SPACES BETWEEN $H^1$ AND $L^1$

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Abstract. In this paper we consider the spaces $X_s$ that lie between $H^1(R^n)$ and $L^1(R^n)$. We discuss their interpolation properties and the behavior of maximal functions and singular integrals acting on them.

Because of their similarities, but mainly because of their differences, it is a matter of interest to determine the relationship between the Hardy space $H^1(R^n)$ and the space of integrable functions $L^1(R^n)$. The purpose of this paper is to gain a better understanding of the gap that separates them. We thus continue with the study of the spaces $X_s(R^n)$ that lie between $H^1$ and $L^1$, and discuss their interpolation properties and the behavior of maximal functions and singular integrals acting on them.

The spaces $X_s$ were introduced by Sweezy; see [10]. They form a nested family that starts at $H^1 = X_1$ and approaches $L^1_0$, the subspace of $L^1$ functions with vanishing integral, as $s \to \infty$. Here we consider the whole range of $X_s$ spaces. First, $X_s = H^1$ for $0 < s \leq 1$; also, $X_\infty = L^1_0$; see [1]. Further we show that, for $f \in X_s$,

$$K(t, f; H^1, L^1) \leq \min(t, t^{1/s'}) \|f\|_{X_s}.$$ 

This estimate allows us to interpolate between $H^1$ and $L^1$. In particular, it gives the fact that $X_s$ is continuously embedded in the Hardy-Lorentz space $H^{1,r}$ consisting of those distributions with non-tangential maximal function in the Lorentz space $L^{1,r}$, for $1 < s < r \leq \infty$. Therefore, the spaces $H^{1,s} \cap L^1$, $1 \leq s < \infty$, also form a nested family of subspaces of $L^1$ that increase from $H^1$ to $L^1$. As for Calderón-Zygmund singular integral operators, they map $X_s$ into $L^{1,r}$ for $1 < s < r \leq \infty$. We conclude the paper by introducing the closely related family of $X^s$ spaces, $0 < s \leq \infty$, that increases towards $L^1$, and then showing how $X_s$ and $X^s$ atoms can be used to build other spaces, including analogues of the local spaces considered in [7], that lie between $H^1$ and $L^1$.

1. Atomic decompositions in Banach spaces

Our first result is of a general nature and will ensure that the various atomic spaces considered below are indeed Banach spaces.

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Let \( A \) be a non-empty subset of the unit ball of a Banach space \((B, \| \cdot \|)\). The atomic space \( A \) spanned by \( A \) consists of all those \( \varphi \in B \) of the form

\[
\varphi = \sum \lambda_j a_j, \quad \sum |\lambda_j| < \infty, a_j \in A.
\]

It is readily seen that, endowed with the atomic norm \( \| \varphi \|_A = \inf \{ \sum_1^\infty |\lambda_j| : \varphi = \sum_1^\infty \lambda_j a_j \} \), \( A \) becomes a normed space. In fact, it is also complete.

**Lemma 1.1.** The atomic space \((A, \| \cdot \|_A)\) is a Banach space.

**Proof.** Since for \( \varphi = \sum_1^\infty \lambda_j a_j \) we have \( \| \varphi \| \leq \sum_1^\infty |\lambda_j| \), it readily follows that \( \| \varphi \| \leq \| \varphi \|_A \) and \( A \) is continuously embedded in \( B \).

To verify that \((A, \| \cdot \|_A)\) is complete, it suffices to show that if \( \{ \varphi_n \} \) is a sequence of elements in \( A \) such that \( \sum_1^\infty \| \varphi_n \|_A < \infty \), then the sum converges to some \( \varphi \) in \( A \), i.e., for some \( \varphi \in A \), \( \lim_{N \to \infty} \| \varphi - \sum_{n=1}^N \varphi_n \|_A = 0 \). First observe that, since \( \sum_1^\infty \| \varphi_n \| < \infty \), also the sum converges to some \( \varphi \) in \( B \). We will show that \( \varphi \in A \), and that the sum also converges to \( \varphi \) in \( A \).

Let \( \varphi_n = \sum_{j=1}^\infty \lambda_{j,n} a_j, n \) where the \( \lambda_{j,n} \)'s satisfy \( \sum_{j=1}^\infty |\lambda_{j,n}| \leq 2 \| \varphi_n \|_A, n = 1, 2, \ldots \). Having fixed these decompositions, we may restrict our attention to the countable set of atoms \( \{a_j\} \). So, we rename these elements \( \{a_j\} \) and, by adding zeroes to the original \( \lambda_{j,n} \)'s as needed, we have \( \varphi_n = \sum_{j=1}^\infty \lambda_{j,n} a_j \) for all \( n \). Clearly \( \sum_{n=1}^\infty \sum_{j=1}^\infty |\lambda_{j,n}| < \infty \). Moreover, if \( \mu_j = \sum_{n=1}^\infty \lambda_{j,n} \), then \( \sum_j |\mu_j| \leq \sum_{j=1}^\infty \sum_{n=1}^\infty |\lambda_{j,n}| < \infty \). Now, since \( \varphi = \sum \varphi_n = \sum_{j=1}^\infty \mu_j a_j, \varphi \in A \). Finally, given \( \varepsilon > 0 \), let \( N_0 \) be such that \( \sum_{n=N}^\infty \sum_{j=1}^\infty |\lambda_{j,n}| \leq \varepsilon \), for \( N \geq N_0 \). Then, for \( N \geq N_0 \),

\[
\| \varphi - \sum_{j=1}^\infty \left( \sum_{n=1}^{N-1} \lambda_{j,n} \right) a_j \|_A = \| \sum_{j=1}^\infty \left( \sum_{n=N}^\infty \lambda_{j,n} \right) a_j \|_A \\
\leq \sum_{n=N}^\infty \sum_{j=1}^\infty |\lambda_{j,n}| \leq \varepsilon,
\]

and we have finished. \( \square \)

## 2. The spaces \( X_s \)

For \( 1 < q \leq 2 \) with conjugate \( 2 \leq p < \infty, 1/p + 1/q = 1 \), and \( 0 < s \leq \infty \), we say that a compactly supported function \( a \) with vanishing integral is a \((q, s)\) atom with defining cube \( Q \) if

\[
supp(a) \subseteq Q, \quad \int_Q a(x) \, dx = 0, \quad p^{1/s} |Q| \left( \frac{1}{|Q|} \int_Q |a(x)|^q \, dx \right)^{1/q} \leq 1.
\]

When \( s = \infty \), \( a \) is a usual \( L^q \) 1-atom.

We denote by \( A_s \) the collection of \((q, s)\) atoms, \( 0 < s \leq \infty \). Since for \( a \in A_s \) we have

\[
\int_Q |a| \leq |Q| \left( \frac{1}{|Q|} \int_Q |a(x)|^q \, dx \right)^{1/q} \leq p^{-1/s} \leq 1,
\]

\( A_s \) is contained in the unit ball of \( L^1 \), and Lemma 1.1 applies. The resulting Banach space is \( X_s(R^n) = X_s \), the space introduced by Sweezy, who also showed that \( X_1 = H^1 \); see [10].

As a first step in determining the relationship between the \( X_s \)'s for the different values of \( s \) we have
Lemma 2.1. Suppose \( a \) is a \((q, s)\) atom. Then,
\[
\|a\|_{X_s} \leq p^{1/u-1/s}, \quad 0 < u \leq \infty.
\]
Proof. Since for a \((q, s)\) atom \( a \) we have
\[
p^{1/u}|Q| \left( \frac{1}{|Q|} \int_Q \left[ \frac{|a(x)|}{p^{1/u-1/s}} \right]^q \, dx \right)^{1/q} \leq 1,
\]
the conclusion follows readily. \( \square \)

From Lemma 2.1 we get \( \|f\|_{X_s} \leq \|f\|_{X_r}, \) \( 0 < r \leq s \leq \infty, \) and the \( X_s \)'s are nested. Also, the norm in \( X_s \) reduces to the atomic \( H^1 \) norm for \( 0 < s < 1. \)

Proposition 2.1. Suppose \( 0 < s < 1. \) Then \( X_s = H^1, \) with equivalent norms.

Proof. We have already noted that \( \|f\|_{H^1} \leq \|f\|_{X_s} \). Now let \( f \in H^1 \) have an atomic decomposition \( f = \sum_j \lambda_j a_j \) in terms of \( L^\infty 1\)-atoms \( a_j. \) Then,
\[
2^{1/s}|Q| \left( \frac{1}{|Q|} \int_Q \left[ \frac{|a_j(x)|}{2^{1/s}} \right]^2 \, dx \right)^{1/2} \leq 1,
\]
a \( \lambda_j(x)/2^{1/s} \) is a \((2, s)\) atom, and \( \|f\|_{X_s} \leq 2^{1/s} \sum_j |\lambda_j| \). Taking the infimum over all possible decompositions of \( f \) it follows that \( \|f\|_{X_s} \leq 2^{1/s} \|f\|_{H^1}, \) and we have finished. \( \square \)

The situation is different for \( s > 1. \) As the \( q_j \)'s approach 1, the \( p_j \)'s tend to \( \infty, \) the sums escape \( H^1, \) and \( X_s \neq H^1. \) In fact, \( X_s \) contains strictly \( X_r, \) \( 1 \leq r < s, \) and, although \( X_r \) is densely embedded in \( X_s \) for \( r < s, \) it is of first category in \( (X_s, \| \cdot \|_{X_s}). \)

Concerning \( s \) large, let \( X = \bigcup_{s<\infty} X_s. \) We introduce a topology in \( X \) that is easier to deal with than the inductive topology there. For \( f \in X, \) let \( \|f\|_{X} = \lim_{s \to \infty} \|f\|_{X_s}. \) It is not hard to see that \( \| \cdot \|_{X} \) is a norm. The homogeneity and triangle inequality follow easily. Moreover, if \( \|f\|_{X} = 0, \|f\|_{1} = 0, \) and \( f = 0 \) a.e.

In fact, for each \( s, \) the inclusion mapping is continuous from \( X_s \) to \((X, \| \cdot \|_{X}),\) and, consequently, \( \lim X_s \) is also continuously included in \( X. \) Similarly, \( (X, \| \cdot \|_{X}) \) is continuously embedded in \( (X_\infty, \| \cdot \|_{X_\infty}), \) but \( X \) and \( X_\infty \) are not the same space. To see this note that \( X_r \) is of first category in \( X_\infty \) for \( r < \infty, \) and the same is true for \( X: \) if \( U \) is open in \( X_\infty, \) it cannot be open in any \( X_r, \) and hence it is not open in \( X. \) Finally, \( X_\infty = L^1_0; \) see [1].

2.1. \( X_s \) as an intermediate space between \( X_{s_1} \) and \( X_{s_2}. \) Recall that the \( K \) functional of \( f \in X_{s_1} + X_{s_2} \) at \( t > 0 \) is defined by
\[
K(t; f; X_{s_1}, X_{s_2}) = \inf_{f = f_1 + f_2} \|f_1\|_{X_{s_1}} + t \|f_2\|_{X_{s_2}},
\]
where \( f = f_1 + f_2, f_1 \in X_{s_1} \) and \( f_2 \in X_{s_2}. \) We begin by estimating the \( K \) functional for \( f \in X_s, 1 \leq s_1 < s < s_2 \leq \infty. \) The reader will have no difficulty in verifying that a similar result holds with \( X \) in place of \( X_\infty. \)

Lemma 2.2. Given \( 1 \leq s_1 < s < s_2 \leq \infty, \) let \( 0 < \eta < 1 \) be given by \( 1/s = (1-\eta)/s_1 + \eta/s_2. \) Then, for \( f \in X_s, \)
\[
K(t; f; X_{s_1}, X_{s_2}) \leq \min \left( t, t^\eta \right) \|f\|_{X_s}.
\]
Proof. Since $X_s \leftrightarrow X_{s_2}$, $K(t, f; X_{s_1}, X_{s_2}) \leq t \|f\|_{X_{s_2}} \leq t \|f\|_{X_s}$. This estimate suffices for $t$ small.

Suppose now that $t$ is large, $t > 1$, say, and let $\alpha > 0$ be given by $1/\alpha = 1/s_1 - 1/s_2$. Let $f \in X_s$ have an atomic decomposition $f = \sum_j \lambda_j a_j$ in terms of $(q_j, s)$ atoms $a_j$, and let $p_j$ denote the conjugate exponent to $q_j$. Finally, put $J_1 = \{j : p_j \leq t^\alpha\}$ and $J_2 = \{j : p_j > t^\alpha\}$. By Lemma 2.1 we have

$$\left\| \sum_{j \in J_1} \lambda_j a_j \right\|_{X_{s_1}} \leq \sum_{j \in J_1} |\lambda_j| \|a_j\|_{X_{s_1}} \leq \sum_{j \in J_1} |\lambda_j| p_j^{1/s_1 - 1/s} \leq t^\alpha \sum_{j \in J_1} |\lambda_j|,$$

$$\left\| \sum_{j \in J_2} \lambda_j a_j \right\|_{X_{s_2}} \leq \sum_{j \in J_2} |\lambda_j| \|a_j\|_{X_{s_2}} \leq \sum_{j \in J_2} |\lambda_j| p_j^{1/s_2 - 1/s} \leq t^\alpha \sum_{j \in J_2} |\lambda_j|.$$ 

Now, since, as is readily seen, $\alpha(1/s_1 - 1/s) = \eta$ and $\alpha(1/s_2 - 1/s) = \eta - 1$, we get

$$K(t, f; X_{s_1}, X_{s_2}) \leq \sum_{j \in J_1} \lambda_j a_j \left\|X_{s_1}\right\| + t \left\| \sum_{j \in J_2} \lambda_j a_j \left\|X_{s_2}\right\| \leq t^\eta \sum_j |\lambda_j|.$$ 

Thus, taking the infimum over the decompositions of $f$ in $X_s$, it follows that

$$K(t, f; X_{s_1}, X_{s_2}) \leq t^\eta \|f\|_{X_s}.$$ 

The conclusion now obtains by combining the estimates for $t$ small and $t$ large. \qed

Corollary 2.1.1. Let $1 < s < \infty$. Then, for $f \in X_s$,

$$K(t, f; H^1, L^1) \leq \min \left( t, t^{1/s} \right) \|f\|_{X_s}.$$ 

Proposition 2.2. Let $f \in X_s$, $1 < s < \infty$. Then $f$ is in the Hardy-Lorentz space $H^{1,r}$, $r > s$, and $\|f\|_{H^{1,r}} \leq c_r \|f\|_{X_s}$, $c_r = O(c/(1/s - 1/r)).$

Proof. Let $f \in X_s$. We will show that the non-tangential maximal function $Nf$ is in $L^{1,r}$ for $r > s$. Since the non-tangential maximal function of a function in $H^1$ is in $L^1$, and that of a function in $L^1$ is in $L^{1,\infty}$, by elementary interpolation considerations and Corollary 2.1.1 it follows that

$$K(t, Nf; L^1, L^{1,\infty}) \leq c K(t, f; H^1, L^1) \leq c \min(t, t^{1/s'}) \|f\|_{X_s}.$$ 

Given $1 < s < r < \infty$, let $1/s' < \theta < 1$ be chosen so that $1/r = 1 - \theta$. Integrating the above inequality we get

$$\|Nf\|_{(L^{1,1,\infty})_u} \leq c \left( \int_0^\infty \left( \frac{\min(t, t^{1/s'})}{t^{q\theta}} \right)^{s} \frac{dt}{t} \right) \|f\|_{X_s} \leq c_r \|f\|_{X_s}.$$ 

Clearly, $c_r \leq c/(1/s - 1/r)$. Furthermore, since $s < r$, we also have (see [2])

$$\|Nf\|_{1,r} \sim \|Nf\|_{(L^{1,1,\infty})_u} \leq c \|Nf\|_{(L^{1,1,\infty})_u} \leq c \|Nf\|_{(L^{1,1,\infty})_u},$$

and the conclusion follows by combining the two estimates. \qed
Lemma 2.2 applies to Calderón-Zygmund singular integrals, and other operators, such as the Marcinkiewicz integral (see [4]) that map $H^1$ into $L^1$, and $L^1$ into weak $L^1$. Thus, these operators also map $X_s$ into the Lorentz space $L^{1,r}$, for $1 < s < r \leq \infty$. It also applies to some Calderón-Zygmund singular integral operators with rough kernels that are known to be of weak-type $(1,1)$ (see [8]) and to map $H^1$ into $L^{1,2}$ (see [9]). Lemma 2.2 then gives that they also map $X_s$ into $L^{1,r}$ for $r > 2s$.

3. Concluding remarks

In order to reach $L^1$ from $H^1$, one more atom, this one with non-vanishing integral, needs to be added to the families $A_s$; the characteristic function of $Q_1$, the cube of sidelength 1 centered at the origin, will do. Let $X^s$ denote the Banach space spanned by $A_s \cup \chi_{Q_1}$, $1 \leq s \leq \infty$, and note that if $f \in X^s$ has an atomic decomposition of the form $f = \sum_j \lambda_j a_j + \lambda \chi_{Q_1}$, $\lambda$ is uniquely determined and is equal to $\int_{R^n} f$. Thus, for $f$ in $X^s$ we have $\|f\|_{X^s} = \inf \sum_j |\lambda_j| + |\lambda|$, where the infimum is taken over the atomic decompositions of $f$. In other words, $X^s = X_s + sp(\chi_{Q_1})$, in the sense of the sum of Banach spaces. Moreover, the family $X^s$ is nested and reaches $L^1$. Other than this important property, the family $X^s$ behaves very much like $X_s$, and, consequently, we only give the description of its dual. It is the space $BMO^s$ consisting of those functions $\varphi(x)$ such that

$$A(\varphi) = \sup_{p > 1} \frac{1}{p^{1/s}} \sup_Q \left( \frac{1}{|Q|} \int_Q |\varphi(x) - \varphi_Q|^p dx \right)^{1/p} < \infty,$$

$$B(\varphi) = \left| \int_{Q_1} \varphi(x) dx \right| < \infty,$$

normed with $\|\varphi\|_{BMO^s} = \max(A(\varphi), B(\varphi))$, $1 \leq s \leq \infty$. We are grateful to the referee for pointing out that the Chang-Wilson-Wolff inequality (see [3]) provides a sufficient condition for a function to be in $BMO^2$. Of course, the dual of $X^\infty$ is $L^\infty$; see [1].

One can also define spaces analogous to the local version of $H^1$ spaces at the origin introduced in [7], with the $H^1$ atoms there replaced by $X_s$ atoms. Let $Q_\delta$ denote the cube of sidelength $\delta$ centered at the origin. The family $C_s$ of central $(q,s)$ atoms consists of those $a \in A_s$ with defining cube $Q_\delta$ for some $\delta > 0$. The atomic space generated by $C_s$ is denoted $HX_s$, for $1 \leq s \leq \infty$.

$HX_1$ is a dense subset of $H^1$ which is embedded continuously in $H^1$ and, as we will see below, it is not $H^1$. The $HX_s$'s form a nested family of subspaces of $L^1_0$ and, for each $s$, $HX_s$ is a dense subset of $X_s$ continuously embedded in $X_s$.

The dual of $HX_s$ is the space $CMO(s)$ which consists of those functions $\varphi$ such that

$$\|\varphi\|_{CMO(s)} = \sup_{p > 1} \sup_{\delta > 0} \left( \frac{1}{|Q_\delta|} \int_{Q_\delta} |\varphi(x) - \varphi_{Q_\delta}|^p dx \right)^{1/p} < \infty, \quad s < \infty,$$

$$\|\varphi\|_{CMO(\infty)} = \sup_{p > 1} \sup_{\delta > 0} \left( \frac{1}{|Q_\delta|} \int_{Q_\delta} |\varphi(x) - \varphi_{Q_\delta}|^p dx \right)^{1/p} < \infty, \quad s = \infty.$$
Now, $\text{CMO}(1)$ strictly contains $BMO$. To see this consider, for $n = 1$,

$$
\varphi(x) = \begin{cases} 
\ln|x+2|, & -\infty < x < -2, \\
0, & -2 \leq x \leq -1, \\
\ln|x+1|, & -1 < x < \infty.
\end{cases}
$$

$\varphi$ is in $\text{CMO}(1)$ but not in $BMO$, which shows that, unlike $X_1$, $HX_1$ is not $H^1$. The space $\text{CMO}(\infty)$ coincides with $L^{\infty,*}$, the dual of $L_0$, as a set of functions, but with a weaker norm.

Finally, let $HX^s$ denote the Banach space spanned by $C_s \cup \chi_{Q_1}$. The $HX^s$'s form an increasing family of subspaces of $L^1$, and $HX^s$ contains strictly $HX_s$. $HX^\infty$ is close to $L^1$, so the $HX^s$'s cover the gap left by the $HX_s$'s. The dual of $HX^s$ is $\text{CMO}^s$, which consists of all functions $\varphi$ in $\text{CMO}(s)$, but normed with

$$
\|\varphi\|_{\text{CMO}^s} = \|\varphi\|_{\text{CMO}(s)} + \left| \int_{Q_1} \varphi(x) \, dx \right|, \quad 1 \leq s < \infty.
$$

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