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A NEW PROOF OF THE INTEGRABILITY OF THE SUBDIFFERENTIAL OF A CONVEX FUNCTION ON A BANACH SPACE

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Abstract. We provide a simple proof of the Moreau-Rockafellar theorem that a proper lower semicontinuous convex function on a Banach space is determined up to a constant by its subdifferential.

1. Introduction

Recall the following famous theorem of Rockafellar; see [8, 9]:

Theorem 1.1 (Rockafellar). *Let X be a Banach space. Let f and g be proper lower semicontinuous convex functions from X to $\mathbb{R} \cup \{+\infty\}$. If*

$$\partial f \subset \partial g, \tag{1.1}$$

then

$$g = f + \text{const}.$$

The reader is referred to Section 3 of Phelps book [7] for a possible context of this theorem and some of its important implications.

The purpose of this note is to present a new and more direct proof of Theorem 1.1.

First of all, however, we present a brief historical overview. The integrability of the subdifferential of proper lower semicontinuous convex functions on Hilbert space is stated and proved by Moreau in [5]. His proof uses Moreau-Yosida regularisation and works also in reflexive Banach space as mentioned on p. 87 of [6]. While the first complete proof of Theorem 1.1 – that of Rockafellar in [9] – uses duality arguments, most of the others rely on approximating the directional derivative and subsequent reduction to the trivial one-dimensional case. This second line is followed by the original Rockafellar proof in [8]. Although there are some gaps in the latter proof, the work of Taylor [10] contains both a remedy and a different proof (cf. [2]). The idea of directional derivative approximation/one-dimensional reduction is most clearly outlined in the proof of Thibault [11]. A different proof using the mean-value theorem of Zagrodny is due to Thibault and Zagrodny [12] (see also [13]).

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The proof we are about to present follows a different route. It is similar to the proof of the classical calculus theorem that a monotone function is Riemann integrable. It uses neither duality nor explicit one-dimensional arguments.

The main step in our proof is to show directly that a proper lower semicontinuous convex function on a Banach space differs by a constant from the *Rockafellar function* (see [1]) of its subdifferential. Namely, we prove the following

Theorem 1.2 (Rockafellar [8, 9])**.** *Let $f : X \to \mathbb{R} \cup \{+\infty\}$ be a proper lower semicontinuous convex function on the Banach space X. Let $x_0 \in \operatorname{dom} \partial f$ and $p_0 \in \partial f(x_0)$ be arbitrary. Then for all $x \in X$,*

$$f(x) = f(x_0) + R_{\partial f,(x_0,p_0)}(x), \tag{1.2}$$

where

$$R_{\partial f,(x_0,p_0)}(x) = \sup\left\{\sum_{i=0}^{n-1} p_i(x_{i+1} - x_i) : x_n = x,\ p_i \in \partial f(x_i),\ n \in \mathbb{N}\right\}; \tag{1.3}$$

that is, the supremum is taken over all finite sequences in $\operatorname{gph} \partial f$, starting at (x_0, p_0).

With the aid of the novel technical Lemma 3.3 we derive first the case $x \in \operatorname{dom} \partial f$ of (1.2). The general case is completed by reference to the following well-known result of Brøndsted and Rockafellar [3]:

Proposition 1.3. *Let f be a proper lower semicontinuous convex function from a Banach space X into $\mathbb{R} \cup \{+\infty\}$. Then for all $x \in X$,*

$$f(x) = \sup\{f(\bar{x}) + \bar{p}(x - \bar{x}) : (\bar{x}, \bar{p}) \in \operatorname{gph} \partial f\}. \tag{1.4}$$

From Theorem 1.2 it readily follows that (1.1) implies

$$f(x) - f(x_0) \le g(x) - g(x_0)$$

for any $x_0 \in \operatorname{dom} \partial f$ and all $x \in X$. In particular, $f - g \equiv \text{const}$ on $\operatorname{dom} \partial f$. The proof of Theorem 1.1 is completed by a reference to lower semicontinuity and graphical density of points of subdifferentiability (see [3] and [2]):

Proposition 1.4. *Let f be a proper lower semicontinuous convex function from a Banach space X into $\mathbb{R} \cup \{+\infty\}$. Then for any $\bar{x} \in \operatorname{dom} f$ and any $\varepsilon > 0$ there exists $x \in \operatorname{dom} \partial f$ such that $\|x - \bar{x}\| + |f(x) - f(\bar{x})| < \varepsilon$.*

The remaining content is organized as follows. After a short section on notation, we state and prove Lemma 3.3 in Section 3. In Section 4 the proofs of Theorem 1.2 and Theorem 1.1 are assembled.

2. Notation

The notation used throughout the paper is fairly standard, so we recall it briefly.

Usually X denotes a real Banach space, that is, a complete normed space over $\mathbb{R}$. The norm of X is denoted by $\|\cdot\|$. The closed unit ball $\{x \in X : \|x\| \le 1\}$ of X is denoted by B_X. The dual space X^* of X is the Banach space of all continuous linear functionals p from X to $\mathbb{R}$. The natural norm of X^* is again denoted by $\|\cdot\|$. The value of $p \in X^*$ at $x \in X$ is denoted by both $p(x)$ and $\langle p, x\rangle$, wherever convenient.

The *effective domain* $\operatorname{dom} f$ of an extended real-valued function $f : X \to \mathbb{R} \cup \{+\infty\}$ is the set of points x where $f(x) \in \mathbb{R}$. The function f is *proper* if $\operatorname{dom} f \neq \emptyset$. It is *lower semicontinuous* if

$$f(\bar{x}) \leq \liminf_{x \to \bar{x}} f(x)$$

for all $\bar{x} \in X$. The *epigraph* of f is the set $\operatorname{epi} f \subset X \times \mathbb{R}$ defined by

$$\operatorname{epi} f = \{(x, r) : x \in \operatorname{dom} f,\ r \geq f(x)\}.$$

It is clear that $\operatorname{epi} f$ is nonempty whenever f is proper, and $\operatorname{epi} f$ is closed iff f is lower semicontinuous.

The *indicator* function of a set $C \subset X$ is the function $\mathrm{I}_C : X \to \mathbb{R} \cup \{+\infty\}$ defined by:

$$\mathrm{I}_C(x) = 0, \text{ if } x \in C; \quad \mathrm{I}_C(x) = \infty, \text{ otherwise.}$$

Obviously, I_C is proper iff C is nonempty and I_C is lower semicontinuous iff C is closed.

If $f : X \to \mathbb{R} \cup \{+\infty\}$ is a convex function, then $\partial f(\bar{x})$ denotes the subdifferential of f at $\bar{x} \in \operatorname{dom} f$ in the sense of Convex Analysis, that is,

$$\partial f(\bar{x}) = \{p \in X^* : \langle p, x - \bar{x} \rangle \leq f(x) - f(\bar{x}),\ \forall x \in X\},$$

and $\partial f = \emptyset$ on $X \setminus \operatorname{dom} f$. It is a well-known and easy consequence of convexity that if $\langle p, x - \bar{x} \rangle \leq f(x) - f(\bar{x})$ holds on some neighbourhood of $\bar{x}$, rather than on the whole space, then again $p \in \partial f(\bar{x})$.

The *domain* $\operatorname{dom} \partial f$ of the multi-valued map $\partial f : X \to X^*$ consists of all points $x \in X$ such that $\partial f(x)$ is nonempty. The *graph* of ∂f is

$$\operatorname{gph} \partial f = \{(x, p) : x \in \operatorname{dom} \partial f,\ p \in \partial f(x)\}.$$

3. Variational arguments

This section is devoted to Lemma 3.3, which can be viewed as a particular enhancement of Proposition 1.4.

Although the technique of proof is standard, we provide full details for the reader's convenience. First, however, we recall the famous Ekeland Variational Principle (e.g. [7], p.45):

Theorem 3.1 (Ekeland Variational Principle)**.** *Let f be a proper lower semicontinuous function from a Banach space X into $\mathbb{R} \cup \{+\infty\}$. Let f be bounded below, $\varepsilon > 0$, and $\bar{y} \in \operatorname{dom} f$ be such that $f(\bar{y}) \leq \inf f + \varepsilon$. Then for each $\lambda > 0$ there is $\bar{x} \in \operatorname{dom} f$ such that:*

(i) $\lambda \|\bar{x} - \bar{y}\| \leq f(\bar{y}) - f(\bar{x})$,
(ii) $\|\bar{x} - \bar{y}\| \leq \varepsilon / \lambda$, *and*
(iii) $\lambda \|x - \bar{x}\| + f(x) > f(\bar{x})$ *for all* $x \neq \bar{x}$.

We also use the following well-known sum theorem (e.g. [7], p.47):

Theorem 3.2 (Moreau-Rockafellar Formula)**.** *Let f be a proper lower semicontinuous convex function from a Banach space X into $\mathbb{R} \cup \{+\infty\}$, and let $g : X \to \mathbb{R}$ be convex and continuous. Then for any $x \in X$,*

$$\partial(f + g)(x) = \partial f(x) + \partial g(x).$$

(*The right-hand side is the usual sum of sets.*)

Lemma 3.3. *Let f be a proper lower semicontinuous convex function from a Banach space X into $\mathbb{R} \cup \{+\infty\}$. Let $(y_i)_{i=1}^k \subset \operatorname{dom} f$. Then for each $\varepsilon > 0$ there are $(x_i, p_i) \in \operatorname{gph} \partial f$ such that*

$$\|x_i - y_i\| \le \varepsilon \text{ and } \|p_j\| \, \|x_i - y_i\| \le \varepsilon, \ \forall i, j = 1, \dots, k. \tag{3.1}$$

Proof. Due to the lower semicontinuity of f there is $\delta \in (0, \varepsilon)$ such that for all $i = 1, \dots, k$,

$$f(y_i + \delta B_X) \ge f(y_i) - \varepsilon.$$

Let $f_i = f + \mathrm{I}_{\{y_i + \delta B_X\}}$. So, y_i is an ε-minimum of the proper lower semicontinuous f_i. The Ekeland Variational Principle applied to f_i (with this ε and $\lambda = 2\varepsilon/\delta$) shows that there is $x_i \in \operatorname{dom} f_i \cap \{y_i + (\varepsilon/\lambda) B_X\}$ such that

$$f_i(x) + \lambda \|x - x_i\| \ge f_i(x_i), \quad \forall x \in X. \tag{3.2}$$

Since $x_i \in y_i + (\varepsilon/\lambda) B_X$ and $\varepsilon/\lambda = \delta/2 < \varepsilon$, we have that

$$\|x_i - y_i\| \le \varepsilon/\lambda = \delta/2 < \varepsilon. \tag{3.3}$$

On the other hand, (3.2) means that the minimum of the proper convex and lower semicontinuous function $f_i(\cdot) + \lambda \| \cdot - x_i\|$ is attained at x_i. Therefore,

$$0 \in \partial \left(f_i(\cdot) + \lambda \| \cdot - x_i \| \right)(x_i).$$

Since $\lambda \| \cdot - x_i\|$ is continuous, from Theorem 3.2 it follows that there is

$$p_i \in \partial f_i(x_i), \text{ such that } -p_i \in \partial \left(\lambda \| \cdot - x_i \| \right)(x_i).$$

By definition $f_i \equiv f$ on $y_i + \delta B_X$, and, due to (3.3), x_i is in the interior of the latter set; hence, $p_i(x - x_i) \le f(x) - f(x_i)$ for all x in a neighbourhood of x_i. That is,

$$p_i \in \partial f(x_i). \tag{3.4}$$

Next, as $-p_i$ is a subdifferential of $\lambda \| \cdot - x_i\|$ at x_i, we get

$$p_i(x_i - x') \le \lambda \|x' - x_i\|, \ \forall x' \in X.$$

So, $\|p_i\| \le \lambda$.

Therefore, for each $i = 1, \dots, k$ we have that $\|p_i\| \le \lambda$ and $\|x_i - y_i\| \le \varepsilon/\lambda$; cf. (3.3). Both of these clearly imply

$$\|p_j\| \, \|x_i - y_i\| \le \lambda \frac{\varepsilon}{\lambda} = \varepsilon, \forall i, j = 1, \dots, k.$$

This, (3.4) and (3.3) are all we had to prove. □

4. Integrability

We are now ready to present the proof mentioned in the title of this note.

Proof of Theorem 1.2. For each finite sequence $(x_i, p_i)_{i=1}^{n-1} \subset \operatorname{gph} \partial f$, and each $x \in X$ we have that

$$f(x) \ge f(x_0) + \sum_{i=0}^{n-1} p_i (x_{i+1} - x_i), \tag{4.1}$$

where $x_n = x$.

Indeed, by the definition of the subdifferential, $f(x_{i+1}) - f(x_i) \ge p_i(x_{i+1} - x_i)$, $i = 0, \dots, n-1$. Summing these, we get $f(x_n) - f(x_0) \ge \sum_{i=0}^{n-1} p_i(x_{i+1} - x_i)$, which is (4.1). So, the left-hand side of (1.2) is greater than its right-hand side.

Assume now that $x \in \operatorname{dom} \partial f$. Fix $p \in \partial f(x)$ and let $(x_n, p_n) = (x, p)$.

Again using the definition of the subdifferential, $f(x_i) - f(x_{i+1}) \geq p_{i+1}(x_i - x_{i+1})$, that is, $f(x_{i+1}) - f(x_i) \leq p_{i+1}(x_{i+1} - x_i)$. So,

$$f(x_{i+1}) - f(x_i) - p_i(x_{i+1} - x_i) \leq \langle p_{i+1} - p_i, x_{i+1} - x_i \rangle ,$$

for $i = 0, \ldots, n-1$. Therefore,

$$0 \leq f(x_n) - f(x_0) - \sum_{i=0}^{n-1} p_i(x_{i+1} - x_i) \leq \sum_{i=0}^{n-1} \langle p_{i+1} - p_i, x_{i+1} - x_i \rangle , \tag{4.2}$$

the first inequality being (4.1).

The job will be done when we find such a sequence $(x_i, p_i)_{i=1}^{n-1} \subset \operatorname{gph} \partial f$ that the right-hand side of (4.2) is arbitrarily small.

Fix $\varepsilon > 0$.

Fix $n \in \mathbb{N}$ so large that

$$n > 2 \langle p - p_0, x - x_0 \rangle / \varepsilon \tag{4.3}$$

and let $(x_n, p_n) = (x, p)$.

Let $y_i = x_0 + \dfrac{i}{n}(x - x_0)$, $i = 0, \ldots, n$. Since f is convex, $y_i \in \operatorname{dom} f$.

Using Lemma 3.3, for each $i = 1, \ldots, n-1$ we get $(x_i, p_i) \in \operatorname{gph} \partial f$ such that

$$\|x_i - y_i\| \leq \frac{\varepsilon}{8n(1 + \|p_0\| + \|p_n\|)} \text{ and } \|p_j\| \|x_i - y_i\| \leq \frac{\varepsilon}{8n}$$

for all $i, j = 1, \ldots, n-1$. Then, using the above and $x_0 = y_0$, $x_n = y_n$,

$$\|p_j\| \|x_i - y_i\| \leq \frac{\varepsilon}{8n}, \forall i, j = 0, \ldots, n. \tag{4.4}$$

Now, for each $i = 0, \ldots, n-1$ we can write

$$\begin{aligned} \langle p_{i+1} - p_i, x_{i+1} - x_i \rangle &\leq \langle p_{i+1} - p_i, y_{i+1} - y_i \rangle + (\|p_i\| + \|p_{i+1}\|)(\|x_i - y_i\| + \|x_{i+1} - y_{i+1}\|) \\ &\leq \langle p_{i+1} - p_i, y_{i+1} - y_i \rangle + \frac{\varepsilon}{2n}, \end{aligned}$$

the second inequality because of (4.4). Therefore,

$$\sum_{i=0}^{n-1} \langle p_{i+1} - p_i, x_{i+1} - x_i \rangle \leq \sum_{i=0}^{n-1} \langle p_{i+1} - p_i, y_{i+1} - y_i \rangle + \frac{\varepsilon}{2}.$$

But, $y_{i+1} - y_i = \dfrac{x - x_0}{n}$, so

$$\begin{aligned} \sum_{i=0}^{n-1} \langle p_{i+1} - p_i, y_{i+1} - y_i \rangle &= \frac{1}{n} \langle \sum_{i=0}^{n-1} p_{i+1} - p_i, x - x_0 \rangle \\ &= \frac{1}{n} \langle p_n - p_0, x - x_0 \rangle = \frac{1}{n} \langle p - p_0, x - x_0 \rangle < \frac{\varepsilon}{2}, \end{aligned}$$

because of (4.3). Therefore, $\sum_{i=0}^{n-1} \langle p_{i+1} - p_i, x_{i+1} - x_i \rangle < \varepsilon$ and, in the case when $x \in \operatorname{dom} \partial f$, (1.2) is established.

Now fix an arbitrary $x \in X$ and a real number r such that $r < f(x)$. By Proposition 1.3 there is $(\bar{x}, \bar{p}) \in \operatorname{gph} \partial f$ such that

$$r < f(\bar{x}) + \bar{p}(x - \bar{x}).$$

Fix $\varepsilon > 0$. Since $\bar{x} \in \operatorname{dom} \partial f$, referring to the case just established we find a finite sequence $(x_i, p_i)_{i=1}^{n-2} \in \operatorname{gph} \partial f$ such that

$$f(\bar{x}) < f(x_0) + \sum_{i=0}^{n-2} p_i(x_{i+1} - x_i) + \varepsilon,$$

where $x_{n-1} = \bar{x}$.

By letting $p_{n-1} = \bar{p}$ and $x_n = x$ and combining the two above we obtain

$$r < f(x_0) + \sum_{i=0}^{n-1} p_i(x_{i+1} - x_i) + \varepsilon.$$

Since $r < f(x)$ and $\varepsilon > 0$ were arbitrary, the proof is completed. □

Proof of Theorem 1.1. Fix $x_0 \in \operatorname{dom} \partial f$ and let $c = g(x_0) - f(x_0)$. From Theorem 1.2 it follows that, provided (1.1),

$$f(x) - f(x_0) \leq g(x) - g(x_0) \iff g(x) \geq f(x) + c, \ \forall x \in X. \tag{4.5}$$

From this it follows that

$$\operatorname{dom} g \subset \operatorname{dom} f, \tag{4.6}$$

and, since we can swap x and x_0 if x is also in $\operatorname{dom} \partial f$,

$$g(x) = f(x) + c, \ \forall x \in \operatorname{dom} \partial f. \tag{4.7}$$

Now let $x \in \operatorname{dom} f$ be arbitrary. From Proposition 1.4 we can find a sequence $(x_n)_{n=1}^{\infty} \subset \operatorname{dom} \partial f$ such that $\lim x_n = x$ and $\lim f(x_n) = f(x)$. The lower semicontinuity of g together with (4.7) gives $g(x) \leq \liminf g(x_n) = \liminf(f(x_n) + c) = f(x) + c$. That is,

$$g(x) \leq f(x) + c, \ \forall x \in \operatorname{dom} f.$$

In particular $\operatorname{dom} g \supset \operatorname{dom} f$, meaning that $\operatorname{dom} g = \operatorname{dom} f$ because of (4.6), and $g = f + c$ on $\operatorname{dom} f$ because of (4.5). Outside of their common domain both functions are equal to ∞ by definition, and thus the proof is completed. □

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