those vectors in $\mathcal{O}(X)$ whose $K$-orbit is contained in a finite-dimensional $K$-submodule where the representation is some multiple of $\delta$.

Harish-Chandra proved that each $f \in \mathcal{O}(X)$ can be uniquely written in the form

$$f = \sum_{\delta \in \hat{K}} f_\delta,$$

where $f_\delta = E_\delta(f) \in \mathcal{O}_\delta(X)$ and the convergence is absolute and uniform on compact subsets in $X$.

Assume now that $L$ is another compact Lie group acting by holomorphic transformations of another complex space $Y$ subject to our assumptions. We will use similar notation for $L$, in particular, $\theta_\epsilon$ will denote the character of $\epsilon \in \hat{L}$ multiplied by the dimension of $\epsilon$. For the representation of $K \times L$ in $\mathcal{O}(X \times Y)$ defined by

$$(k,l) \cdot f(x,y) = f(k^{-1}x, l^{-1}y),\ x \in X,\ y \in Y,\ k \in K,\ l \in L,$$

the type of an isotypic component is determined by a pair $\delta \in \hat{K},\ \epsilon \in \hat{L}$. The corresponding isotypic component will be denoted $\mathcal{O}_{\delta,\epsilon}(X \times Y)$. Of course, the tensor product $\mathcal{O}_\delta(X) \otimes \mathcal{O}_\epsilon(Y)$ is contained in $\mathcal{O}_{\delta,\epsilon}(X \times Y)$. We will need the following lemma.

**Lemma 2.1.** *If* $\dim \mathcal{O}_\delta(X) < \infty$ *for some* $\delta \in \hat{K}$, *then* $\mathcal{O}_{\delta,\epsilon}(X \times Y) = \mathcal{O}_\delta(X) \otimes \mathcal{O}_\epsilon(Y)$ *for all* $\epsilon \in \hat{L}$.

*Proof.* Let $f \in \mathcal{O}_{\delta,\epsilon}(X \times Y)$. Then

$$\begin{aligned} f(x,y) &= \int_{K \times L} \overline{\chi_\delta(k)\theta_\epsilon(l)} \cdot f(k^{-1}x, l^{-1}y) \cdot dk dl \\ &= \int_L \overline{\theta_\epsilon(l)} \left( \int_K \overline{\chi_\delta(k)} \cdot f(k^{-1}x, l^{-1}y) \cdot dk \right) dl \end{aligned}$$

by Fubini's theorem. The function

$$x \mapsto \int_K \overline{\chi_\delta(k)} \cdot f(k^{-1}x, y) \cdot dk$$

is in $\mathcal{O}_\delta(X)$ for all $y \in Y$. Let $\{\varphi_i\}_{i=1,\dots,N}$ be a basis of $\mathcal{O}_\delta(X)$. Then

$$\int_K \overline{\chi_\delta(k)} \cdot f(k^{-1}x, y) \cdot dk = \sum_{i=1}^N c_i(y)\varphi_i(x)$$

with some $c_i \in \mathcal{O}(Y)$. Replace $y$ by $l^{-1}y$ in this equality, multiply it by $\overline{\theta_\epsilon(l)}$, and integrate over $L$ against the Haar measure $dl$. Then we get

$$f(x,y) = \sum_{i=1}^N \varphi_i(x)\psi_i(y),$$

where

$$\psi_i(y) = \int_L \overline{\theta_\epsilon(l)} \cdot c_i(l^{-1}y) \cdot dl \ \in \mathcal{O}_\epsilon(Y). \qquad \square$$

## 3. $K$-action and complex conjugation

As in the previous section, $X$ is an irreducible reduced complex space and $K$ is a compact group acting on $X$ by holomorphic transformations.

**Lemma 3.1.** *Let* $W \subset \mathcal{O}(X)$ *be a finite-dimensional* $K$*-submodule. Introduce a* $K$*-invariant Hermitian inner product and choose a unitary basis* $\{f_1, \dots, f_N\}$ *in* $W$. *The function* $F := \sum_{j=1}^N f_j \overline{f_j}$ *is* $K$*-invariant. Furthermore,* $F$ *does not depend on the choice of basis.*

*Proof.* Let $\{g_1, \dots, g_N\}$ be another unitary basis of $W$. There is a unitary transformation $A : W \to W$ such that $A(f_j) = g_j = \sum_{l=1}^N a_{lj} f_l$. We have

$$\sum_{j=1}^N g_j \overline{g_j} = \sum_{j=1}^N \sum_{l,l'=1}^N a_{lj} \overline{a_{l'j}} f_l \overline{f_{l'}} = \sum_{l,l'=1}^N \delta_{ll'} f_l \overline{f_{l'}} = F.$$

Now, $k \cdot F = \sum_{j=1}^N (k \cdot f_j)\overline{(k \cdot f_j)}$ for $k \in K$. But $\{k \cdot f_1, \dots, k \cdot f_N\}$ is another unitary basis of $W$. Since $F$ does not depend on the choice of basis, it follows that $k \cdot F = F$ for any $k \in K$. □

**Lemma 3.2.** *If $W \subset \mathcal{O}(X)$ is a finite-dimensional $K$-submodule, then $\overline{W} \subset \overline{\mathcal{O}(X)}$ is also a $K$-submodule, which is isomorphic to the dual module $W^*$.*

*Proof.* Let $(f, g)$ be a $K$-invariant Hermitian product on $W$. For $f \in W$, $\phi \in \overline{W}$ we have the bilinear pairing

$$\langle f, \phi \rangle = (f, \overline{\phi}),$$

which is obviously $K$-invariant and non-degenerate. This shows that $\overline{W}$ is isomorphic to $W^*$. □

Let $\mu : X \to X$ be an antiholomorphic involution. Then, by definition, the function $\mu f(x) = f(\mu x)$ is antiholomorphic for any $f \in \mathcal{O}(X)$. We want to give a simple proof of the theorem of J. Faraut and E. G. F. Thomas in our setting.

**Proposition 3.3.** *If the $K$-action on $X$ satisfies* (FT)*, then $\mathcal{O}(X)$ is a multiplicity free $K$-module.*

*Proof.* Assume the contrary. Let $W$, $W' \subset \mathcal{O}(X)$ be two irreducible isomorphic $K$-submodules such that $W \neq W'$. Define $f_1, \dots, f_N \in W$ as in Lemma 3.1. Fix a $K$-equivariant isomorphism $\phi : W \to W'$ and let $f_i' = \phi(f_i)$. By Lemma 3.1 the function $F = \sum f_i \overline{f_i}$ is $K$-invariant. The same proof shows that the function $G = \sum f_i \overline{f_i'}$ is also $K$-invariant. By (FT) we have $\mu F = F$ and $\mu G = G$. Since the multiplication map $\mathcal{O}(X) \otimes \overline{\mathcal{O}(X)} \to \mathcal{O}(X) \cdot \overline{\mathcal{O}(X)}$ is an isomorphism of vector spaces, it follows that

$$\sum_i \overline{\mu f_i} \otimes \mu f_i = \sum_i f_i \otimes \overline{f_i}$$

and

$$\sum_i \overline{\mu f_i'} \otimes \mu f_i = \sum_i f_i \otimes \overline{f_i'}.$$

Therefore the linear span of $\overline{\mu f_1}, \dots, \overline{\mu f_N}$ coincides with the linear span of $f_1, \dots, f_N$ and with the linear span of $\overline{\mu f_1'}, \dots, \overline{\mu f_N'}$. Thus $\overline{\mu W} = \overline{\mu W'}$ and $W = W'$, contrary to our assumption. □

From now on the compact group $K$ is assumed to be connected. An involutive automorphism $\theta : K \to K$ is called a *Weyl involution* if there exists a maximal torus $T \subset K$ such that $\theta(t) = t^{-1}$ for $t \in T$. It is known that Weyl involutions exist and that they are all conjugated by inner automorphisms. If $\theta$ is a Weyl involution and $\rho$ is a linear representation of $K$, then $\rho \circ \theta$ is the dual representation. Recall that an antiholomorphic involution $\mu : X \to X$ is called $\theta$-equivariant if $\mu(kx) = \theta(k)\mu(x)$ for all $x \in X$, $k \in K$.

**Lemma 3.4.** *Let $\theta$ be a Weyl involution of $K$ and let $\mu : X \to X$ be a $\theta$-equivariant antiholomorphic involution of $X$. If $W \subset \mathcal{O}(X)$ is a finite-dimensional $K$-submodule, then $\mu W$ is also a $K$-submodule. Furthermore, $\overline{W}$ and $\mu W$ are isomorphic $K$-modules.*

*Proof.* Introduce a $K$-invariant Hermitian inner product and choose a unitary basis $\{f_1, \dots, f_N\}$ in $W$. Denote the representation in $W$ by $\rho$. The condition $\mu(kx) = \theta(k)\mu(x)$ implies that

$$k \cdot \mu f(x) = \mu f(k^{-1}x) = f(\mu(k^{-1}x)) = f(\theta(k)^{-1}\mu(x)) = \theta(k)f(\mu x) = \mu\theta(k)f(x).$$

Hence $\mu W$ is indeed a $K$-submodule with the representation $\rho \circ \theta$. Since $\theta$ is a Weyl involution, this representation is dual to $\rho$. But the representation in $\overline{W}$ is also dual to $\rho$ by Lemma 3.2, and our assertion follows. □

**Lemma 3.5.** *Keep the notation of Lemma 3.4 and assume in addition that $W$ is irreducible and $\mu W = \overline{W}$. Then for a $K$-invariant Hermitian inner product on $W$ one has*

$$(\overline{\mu f_1}, \overline{\mu f_2}) = (f_1, f_2),$$

*where $f_1, f_2 \in W$.*

*Proof.* The new Hermitian inner product $\{f_1, f_2\} := (\overline{\mu f_1}, \overline{\mu f_2})$ on $W$ is also $K$-invariant. Since $W$ is an irreducible $K$-module, it follows that $\{f_1, f_2\} = c(f_1, f_2)$, where $c > 0$. But then

$$\{\overline{\mu f_1}, \overline{\mu f_2}\} = c(\overline{\mu f_1}, \overline{\mu f_2})$$

and, on the other hand,

$$\{\overline{\mu f_1}, \overline{\mu f_2}\} = (f_1, f_2)$$

because $\mu$ is an involution. Thus

$$c(\overline{\mu f_1}, \overline{\mu f_2}) = (f_1, f_2) = c^{-1}(\overline{\mu f_1}, \overline{\mu f_2});$$

hence $c^2 = 1$ and $c = 1$. □

## 4. Holomorphically separable spaces

Since we assume that $K$ is connected, the irreducible representations of $K$ are determined by their highest weights. We denote by $W_\lambda$ an irreducible $K$-module with highest weight $\lambda$ and we write $\mathcal{O}_\lambda(X)$ instead of $\mathcal{O}_\delta(X)$, where $\delta \in \hat{K}$ and $\lambda = \lambda(\delta)$ is the highest weight of $\delta$. Those highest weights $\lambda$, for which $W_\lambda$ occurs in our $K$-module $\mathcal{O}(X)$, form an additive semigroup, to be denoted by $\Lambda(X)$. In other words, $\Lambda(X)$ is the set of highest weights such that $\mathcal{O}_\lambda(X) \neq \{0\}$. The subspace of fixed vectors of a $K$-module $W$ is denoted by $W^K$. We remark that if $A$ is an algebra on which $K$ acts as a group of automorphisms, then $A^K$ is a subalgebra of $A$.

**Theorem 4.1.** *Let $X$ be a holomorphically separable irreducible reduced complex space, $K$ a connected compact Lie group acting on $X$ by holomorphic transformations, $\theta : K \to K$ a Weyl involution, and $\mu : X \to X$ a $\theta$-equivariant antiholomorphic involution of $X$. Then $\mathcal{O}(X)$ is multiplicity free if and only if $\mu(x) \in Kx$ for all $x \in X$.*

*Proof.* If $\mu(x) \in Kx$ for all $x \in X$, then (FT) guarantees that the $K$-action on $\mathcal{O}(X)$ is multiplicity free; see the Introduction and Proposition 3.3.

We now prove the converse. Let $\mathcal{A}(X) = \mathcal{O}(X) \cdot \overline{\mathcal{O}(X)}$. Since $X$ is holomorphically separable, the algebra $\mathcal{A}(X)$ separates points of $X$. By Stone-Weierstrass theorem $\mathcal{A}(X)$ is dense in the algebra $\mathcal{C}(X)$ of continuous functions on $X$. The standard averaging argument shows that $\mathcal{A}(X)^K$ is dense in $\mathcal{C}(X)^K$. Now, if $Kx$ and $Ky$ are two different $K$-orbits in $X$, then there is a $K$-invariant continuous function $f \in \mathcal{C}(X)$ which separates these orbits. Since this function can be approximated by $K$-invariant functions from $\mathcal{A}(X)$, it follows that $\mathcal{A}(K)^K$ separates $K$-orbits. Let $\lambda \in \Lambda(X)$ be a highest weight which occurs in the decomposition of the $K$-algebra $\mathcal{O}(X)$. Since $\mathcal{O}(X)$ is multiplicity free, the isotypic component $\mathcal{O}_\lambda(X)$ is irreducible. We can identify this isotypic component with $W_\lambda$, and so we write $W_\lambda = \mathcal{O}_\lambda(X)$. Now apply Lemma 3.1 to construct a $K$-invariant function in $W_\lambda \cdot \overline{W_\lambda}$. Call this function $F_\lambda$. We claim that the family $\{F_\lambda\}_{\lambda \in \Lambda(X)}$ also separates $K$-orbits in $X$.

To prove the claim, it is enough to present each $F \in \mathcal{A}(X)^K$ as the sum of a series

$$F = \sum_{\lambda \in \Lambda(X)} c_\lambda F_\lambda,$$

where the convergence is absolute and uniform on compact subsets in $X$. In order to prove this decomposition, consider the complex space $\overline{X}$ with the conjugate complex structure. There is a natural $K$-action on $\overline{X}$, and so we obtain an action of $K \times K$ on $X \times \overline{X}$. Since $\mathcal{O}(\overline{X}) = \overline{\mathcal{O}(X)}$, the isotypic components of the $K$-module $\mathcal{O}(\overline{X})$ are just the submodules $\overline{W_\lambda}$. By Lemma 2.1 the isotypic components of the $(K \times K)$-module $\mathcal{O}(X \times \overline{X})$ are the tensor products $W_\lambda \otimes \overline{W_{\lambda'}}$.

For any $F \in \mathcal{O}(X \times \overline{X})$ the theorem of Harish-Chandra yields the decomposition

$$F = \sum F_{\lambda\lambda'} \quad \text{with } F_{\lambda\lambda'} \in W_\lambda \otimes \overline{W_{\lambda'}},$$

where the convergence is absolute and uniform on compact subsets in $X \times \overline{X}$. In particular, if $F \in (\mathcal{O}(X) \otimes \overline{\mathcal{O}(X)})^K$, then all summands are $K$-invariant. But $\overline{W_\lambda}$ is dual to $W_\lambda$ by Lemma 3.2; hence $F_{\lambda\lambda'} = 0$ for $\lambda' \neq \lambda$ by Schur's lemma. The remaining summands $F_{\lambda\lambda}$ are $K$-invariant elements in $W_\lambda \otimes \overline{W_\lambda}$. But the space $(W_\lambda \otimes \overline{W_\lambda})^K$ is one-dimensional, again by Schur's lemma. Therefore, restricting $F_{\lambda\lambda}$ to the diagonal in $X \times \overline{X}$, we get the functions proportional to the $F_\lambda$'s defined above.

Now, because $\mathcal{O}(X)$ is multiplicity free, it follows from Lemma 3.4 that $\overline{W_\lambda} = \mu W_\lambda$. Furthermore, Lemma 3.5 shows that the composition of $\mu$ with complex conjugation preserves a $K$-invariant Hermitian product on $W_\lambda$. Therefore $\mu F_\lambda = F_\lambda$ by Lemma 3.1. Since the family of functions $F_\lambda$ separates $K$-orbits, $\mu$ must preserve each of them or, equivalently, $\mu x \in Kx$ for all $x \in X$. $\square$

*Remark.* For the torus $T = (S^1)^m$ the Weyl involution is given by $\theta(t) = t^{-1}$. Suppose that $T$ acts on $\mathbb{P}_n$ by $t \cdot (z_0 : \ldots : z_n) = (\chi_0(t)z_0 : \ldots : \chi_n(t)z_n)$ with some characters $\chi_i : T \to S^1$, $i = 0, \ldots, n$, and $\mu : \mathbb{P}_n \to \mathbb{P}_n$ is given by $\mu(z_0 : \ldots : z_n) = (\overline{z_0} : \ldots : \overline{z_n})$. Then $\mu$ is obviously $\theta$-equivariant. However, if $m < n$, then $\mu$ cannot map each $T$-orbit onto itself. This shows that holomorphic separability of $X$ in Theorem 4.1 is essential.

Let $\mathfrak{k}$ be the Lie algebra of $K$ and let $\mathfrak{g} = \mathfrak{k}^{\mathbb{C}}$ be its complexification. An irreducible reduced complex space $X$ is called *spherical* under the action of a compact connected Lie group $K$ if $X$ is normal and there exists a point $x \in X$ such that the tangent space $T_xX$ is generated by the elements of a Borel subalgebra of $\mathfrak{g}$; see [AH].

**Theorem 4.2.** *Let $X$ be a normal Stein space, $K$ a connected compact Lie group acting on $X$ by holomorphic transformations, $\theta : K \to K$ a Weyl involution, and $\mu : X \to X$ a $\theta$-equivariant antiholomorphic involution of $X$. Then $X$ is spherical if and only if $\mu(x) \in Kx$ for all $x \in X$.*

*Proof.* It is known that a normal Stein space $X$ is spherical if and only if $\mathcal{O}(X)$ is a multiplicity free $K$-module [AH]. The result follows from Theorem 4.1. □

## 5. Weyl involution and Cartan involution

Throughout this section, except in Theorem 5.5, the word *involution* means an involutive automorphism of a group. This notion will be used for complex algebraic groups and for Lie groups. Let $G$ be a connected reductive algebraic group over $\mathbb{C}$ and let $K$ be a connected compact Lie group. So far we considered Weyl involutions of $K$, but they can also be defined for $G$. Namely, an involution $\theta : G \to G$ is called a *Weyl involution* if there exists a maximal algebraic torus $T \subset G$ such that $\theta(t) = t^{-1}$ for $t \in T$.

**Lemma 5.1.** *Let $G$ be a connected reductive complex algebraic group and let $K \subset G$ be a maximal compact subgroup. Any Weyl involution $\theta$ of $K$ extends uniquely to a Weyl involution of $G$.*

*Proof.* By Theorem 5.2.11 in [OV] any differentiable automorphism $K \to K$ extends uniquely to a polynomial automorphism $G \to G$. Let $\theta : K \to K$ be a Weyl involution and let $T \subset K$ be a maximal torus of $K$ on which $\theta(t) = t^{-1}$. Now, the complexification $T^{\mathbb{C}}$ of $T$ is a maximal torus of $G$. The extension of $\theta$ to $G$, which we again denote by $\theta$, is a Weyl involution of $G$ because $\theta(t) = t^{-1}$ for all $t \in T^{\mathbb{C}}$. □

An algebraic subgroup $H \subset G$ is called *spherical* if $G/H$ is a spherical variety, i.e., if a Borel subgroup of $G$ acts on $G/H$ with an open orbit. A reductive algebraic subgroup $H \subset G$ is called *adapted* if there exists a Weyl involution $\theta : G \to G$ such that $\theta(H) = H$ and $\theta|_{H^\circ}$ is a Weyl involution of the connected component $H^\circ$. A similar definition is used for compact subgroups of connected compact Lie groups.

**Proposition 5.2.** *Any spherical reductive subgroup $H \subset G$ is adapted.*

*Proof.* See [AV], Proposition 5.10. □

**Proposition 5.3.** *Let $H \subset G$ be an adapted algebraic subgroup, $K \subset G$ and $L \subset H$ maximal compact subgroups, and $L \subset K$. Then $L$ is adapted in $K$.*

*Proof.* See [AV], Proposition 5.14. □

**Theorem 5.4.** *Let $G$ be a connected reductive algebraic group and let $H \subset G$ be a reductive spherical subgroup. Then there exist a Weyl involution $\theta : G \to G$ and a Cartan involution $\tau : G \to G$ such that $\theta\tau = \tau\theta$, $\theta(H) = H$, and $\tau(H) = H$.*

*Proof.* Let $L$ be a maximal compact subgroup of $H$ and let $K$ be a maximal compact subgroup of $G$ that contains $L$. Then $K$ is the fixed point subgroup $G^\tau$ of some Cartan involution $\tau$. Since $H$ is adapted in $G$, there is a Weyl involution $\theta : K \to K$ such that $\theta(L) = L$. For any $k \in K$, we have $\theta\tau(k) = \theta(k) = \tau\theta(k)$ by the definition of $\tau$. Denote again by $\theta : G \to G$ the unique extension to $G$ of the given Weyl involution of $K$. Since $G$ is connected and the relation $\theta\tau(g) = \tau\theta(g)$ holds on $K$, it also holds on $G$. □

**Theorem 5.5.** *Let $X = G/H$ be an affine homogeneous space of a connected reductive algebraic group $G$. Let $K$ be a maximal compact subgroup of $G$. Then $X$ is spherical if and only* (FT) *is satisfied for the action of $K$ on $X$. Moreover, if $X$ is spherical, one can choose $\mu$ in* (FT) *to be $\theta$-equivariant, where $\theta$ is a Weyl involution of $K$.*

*Proof.* (FT) implies that $\mathcal{O}(X)$ is multiplicity free or, equivalently, that $X$ is spherical. Conversely, assume that $X = G/H$ is a spherical variety. Since $X$ is affine, $H$ is a reductive subgroup by the Matsushima-Onishchik theorem. Define $\theta$ and $\tau$ as in Theorem 5.4 and put $\mu(g \cdot H) = \theta\tau(g) \cdot H$. The map $\mu : X \to X$ is well defined because $\theta\tau(H) = H$. The lift of $\mu$ to $G$ is an antiholomorphic involutive automorphism, so it is obvious that $\mu$ is an antiholomorphic involution of $X$. Since $\theta\tau = \tau\theta$, it follows that $\theta(K) = K$. Therefore, for any $x = gH \in X$ one has

$$\mu(kx) = \theta\tau(kg) \cdot H = \theta(k)\theta\tau(g) \cdot H = \theta(k)\mu(x)$$

for all $k \in K$. From Theorem 4.2 it follows that $\mu(x) \in Kx$ for all $x \in X$. □

## 6. Non-spherical spaces: an example

We keep the notation of the previous section. For a spherical affine homogeneous space $X = G/H$, we constructed a $\theta$-equivariant antiholomorphic involution $\mu$. In this section we want to show that the sphericity assumption is essential.

**Lemma 6.1.** *Let $X$ be an irreducible reduced complex space with a holomorphic action of $G$. Let $\theta$ be any algebraic automorphism of $G$ preserving a maximal compact subgroup $K \subset G$. Denote by $\tau$ the Cartan involution with fixed point subgroup $K$. If $\mu$ is an antiholomorphic involution of $X$ satisfying $\mu(kx) = \theta(k)\mu(x)$ for all $x \in X,\ k \in K$, then one has $\mu(gx) = \theta\tau(g)\mu(x)$ for all $x \in X,\ g \in G$.*

*Proof.* For every fixed $x \in X$ consider two antiholomorphic maps $\varphi_x : G \to X$ and $\psi_x : G \to X$, defined by $\varphi_x(g) = \mu(gx)$ and $\psi_x(g) = \theta\tau(g)\mu(x)$. Since the required identity holds for $g \in K$, the maps $\varphi_x$ and $\psi_x$ coincide on $K$. But $K$ is a maximal totally real submanifold in $G$, so $\varphi_x$ and $\psi_x$ must coincide on $G$. □

**Lemma 6.2.** *Let $X = G/H$, where $H \subset G$ is an algebraic reductive subgroup, $\theta$ a Weyl involution of $G$ preserving $K$, and $\mu : X \to X$ an antiholomorphic involution of $X$ satisfying $\mu(kx) = \theta(k)\mu(x)$. Then $H$ and $\theta(H)$ are conjugate by an inner automorphism of $G$.*

*Proof.* Assume first that $\tau(H) = H$. Let $x_0 = e \cdot H$ and $h \in H$. Then $\theta(h)\mu(x_0) = \theta(\tau(\tau(h)))\mu(x_0) = \mu(\tau(h)x_0) = \mu(x_0)$ by Lemma 6.1. It follows that $\theta(H)$ is the stabilizer of $\mu(x_0)$, so $H$ and $\theta(H)$ are conjugate.

To remove the above assumption, take a maximal compact subgroup $L \subset H$ and a maximal compact subgroup $K_1 \subset G$, such that $L \subset K_1$. Then $K_1 = gKg^{-1}$ for some $g \in G$. The fixed point subgroup of the Cartan involution

$\tau_1 := \mathrm{Ad}(g)\tau\mathrm{Ad}(g)^{-1}$ is exactly $K_1$, so $\tau_1$ is the identity on $L$ and, consequently, $\tau_1(H) = H$. Let $H_1 := g^{-1}Hg$, then $\tau(H_1) = (\mathrm{Ad}g)^{-1}\tau_1(\mathrm{Ad}g)(H_1) = (\mathrm{Ad}g)^{-1}\tau_1(H) = (\mathrm{Ad}g)^{-1}(H) = g^{-1}Hg = H_1$. Replacing $H$ by $H_1$, we can apply the above argument. □

**Proposition 6.3.** *Let $G = \mathrm{SO}_{10}(\mathbb{C})$ and let $H \subset G$ be the adjoint group of $\mathrm{SO}_5(\mathbb{C})$. Let $\theta$ be a Weyl involution of the maximal compact subgroup $K = \mathrm{SO}_{10}(\mathbb{R})$. Then an antiholomorphic involution of $X = G/H$ cannot be $\theta$-equivariant.*

*Proof.* Extend $\theta$ holomorphically to $G$. In view of Lemma 6.2 it suffices to show that $\theta(H)$ and $H$ are not conjugate by an inner automorphism of $G$. Assume that $\theta(H) = g_0Hg_0^{-1}$. Then there is an automorphism $\phi : H \to H$, such that $\theta(h) = g_0\phi(h)g_0^{-1}$ for $h \in H$. All automorphisms of $H$ are inner, so $\phi(h) = h_0hh_0^{-1}$ for some $h_0 \in H$. Therefore $\theta(h) = g_1hg_1^{-1}$ for all $h \in H$, where $g_1 = g_0h_0$. Define an automorphism of $\alpha : G \to G$ by $\alpha := (\mathrm{Ad}(g_1))^{-1} \cdot \theta$ and note that $H \subset G^\alpha$, where $G^\alpha$ is the fixed point subgroup of $\alpha$.

Recall that E. B. Dynkin classified maximal subgroups of classical groups in [D]. Since $B_2$ does not occur in his Table 1, every irreducible representation of $B_2$ defines a maximal subgroup by Theorem 1.5 in [D]. In particular, $H$ is a maximal connected subgroup in $G$. Therefore, either (i) $H$ is the connected component of $G^\alpha$ or (ii) $\alpha = \mathrm{Id}$. Now, (ii) implies that $\mathrm{Ad}(g_1) = \theta$, thus $\theta$ is an inner automorphism of $G$, which is not the case. So we are left with (i). Applying the same argument to $\beta = \alpha^2$, we see that either (1) $H$ is the connected component of $G^\beta$ or (2) $\beta = \mathrm{Id}$. Since $\beta$ is certainly an inner automorphism, (1) would imply that $H$ is the centralizer of an element of $G$. However, all centralizers have even codimension in $G$ and $\mathrm{codim}(H) = 35$. On the other hand, if (2) were true, then $H$ would be a symmetric subgroup in $G$. The list of symmetric spaces shows that this is not the case. The contradiction just obtained completes the proof. □

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