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AN EXTREMAL PROPERTY OF JACOBI POLYNOMIALS IN TWO-SIDED CHERNOFF-TYPE INEQUALITIES FOR HIGHER ORDER DERIVATIVES

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Abstract. For a weight function generating the classical Jacobi polynomials, the sharp double estimate of the distance from the subspace of all polynomials of an arbitrary fixed order is established.

1. Introduction

Let $\Delta := (-1, 1)$ and let $\omega(x) > 0$ be a Lebesgue integrable weight function. Denote by $L^2(\omega)$ the real Hilbert space with the inner product $(f, g)_{L^2(\omega)} := \int_\Delta fg\omega$ and the norm $\|f\|_{L^2(\omega)} := (f, f)^{1/2}_{L^2(\omega)}$. Let $\mathbf{P}$ be the set of all polynomials and let $\mathbf{P_k} \subset \mathbf{P}$ be the set of all polynomials of degree $\leq k$.

We study the problem on a sharp two-sided estimate of the distance

$$D_{k,\omega}(G) := \left[\mathrm{dist}_{L^2(\omega)}(G, \mathbf{P_{k-1}})\right]^2 := \inf_{c_0,\dots,c_{k-1}} \int_\Delta \Big| G(x) - \sum_{j=0}^{k-1} c_j x^j \Big|^2 \omega(x)dx$$

by an analog with the well-known two-sided inequality from the probability theory

$$[\mathbf{E}G'(X)]^2 \leq \mathbf{D}[G(X)] \leq \mathbf{E}[G'(X)]^2, \tag{1.1}$$

which is valid for any absolutely continuous function G with a finite right-hand side (1.1) and the Gaussian normal random variable X. Moreover, both inequalities become the equalities for $G(x) = \mathrm{const} \cdot x$. The right-hand side of (1.1), proved in [1], is often cited as the *Chernoff inequality*; however, in a form even more general than (1.1), it can be derived from the earlier work [2]. An accurate piece of the history of the problem and the extension of (1.1) for the L^p-deviation, $p \geq 1$, can be found in [3]. Unlike [3] and many other works, we extend (1.1) in a different way, increasing the order of derivative, but saving the L^2-metric. In this case the role of the extremal functions, when both inequalities become the equalities, is to

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play the orthogonal polynomials. A natural extension of (1.1) for the kth derivative found in [4] has a form

$$\frac{1}{k!}\left[\int_{\mathbb{R}} G^{(k)}dF\right]^2 \le D_{k,F}(G) \le \frac{1}{k!}\int_{\mathbb{R}} |G^{(k)}|^2 dF, \tag{1.2}$$

where $dF(x) := \frac{1}{\sqrt{2\pi}}\, e^{-x^2/2}dx$. Moreover, the double equality in (1.2) takes place for the Chebyshev-Hermite polynomials of degree k.

In the present paper we consider an analog of (1.1) for the case of a finite interval Δ and the weight

$$\omega(x) = (1-x)^{\alpha}(1+x)^{\beta}, \quad \alpha > -1, \beta > -1, \tag{1.3}$$

when the extremal functions are the classical Jacobi polynomials. Similar to [4], we obtain the result with the help of weighted differential operators for which the Jacobi polynomials are the eigenfunctions (Section 3). Section 2 is devoted to analysis of the case with a general weight ω, and we show that the lower bound is always true with the extremal functions as the orthogonal polynomials with weight ω and the upper bound with the same extremal functions is valid, roughly speaking, if a polynomial of the second order is the eigenfunction of a related differential operator of the second order, which in turn implies that the weight function ω has the form (1.3).

We use the symbols := and =: for a definition of new variables, $\mathbb{N}$ is the set of all natural integers, $\mathbb{N}_0 := \{0\} \cup \mathbb{N}$, $c_i, i = 1, 2, ...$ are positive constants possibly different from place to place.

2. General case

Let $\omega(x) > 0$, $x \in \Delta$, be a weight function such that $\int_\Delta \omega(x)dx = 1$. Then the orthonormal system of polynomials $\{\varphi_k\}$, $k \in \mathbb{N}_0$, such that $\varphi_0(x) = 1$,

$$\varphi_k(x) = (-1)^k a_k x^k + \ldots + a_0, \quad k \ge 1,$$

with $a_k > 0$ and

$$\int_\Delta \varphi_k^2 \omega = 1$$

is complete in the real Hilbert space $\mathbb{H} := L^2(\omega)$ (see [6], § 1.3.2).

We shall need the Riemann-Liouville transform

$$R_k g(x) := \frac{1}{(k-1)!}\int_{\mathbb{R}} r_k(x,t)g(t)dt,$$

where $\mathbb{R} := (-\infty, \infty)$ and

$$r_k(x,t) := \begin{cases} (x-t)^{k-1}, & 0 \le t \le x < \infty, \\ (-1)^k (t-x)^{k-1}, & -\infty < x \le t < 0, \\ 0, & \text{otherwise.} \end{cases}$$

Let $AC^{(k)}(\Delta)$ denote the class of all functions on Δ with absolutely continuous $(k-1)$-th derivative. It easy to see that for any $G \in AC^{(k)}(\Delta)$ the following is

true:

$$G(x) = R_k G^{(k)}(x) + \sum_{j=0}^{k-1} \frac{x^j}{j!} G^{(j)}(0), \tag{2.1}$$

$$G - \sum_{j=0}^{k-1} (G, p_j)_H p_j = R_k G^{(k)} - \sum_{j=0}^{k-1} (R_k G^{(k)}, p_j)_H p_j. \tag{2.2}$$

Define

$$\rho_k(x) := \frac{1}{(k-1)!} \int_{-1}^{x} (x-t)^{k-1} \varphi_k(t) \omega(t) dt, \quad k \in \mathbb{N}.$$

Then

$$\rho_k^{(n)}(\pm 1) = 0, \quad n = 0, 1, \ldots, k-1. \tag{2.3}$$

Moreover, $\rho_k(x)$ can have alternating signs, but

$$\lambda_k^{1/2} := \int_\Delta \rho_k(x) dx = \frac{1}{a_k k!} > 0, \quad k \in \mathbb{N}.$$

If we put

$$\mu_k(x) := \lambda_k^{-1/2} \rho_k(x), \quad k \in \mathbb{N},$$

then for

$$D_{k,\omega}(G) := [\mathrm{dist}_{\mathbb{H}}(G, P_{k-1})]^2, \quad k \in \mathbb{N}$$

the lower bound

$$\lambda_k \left[\int_\Delta G^{(k)}(x) \mu_k(x) dx \right]^2 \le D_{k,\omega}(G) \tag{2.4}$$

holds (see [4], Lemma 1) provided $G \in AC^{(k)}(\Delta)$ and $G^{(k)} \in L^2(\overline{\rho}_k)$, where

$$\overline{\rho}_k(t) := \int_\Delta |r_k(x,t) \varphi_k(x)| \omega(x) dx.$$

Moreover, (2.4) turns into an equality for $G = \mathrm{const} \cdot \varphi_k$.

Now, suppose that $\rho_k(x) \ge 0$ and for all $2k$ times differentiable functions $y(x)$ we define differential operators

$$L_k y := (-1)^k D^k [\rho_k D^k y].$$

Then

$$L_k \varphi_k = (-1)^k D^k [\rho_k D^k \varphi_k] = a_k k! \rho_k^{(k)} = \lambda_k^{-1/2} \varphi_k \omega,$$

so φ_k is an eigenfunction of the weighted differential operator L_k.

Suppose $\varphi_n, n > k$, are the eigenfunctions of L_k too, that is,

$$L_k \varphi_n = \sigma_{k,n} \varphi_n \omega, \quad n > k \ge 1, \tag{2.5}$$

and, moreover,

$$\lambda_k^{-1/2} = \sigma_{k,k} \le \sigma_{k,n}, \quad n > k \ge 1. \tag{2.6}$$

Then for all polynomials $y \in \mathbf{P}$ such that

$$(y, \varphi_n)_{\mathbb{H}} = 0, \quad n = 0, 1, \ldots, k-1,$$

the inequality

$$\int_\Delta y L_k y \ge \lambda_k^{-1/2} \int_\Delta y^2 \omega$$

holds and becomes the equality for $y = \text{const} \cdot \varphi_k$. By the density $\mathbf{P} \subset L^2(\mu_k)$ it implies the inequality

$$D_{k,\omega}(G) \le \lambda_k \int_\Delta |G^{(k)}|^2 \mu_k \tag{2.7}$$

for all $G \in AC^{(k)}(\Delta)$ such that $G^{(k)} \in L^2(\mu_k)$ (see the end of the proof of Theorem 3.1 for details).

Thus, we obtain the following.

Theorem 2.1. *Let $k \in \mathbb{N}$ and let a weight function $\omega(x) > 0$, $x \in \Delta$, be such that $\int_\Delta \omega = 1$. Let $\{\varphi_n\}$, $n \in \mathbb{N}$, be the above constructed orthonormal sequence of polynomials in $\mathbb{H} = L^2(\omega)$. Then for any function $G \in AC^{(k)}(\Delta)$ such that $G^{(k)} \in L^2(\overline{\rho}_k)$, the inequality* (2.4) *holds. Moreover, if the sequence $\{\varphi_n\}$ satisfies* (2.5), (2.6), *and $\rho_k(x) \ge 0$, then for any function $G \in AC^{(k)}(\Delta)$ such that $G^{(k)} \in L^2(\mu_k)$, the inequality* (2.7) *holds, and both inequalities* (2.4) *and* (2.7) *simultaneously become the equalities, when $G = \text{const} \cdot \varphi_k$.*

Naturally, the sequence of conditions (2.5), (2.6) looks much too restrictive. To this end we consider the first of the equations, (2.5):

$$L_1\varphi_2 = \sigma_{1,2}\varphi_2\omega =: \sigma\varphi_2\omega, \tag{2.8}$$

where $\sigma > 0$ is some number and $\varphi_2(x) = b_2x^2 + b_1x + b_0, b_2 > 0$. Taking into account that $\varphi_1 \in \mathbf{P}_1$, $\varphi_2 \in \mathbf{P}_2$, and

$$\rho_1(x) = \int_{-1}^{x} \varphi_1\omega,$$

we obtain

$$L_1\varphi_2 = -(\rho_1\varphi_2')' = -\varphi_1\varphi_2'\omega - b_2\rho_1 = \sigma\varphi_2\omega.$$

Consequently,

$$\rho_1 b_2 = [-\varphi_1\varphi_2' - \sigma\varphi_2]\omega =: q_2\omega, \tag{2.9}$$

where $q_2 \in \mathbf{P}_2$. It follows from (2.3) and (2.9) that

$$\lim_{x\to\pm 1} q_2(x)\omega(x) = 0. \tag{2.10}$$

Moreover, it follows from (2.9) that ω is differentiable on the interval Δ and satisfies the Pirson equation

$$\frac{\omega'}{\omega} = \frac{b_2\varphi_1 - q_2'}{q_2} =: \frac{q_1}{q_2}, \tag{2.11}$$

where $q_1 \in \mathbf{P}_1$. It is known (see [5], Chapter II, § 1) that, under the conditions $\omega(x) > 0$, $x \in \Delta$, $\int_\Delta \omega < \infty$ and (2.10), the equation (2.11) has the only solutions of the form

$$\omega(x) = \text{const}(1-x)^\alpha(1+x)^\beta,$$

where $\alpha > -1$, $\beta > -1$. To close the circle, we show in the next section that polynomials generated by this weight (Jacobi polynomials) satisfy (2.5) and (2.6) as well as $\rho_k(x) \ge 0$ for this case, and the condition $G^{(k)} \in L^2(\overline{\rho}_k)$ of Theorem 2.1 can be weakened up to $G^{(k)} \in L^2(\mu_k)$.

3. Chernoff-type inequality

Again let $H := L^2(\omega)$ be the Hilbert space with the inner product $(f,g)_{L^2(\omega)} := \int_\Delta fg\omega$ and the norm $\|f\|_{L^2(\omega)} := (f,f)^{1/2}_{L^2(\omega)}$, where the weight function $\omega(x)$ has the form (1.3). Denote by $P_n(x)$, $n \in \mathbb{N}_0$, the orthogonal sequence of Jacobi polynomials. It is known that $\{P_n(x)\}$ is complete in H. By Rodriges formula (see [5], Chapter VII, § 1), we have

$$P_n(x) = \frac{(-1)^n}{2^n n!}(1-x)^{-\alpha}(1+x)^{-\beta}D^n\left[(1-x)^{\alpha+n}(1+x)^{\beta+n}\right], \tag{3.1}$$

where $D := \frac{d}{dx}$. It is known (see [5], Chapter VII, § 1) that

$$\eta_n^2 := \|P_n\|_H^2 = \frac{2^{\alpha+\beta+1}\Gamma(\alpha+n+1)\Gamma(\beta+n+1)}{n!(\alpha+\beta+2n+1)\Gamma(\alpha+\beta+n+1)}, \tag{3.2}$$

where $\Gamma(\cdot)$ denotes the gamma function. Let

$$p_n := P_n/\|P_n\|_H \tag{3.3}$$

be the orthogonal sequence of Jacobi polynomials. Then

$$D_{k,\omega}(G) = \|G\|_H^2 - \sum_{j=0}^{k-1}(G,p_j)_H^2. \tag{3.4}$$

For $n \in \mathbb{N}$ we put

$$\beta_n := \int_\Delta (1-t)^{\alpha+n}(1+t)^{\beta+n}dt = 2^{\alpha+\beta+2n+1}\frac{\Gamma(\alpha+n+1)\Gamma(\beta+n+1)}{\Gamma(\alpha+\beta+2n+2)}, \tag{3.5}$$

$$\nu_n(x) := \frac{(1-x)^{\alpha+n}(1+x)^{\beta+n}}{\beta_n}, \tag{3.6}$$

$$\gamma_n := \frac{\beta_n^2}{\eta_n^2 2^{2n}[n!]^2}. \tag{3.7}$$

Theorem 3.1. *Let $k \in \mathbb{N}$. Then for any function $G \in AC^{(k)}(\Delta)$ such that $G^{(k)} \in L^2(\nu_k)$, the inequalities*

$$\gamma_k\left[\int_\Delta G^{(k)}\nu_k\right]^2 \le D_{k,\omega}(G) \le \gamma_k\int_\Delta |G^{(k)}|^2\nu_k \tag{3.8}$$

hold. Moreover, both inequalities in (3.8) *become the equalities for $G = \mathrm{const}\cdot P_k$.*

Proof. Observe that if $G \in AC^{(k)}(\Delta)$ and $G^{(k)} \in L^2(\nu_k)$, then $G \in H$. Indeed, it follows from (2.1) that

$$\int_\Delta G^2\omega \le c_1\int_\Delta |R_kG^{(k)}|^2\omega + c_2\sum_{j=0}^{k-1}|G^{(j)}(0)|^2\int_\Delta x^{2j}\omega(x)dx.$$

Applying Schwarz's inequality, we obtain

$$|R_kG^{(k)}(x)|^2 \le \int_\Delta |r_k(x,t)|dt\int_\Delta |r_k(x,t)||G^{(k)}(t)|^2dt \le c_3\int_\Delta |r_k(x,t)||G^{(k)}(t)|^2dt.$$

Thus,

$$\int_\Delta G^2\omega \le c_4 + c_5\int_\Delta \omega(x)dx\int_\Delta |r_k(x,t)||G^{(k)}(t)|^2dt$$
$$= c_4 + c_5\int_\Delta |G^{(k)}(t)|^2dt\int_\Delta |r_k(x,t)|\omega(x)dx \le c_4 + c_6\int_\Delta |G^{(k)}(t)|^2\nu_k(t)dt < \infty.$$

First we prove the left-hand side of inequality (3.8). It follows from (3.4) and the Bessel inequality that

$$D_{k,\omega}(G) \ge (G,p_k)_H^2$$

(applying (3.1), (3.2) and (3.3))

$$= \frac{1}{\eta_k^2 2^{2k}[k!]^2}\left[\int_\Delta G(x)D^k\left[(1-x)^{\alpha+k}(1+x)^{\beta+k}\right]dx\right]^2$$

(using (2.1) and the orthogonality of p_k to all polynomials of degree $\le k-1$)

$$= \frac{1}{\eta_k^2 2^{2k}[k!]^2}\left[\int_\Delta R_kG^{(k)}(x)D^k\left[(1-x)^{\alpha+k}(1+x)^{\beta+k}\right]dx\right]^2. \tag{3.9}$$

From the elementary upper bound

$$|D^k[(1-x)^{\alpha+k}(1+x)^{\beta+k}]| \le c_7\omega(x), \quad x\in\Delta,$$

applying Schwarz's inequality twice, we obtain

$$\int_\Delta\int_\Delta |r_k(x,t)G^{(k)}(t)|dt|D^k[(1-x)^{\alpha+k}(1+x)^{\beta+k}]|dx$$
$$\le c_8\left(\int_\Delta |G^{(k)}(t)|^2dt\int_\Delta |r_k(x,t)|\omega(x)dx\right)^{1/2} \le c_9\left(\int_\Delta |G^{(k)}(t)|^2\nu_k(t)dt\right)^{1/2} < \infty.$$

Therefore, by Fubini's theorem, we find from (3.9)

$$D_{k,\omega}(G) \ge \frac{1}{\eta_k^2 2^{2k}[k!]^2}\left[\int_\Delta G^{(k)}(t)dt\int_\Delta r_k(x,t)D^k[(1-x)^{\alpha+k}(1+x)^{\beta+k}]dx\right]^2$$
$$= \frac{1}{\eta_k^2 2^{2k}[k!]^2}\left[\int_\Delta G^{(k)}(t)(1-t)^{\alpha+k}(1+t)^{\beta+k}dt\right]^2$$

(using the definitions (3.5), (3.6) and (3.7))

$$= \gamma_k\left[\int_\Delta G^{(k)}\nu_k\right]^2.$$

Consequently, the left-hand side of inequality (3.8) is proved. If $G = P_k$, then

$$\int_\Delta G^{(k)}\nu_k = \int P_k^{(k)}\nu_k = \frac{1}{\beta_k}\int_\Delta P^{(k)}(x)(1-x)^{\alpha+k}(1+x)^{\beta+k}dx$$

(integrating by parts)

$$= \frac{(-1)^k}{\beta_k}\int_\Delta P_k(x)D^k[(1-x)^{\alpha+k}(1+x)^{\beta+k}]dx$$

(applying (3.1) and then (3.7))

$$= \frac{2^k k!}{\beta_k}\int_\Delta P_k^2\omega = \frac{2^k k!\eta_k^2}{\beta_k} = \frac{\eta_k}{\gamma_k^{1/2}}.$$

Thus, we have

$$\gamma_k \left[\int_\Delta P_k^{(k)} \nu_k \right]^2 = \eta_k^2 = D_{k,\omega}(P_k), \tag{3.10}$$

that is, the left-hand side of inequality (3.8) becomes the equality for $G = \text{const} \cdot P_k$.

Now we show the required upper bound. To this end we observe first that the Jacobi polynomials $P_n(x)$ satisfy the differential equations

$$\begin{aligned} &(1-x^2)P_n^{(k+1)}(x) + (\beta - \alpha - (\alpha+\beta+2k)x)P_n^{(k)}(x) \\ &+(\alpha+\beta+n+k)(n-k+1)P_n^{(k-1)}(x) = 0, \quad k \in \mathbb{N}. \end{aligned} \tag{3.11}$$

It is well known (see [5], Chapter VII, § 3) that every Jacobi polynomial $P_n(x)$ satisfies the differential equation of the second order

$$(1-x^2)P_n''(x) + (\beta - \alpha - (\alpha+\beta+2)x)P_n'(x) + n(\alpha+\beta+n+1)P_n(x) = 0. \tag{3.12}$$

Also we have

$$\begin{aligned} D^{k-1}[(1-x^2)P_n''(x)] &= (1-x^2)P_n^{(k+1)}(x) - 2(k-1)xP_n^{(k)}(x) \\ &\quad - (k-1)(k-2)P_n^{(k-1)}(x), \\ D^{k-1}[(\beta-\alpha-(\alpha+\beta+2)x)P_n'(x)] &= (\beta-\alpha-(\alpha+\beta+2)x)P_n^{(k)}(x) \\ &\quad - (\alpha+\beta+2)(k-1)P_n^{(k-1)}(x). \end{aligned}$$

Differentiating (3.12) $k-1$ times and applying these equations, we obtain (3.11). For any natural integer k we define the differential operation

$$L_k y := (-1)^k D^k[(1-x)^{\alpha+k}(1+x)^{\beta+k}D^k y]. \tag{3.13}$$

Denote $\widetilde{\nu}_k(x) := (1-x)^{\alpha+k}(1+x)^{\beta+k}$. Let us show that for $n \geq k \geq 1$, the equalities

$$L_1 P_n(x) = n(\alpha+\beta+n+1)P_n(x)\omega(x), \tag{3.14}$$

$$L_2 P_n(x) = (\alpha+\beta+n+2)(\alpha+\beta+n+1)n(n-1)P_n(x)\omega(x),$$

$$\dots$$

$$L_n P_n(x) = (\alpha+\beta+2n)\dots(\alpha+\beta+n+1)n!P_n(x)\omega(x) \tag{3.15}$$

hold. It is easy to see that (3.14) follows from (3.12). For the proof of the other equalities it is sufficient to establish the recurrent formula

$$L_{k+1}P_n = (\alpha+\beta+n+k+1)(n-k)L_k P_n. \tag{3.16}$$

Write

$$L_{k+1}P_n(x) = (-1)^{k+1}D^k \cdot D\left[\widetilde{\nu}_k(x)(1-x^2)P_n^{(k+1)}(x)\right]. \tag{3.17}$$

We have

$$D\widetilde{\nu}_k(x) = \widetilde{\nu}_k(x)(1-x^2)^{-1}(\beta-\alpha-(\alpha+\beta+2k)x), \tag{3.18}$$

and also it follows from (3.11) that

$$\begin{aligned} D\left[(1-x^2)P_n^{(k+1)}(x)\right] &= -(\alpha+\beta+n+k+1)(n-k)P_n^{(k)}(x) \\ &\quad -(\beta-\alpha-(\alpha+\beta+2k)x)P_n^{(k+1)}(x). \end{aligned}$$

From this, (3.17) and (3.18) follow (3.16) and (3.15). Note that if the equalities (3.14) and (3.15) are rewritten in the form

$$L_k P_n = \sigma_{k,n} P_n \omega, \quad n \geq k \geq 1, \tag{3.19}$$

then

$$\sigma_{k,n} \geq \sigma_{k,k} = (\alpha+\beta+2k)\dots(\alpha+\beta+k+1)k! \tag{3.20}$$

Now we show the upper estimate (3.8). Note that it trivially holds for a polynomial of degree $< k$, and for polynomials $y \in \mathbf{P}$ of degree $\geq k$ it is equivalent to the inequality

$$\gamma_k \int_\Delta |y^{(k)}|^2 \nu_k \geq \int_\Delta y^2 \omega, \tag{3.21}$$

provided

$$(y, p_n)_H = 0, \quad n = 0, 1, \dots, k-1. \tag{3.22}$$

It follows from (3.22) that there exists $n_y \in \mathbb{N}$ such that

$$y(x) = \sum_{n=k}^{n_y} (y, p_n)_H p_n(x).$$

Hence,

$$\gamma_k \int_\Delta |y^{(k)}|^2 \nu_k = \frac{\gamma_k}{\beta_k} \int_\Delta y L_k y = \frac{\gamma_k}{\beta_k} \int_\Delta \left[\sum_{n=k}^{n_y} (y, p_n)_H p_n\right] \left[\sum_{n=k}^{n_y} (y, p_n)_H L_k p_n\right]$$

(because of (3.19))

$$= \frac{\gamma_k}{\beta_k} \sum_{n=k}^{n_y} (y, p_n)_H^2 \sigma_{k,n} \int_\Delta p_n^2 \omega$$

(applying (3.20) and Parseval's equality)

$$\geq \frac{\gamma_k}{\beta_k} \sigma_{k,k} \sum_{n=k}^{n_y} (y, p_n)^2 = \frac{\gamma_k}{\beta_k} (\alpha+\beta+2k)\dots(\alpha+\beta+k+1)k! \int_\Delta y^2 \omega$$

(inserting the values of γ_k and β_k)

$$= \int_\Delta y^2 \omega,$$

and (3.21) is proved.

It is well known that Jacobi's polynomials $\{p_n\}$, $n \in \mathbb{N}_0$, form the complete orthonormal system in H. Therefore, $\mathbf{P}$ is dense in H and, since $\widetilde{\nu}_k(x)$ has the form (1.3), then $\mathbf{P}$ is dense in $L^2(\nu_k)$. Let $G \in AC^{(k)}(\Delta)$ be such that $G^{(k)} \in L^2(\nu_k)$. For any $\varepsilon > 0$ there exists such a polynomial $h_\varepsilon \in P$ that

$$\|G^{(k)} - h_\varepsilon\|_{L^2(\nu_k)} \leq \varepsilon.$$

Let $g_\varepsilon := G^{(k)} - h_\varepsilon$, $g := G^{(k)} = h_\varepsilon + g_\varepsilon$. By Schwarz's inequality we find

$$\begin{aligned} \int_\Delta |R_k g_\varepsilon|^2 \omega &\leq \int_\Delta \omega(x) dx \int_\Delta |r_k(x,s)| ds \int_\Delta |r_k(x,t)| |g_\varepsilon(t)|^2 dt \\ &\leq c_{10} \int_\Delta |g_\varepsilon(t)|^2 dt \int_\Delta |r_k(x,t)| \omega(x) dx \leq c_{11} \int_\Delta |g_\varepsilon|^2 \nu_k \leq c_{11} \varepsilon^2. \end{aligned} \tag{3.23}$$

Put

$$y_\varepsilon := R_k h_\varepsilon - \sum_{j=0}^{k-1} (R_k h_\varepsilon, p_j)_H p_j.$$

Then $y_\varepsilon \in P$ and satisfies the condition (3.22). Using (2.2) we find

$$G - \sum_{j=0}^{k-1}(G,p_j)_H p_j = R_k g - \sum_{j=0}^{k-1}(R_k g, p_j)_H p_j = y_\varepsilon + R_k g_\varepsilon - \sum_{j=0}^{k-1}(R_k g_\varepsilon, p_j)_H p_j.$$

Hence,

$$[D_{k,\omega}(G)]^{1/2} \le \|y_\varepsilon\|_H + \|R_k g_\varepsilon\|_H$$

(then (3.21) and (3.23) imply)

$$\le \gamma_k^{1/2}\|y_\varepsilon^{(k)}\|_{L^2(\nu_k)} + c_{12}\cdot\varepsilon = \gamma_k^{1/2}\|G^{(k)} - g_\varepsilon\|_{L^2(\nu_k)} + c_{12}\cdot\varepsilon$$
$$\le \gamma_k^{1/2}\|G^{(k)}\|_{L^2(\nu_k)} + (\gamma_k^{1/2} + c_{12})\varepsilon$$

and the right-hand side of (3.8) follows by $\varepsilon \to 0$. If $G = P_k$, then, integrating by parts, we find

$$\gamma_k \int_\Delta |P_k^{(k)}|^2 \nu_k = \frac{\gamma_k}{\beta_k}\int_\Delta P_k L_k P_k$$
$$= \frac{\gamma_k}{\beta_k}(\alpha+\beta+2k)\dots(\alpha+\beta+k+1)k!\int_\Delta P_k^2\omega = D_{k,\omega}(P_k). \qquad \square$$

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