ON THE SUM OF THE INDEX OF A PARABOLIC SUBALGEBRA AND OF ITS NILPOTENT RADICAL

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Abstract. In this short note, we investigate the following question of Panyushev stated in 2003: “Is the sum of the index of a parabolic subalgebra of a semisimple Lie algebra $g$ and the index of its nilpotent radical always greater than or equal to the rank of $g$?” Using the formula for the index of parabolic subalgebras conjectured by Tauvel and the author and proved by Fauquant-Millet and Joseph in 2005 and Joseph in 2006, we give a positive answer to this question. Moreover, we also obtain a necessary and sufficient condition for this sum to be equal to the rank of $g$. This provides new examples of direct sum decomposition of a semisimple Lie algebra verifying the “index additivity condition” as stated by Raïs.

1. Introduction

Let $\mathfrak{g}$ be a Lie algebra over an algebraically closed field $\mathbb{k}$ of characteristic zero. For $f \in \mathfrak{g}^*$, we denote by $\mathfrak{g}^f = \{ X \in \mathfrak{g}; f([X,Y]) = 0 \text{ for all } Y \in \mathfrak{g} \}$, the annihilator of $f$ for the coadjoint representation of $\mathfrak{g}$. The index of $\mathfrak{g}$, denoted by $\chi(\mathfrak{g})$, is defined to be

$$\chi(\mathfrak{g}) = \min_{f \in \mathfrak{g}^*} \dim \mathfrak{g}^f.$$ 

It is well known that if $\mathfrak{g}$ is an algebraic Lie algebra and $G$ denotes its algebraic adjoint group, then $\chi(\mathfrak{g})$ is the transcendence degree of the field of $G$-invariant rational functions on $\mathfrak{g}^*$.

The index of a semisimple Lie algebra $\mathfrak{g}$ is equal to the rank of $\mathfrak{g}$. This can be obtained easily from the isomorphism between $\mathfrak{g}$ and $\mathfrak{g}^*$ via the Killing form. There has been quite a lot of recent work on the determination of the index of certain subalgebras of a semisimple Lie algebra: parabolic subalgebras and related subalgebras ([2], [9], [13], [8]), centralizers of elements and related subalgebras ([10], [1], [15], [7]).

Let $\mathfrak{g}$ be a semisimple Lie algebra, $\mathfrak{p}$ a parabolic subalgebra of $\mathfrak{g}$ and $\mathfrak{u}$ (resp. $\mathfrak{l}$) the nilpotent radical (resp. a Levi factor) of $\mathfrak{p}$. In [10, Corollary 1.5 (i)], Panyushev showed that

$$\chi(\mathfrak{p}) + \chi(\mathfrak{u}) \leq \dim \mathfrak{l}.$$
He then suggested [10, Remark (ii) of Section 6] that

\[
\chi(p) + \chi(u) \geq rk g.
\]

For example, it is well known that if \( b \) is a Borel subalgebra of \( g \) and \( n \) is its nilpotent radical, then \( \chi(b) + \chi(n) = rk g \) (see for example [12], [14, Chapter 40]). It is therefore also interesting to characterise parabolic subalgebras where equality holds in (2). Indeed, in [11], Raïs looked for examples of direct sum decompositions \( g = m \oplus n \) verifying the “index additivity condition”, namely \( m \) and \( n \) are Lie subalgebras of \( g \) and

\[
\chi(g) = \chi(m) + \chi(n).
\]

If \( u_- \) denotes the nilpotent radical of the opposite parabolic subalgebra \( p_- \) of \( p \), then \( g = p \oplus u_- \) and the Lie algebras \( u \) and \( u_- \) are isomorphic. Thus parabolic subalgebras such that equality holds in (2) would provide examples of direct sum decompositions verifying the index additivity condition.

Using the formula conjectured in [13] and proved in [3, 6] for the index of parabolic subalgebras, we obtain a formula for the sum \( \chi(p) + \chi(u) \). By a careful analysis of root systems, we prove inequality (2) and give a necessary and sufficient condition for the equality to hold in (2) (see Theorem 2.2).

To describe the index of a parabolic subalgebra and the index of its nilpotent radical, we need to recall Kostant’s cascade construction of pairwise strongly orthogonal roots ([4], [5], [14]).

Let us fix a Cartan subalgebra \( h \) of \( g \) and a Borel subalgebra \( b \) of \( g \) containing \( h \). Denote by \( R \), \( R^+ \) and \( \Pi = \{\alpha_1, \ldots, \alpha_t\} \) respectively the set of roots, positive roots and simple roots with respect to \( h \) and \( b \). For any \( \alpha \in R \), let \( g_\alpha \) be the root subspace associated to \( \alpha \). Choose \( X_\alpha \in g_\alpha \) such that \( \alpha([X_\alpha, X_-\alpha]) = 2 \). We shall write \( \alpha^\vee = [X_\alpha, X_-\alpha] \in h \), and for \( \lambda \in h^* \), \( \langle \lambda, \alpha^\vee \rangle = \lambda(\alpha^\vee) \). For \( S \subset \Pi \), we denote by \( R_S = R \cap ZS, R^+_S = R_S \cap R^+ \). If \( S \) is connected, then we shall denote by \( \varepsilon_S \) the highest root of \( R_S \).

Let \( S \subset \Pi \). We define inductively a set \( K(S) \) whose elements are subsets of \( \Pi \) as follows:

a) \( K(\emptyset) = \emptyset \).

b) If \( S_1, \ldots, S_r \) are the connected components of \( S \), then \( K(S) = K(S_1) \cup \cdots \cup K(S_r) \).

c) If \( S \) is connected, then \( K(S) = \{S\} \cup K(\bar{S}) \) where \( \bar{S} = \{\alpha \in S; \langle \alpha, \varepsilon_S \rangle = 0\} \).

It is well known that (see for example [14, Chapter 40]) elements of \( K(S) \) are connected subsets of \( S \). Moreover, if we denote by \( R(S) = \{\varepsilon_K; K \in K(S)\} \), then \( R(S) \) is a maximal set of pairwise strongly orthogonal roots in \( R_S \).

Let us also recall the following properties of \( K(S) \) which are easy consequences from the definition (see for example [14, Chapter 40]):

**Lemma 1.1.** Let \( S \) be a subset of \( \Pi \), \( K, K' \in K(S) \) and set

\[
\Gamma^K = \{\alpha \in R_K; \langle \alpha, \varepsilon^K \rangle > 0\}
\]

\[
= \{\alpha = \sum_{\beta \in K} n_\beta \beta \in R^K_+; n_\beta > 0 \text{ for some } \beta \in K \setminus \hat{K}\}
\]

i) We have either \( K \subset K' \) or \( K' \subset K \) or \( K \) and \( K' \) are connected components of \( K \cup K' \).

ii) \( \Gamma^K = R^K_+ \setminus \{\beta \in R^K_+; \langle \beta, \varepsilon^K \rangle = 0\} \). In particular, \( R^K_+ \) is the disjoint union of \( \Gamma^K \)'s, \( K \in K(S) \).
The subset $E$ of $\Gamma(\Pi)$ is listed in Table 1 for an irreducible root system $R$, where for any $x \in \mathbb{Q}$, $[x]$ is a unique integer such that $[x] \leq x < [x] + 1$.

**Examples 1.2.** Let $R$ be an irreducible root system. We shall use the numbering of simple roots in [14, Chapter 18]. Set $k = \sharp \mathcal{K}(\Pi)$.

1) Let $R$ be of type $A_{\ell}$. Then $\mathcal{K}(\Pi) = \bigcup_{i = 1}^{k} \{K_i\}$ where $K_i = \{\alpha_i, \ldots, \alpha_{\ell+1-i}\}$. For $1 \leq i \leq k$,

$$\Gamma_{K_i} = \{\alpha_i + \cdots + \alpha_{r}, \alpha_{r+1-i} + \cdots + \alpha_{\ell+1-i}; 0 \leq r \leq \ell - 2i\} \cup \{\varepsilon_{K_i}\}.$$

2) Let $R$ be of type $D_{2n+1}$. Then $k = 2n$,

$$\mathcal{K}(\Pi) = \bigcup_{i = 1}^{n} \{K_i\} \cup \{L_i\},$$

where $K_i = \{\alpha_{2i-1}, \ldots, \alpha_{2n+1}\}$ and $L_i = \{\alpha_{2i-1}\}$. For $1 \leq i \leq n$,

$$\Gamma_{K_i} = \left\{ \sum_{j=2i-1}^{\ell} m_j \alpha_j; m_{2i} \neq 0 \right\}, \quad \Gamma_{L_i} = \{\alpha_{2i-1}\}.$$

3) Let $R$ be of type $E_6$. Then

$$\mathcal{K}(\Pi) = \{\Pi\} \cup \{\{\alpha_1, \alpha_3, \alpha_4, \alpha_5, \alpha_6\}\} \cup \{\{\alpha_3, \alpha_4, \alpha_5\}\} \cup \{\{\alpha_4\}\}.$$  

2. MAIN RESULT

Recall that for any subset $S \subset \Pi$,

$$p_S = \mathfrak{h} \oplus \bigoplus_{\alpha \in R_S \cup R^+} g_\alpha$$

is a (standard) parabolic subalgebra of $g$. Any parabolic subalgebra of $g$ is conjugated to a standard parabolic subalgebra. The Lie subalgebra

$$u_S = \bigoplus_{\alpha \in R^+ \setminus R_S} g_\alpha$$

is the nilpotent radical of $p_S$.

For $S \subset \Pi$, denote by $V_S$ the vector subspace of $\mathfrak{h}^*$ spanned by the elements of $R(S)$ and $R(\Pi)$. Set

$$E_S = \{K \in \mathcal{K}(\Pi); X_{\varepsilon_K} \in u_S\} = \{K \in \mathcal{K}(\Pi); \varepsilon_K \notin R_S\}$$

and $Q_S = \left( \bigcup_{K \in E_S} \Gamma_K \right) \cap R^+_S$.

The subset $E_S$ has the following simple characterisation.
Lemma 2.1. Let $T_S$ be the union of the subsets $K \in \mathcal{K}(\Pi)$ verifying $K \subset S$. Then $E_S = \mathcal{K}(\Pi) \setminus \mathcal{K}(T_S)$.

Proof. This is straightforward. \hfill \square

Our main result is the following theorem.

Theorem 2.2. Let $S \subset \Pi$. Then

\[(3) \quad \chi(p_S) + \chi(u_S) = \text{rk} \mathfrak{g} + \sharp \mathcal{K}(S) - \sharp \mathcal{K}(T_S) + 2(\sharp \mathcal{K}(\Pi) - \dim V_S) + \sharp \mathcal{Q}_S.\]

We have $\chi(p_S) + \chi(u_S) \geq \text{rk} \mathfrak{g}$, and the equality holds if and only if the following conditions are satisfied:

i) $\sharp (\mathcal{K}(S) \cup \mathcal{K}(\Pi)) = \dim V_S$.

ii) For any connected component $S'$ of $S$, we have either $S' \in \mathcal{K}(\Pi)$ or $\sharp (S' \setminus T_S) = 1$.

Proof. We may clearly assume that $\mathfrak{g}$ is simple.

The formula for the sum $\chi(p_S) + \chi(u_S)$ is a direct consequence of the formula of the index of parabolic subalgebras conjectured in [13] and proved in [3, 6]:

\[(4) \quad \chi(p_S) = \text{rk} \mathfrak{g} + \sharp \mathcal{K}(\Pi) + \sharp \mathcal{K}(S) - 2 \dim V_S,\]

and the formula for the index of $u_S$ (see for example [14, Chapter 40]), which, in view of Lemma 1.1, can be expressed in the following way:

\[(5) \quad \chi(u_S) = \sharp E_S + \sum_{K \in E_S} \sharp \Gamma^K - \dim u_S = \sharp E_S + \sharp \mathcal{Q}_S.\]

Observe that

\[(6) \quad \sharp \mathcal{K}(S) - \sharp \mathcal{K}(T_S) \geq 0.\]

1) Let $S_1, \ldots, S_r$ be the connected components of $S$. For each $i$, there is a unique $K_i \in \mathcal{K}(\Pi)$ (see Lemma 1.1) such that $\varepsilon_{S_i} \in \Gamma^{K_i}$. If $K_i = S_i$, then $S_i$ is a connected component of $T_S$. Otherwise $K_i \in E_S$, and we have

$\varepsilon_{S_i} \in \Gamma^{K_i} \cap \Gamma^{S_i} \subset \mathcal{Q}_S$.

2) It follows from Point 1) that $\mathcal{Q}_S = \emptyset$ if and only if $\mathcal{K}(S) \subset \mathcal{K}(\Pi)$ (or equivalently $S = T_S$).

3) Note that the connected components of $T_S$ are the connected components of $T_S \cap S_i$. It follows again from Point 1) that

$$\sharp \mathcal{K}(S) - \sharp \mathcal{K}(T_S) = \sum_{i=1}^{r} (\sharp \mathcal{K}(S_i) - \sharp \mathcal{K}(T_S \cap S_i))$$

$$= \sum_{K_i \in E_S} (\sharp \mathcal{K}(S_i) - \sharp \mathcal{K}(T_S \cap S_i)).$$

4) From Table 1, we have $\dim V_S = \sharp \mathcal{K}(\Pi) = \text{rk} \mathfrak{g}$ in the cases where $\mathfrak{g}$ is of type $B \ell, C \ell, D_{2n}, E_7, E_8, F_4$ and $G_2$. The inequality follows immediately from (3) and (6), and by Point 2) the equality holds if and only if $\mathcal{K}(S) \subset \mathcal{K}(\Pi)$.

On the other hand, since $\dim V_S = \sharp \mathcal{K}(\Pi)$ in these cases, condition i) is equivalent to $\mathcal{K}(S) \subset \mathcal{K}(\Pi)$. Finally, if condition i) is verified, then $S = T_S$, and condition ii) is automatically verified. So we have the result in these cases.

5) Type $A \ell$.\hfill \square
For any \( i \) verifying \( S_i \neq K_i \), by Point 1), Lemma 1.1 and Examples 1.2, half of \( \Gamma^S \setminus \{ \varepsilon_S \} \) belongs to \( Q_S \). Since \( \sharp(\Gamma^S) = 2\sharp(S) - 1 \) (Examples 1.2), such an \( S_i \) contributes \( \sharp(S_i) \) elements of \( Q_S \).

Again, since we are in type \( A_\ell \), \( K(\Pi) \) is totally ordered by inclusion. It follows that \( T_S \) is connected. Without loss of generality, we may assume that \( T_S \subset S_1 \).

Suppose that \( S_1 = T_S \). Then from the previous discussion, we deduce that

\[
\chi(p_S) + \chi(u_S) \geq \text{rk } g + \sharp(K(S \setminus S_1)) + 2(\sharp(K(\Pi)) - \dim V_S) + \sharp(S \setminus S_1).
\]

But our hypothesis implies that

\[
\dim V_S \leq \sharp(K(\Pi)) + \sharp(K(S \setminus S_1)) = \sharp(K(\Pi) \cup K(S)),
\]

so

\[
\chi(p_S) + \chi(u_S) \geq \text{rk } g + \sharp(S \setminus S_1) - \sharp(K(S \setminus S_1)).
\]

Hence \( \chi(p_S) + \chi(u_S) \geq \text{rk } g \). For the equality to hold, we must have the equality in (7) and

\[
\sharp(S \setminus S_1) = \sharp(K(S \setminus S_1)).
\]

This latter is only possible if \( \sharp(S_i) = 1 \) for \( i \geq 2 \), so we have conditions (i) and (ii). Conversely, suppose that conditions (i) and (ii) are verified; then \( \sharp(S_i) = 1 \) for \( i \geq 2 \). Consequently \( Q_S = S \setminus S_1 \) by Point 1) and the definition of \( Q_S \).

Suppose that \( S_1 \supseteq T_S \) (this includes the case \( T_S = \emptyset \)). Then \( K(S) \cap K(\Pi) = \emptyset \). Thus

\[
\dim V_S \leq \sharp(K(\Pi)) + \sharp(K(S)) = \sharp(K(\Pi) \cup K(S)).
\]

We deduce from Point 1) and the remark in the first paragraph of Point 5) that

\[
\chi(p_S) + \chi(u_S) \geq \text{rk } g + \sharp(K(S)) - \sharp(K(T_S)) + 2(\sharp(K(\Pi)) - \dim V_S) + \sharp(S) \geq \text{rk } g + \sharp(S) - \sharp(K(S)) - \sharp(K(T_S)).
\]

Hence

\[
\chi(p_S) + \chi(u_S) \geq \text{rk } g + \sum_{i=1}^{r} \left( \sharp(S_i) - \left\lfloor \frac{\sharp(S_i) + 1}{2} \right\rfloor \right) - \left\lfloor \frac{\sharp(T_S) + 1}{2} \right\rfloor.
\]

Since \( T_S \subset S_1 \), we deduce from Table 1 that

\[
\sharp(S_1) - \left\lfloor \frac{\sharp(S_1) + 1}{2} \right\rfloor - \left\lfloor \frac{\sharp(T_S) + 1}{2} \right\rfloor \geq 0.
\]

So we have our inequality \( \chi(p_S) + \chi(u_S) \geq \text{rk } g \).

Now for the equality \( \chi(p_S) + \chi(u_S) = \text{rk } g \) to hold, we must have the equality in (8) and \( \sharp Q_S = \sharp(S) \),

\[
\sharp(S_1) - \left\lfloor \frac{\sharp(S_1) + 1}{2} \right\rfloor - \left\lfloor \frac{\sharp(T_S) + 1}{2} \right\rfloor = 0 \text{ and } \sharp(S_i) - \left\lfloor \frac{\sharp(S_i) + 1}{2} \right\rfloor = 0
\]

for \( i \geq 2 \). This implies that \( \sharp(S_1) = \sharp(T_S) + 1 \), and \( \sharp(S_i) = 1 \) for \( i \geq 2 \). So we have conditions (i) and (ii). Conversely, if conditions (i) and (ii) are verified, then \( \sharp(S_i) = 1 \) for \( i \geq 2 \), and \( \sharp(S_1) = \sharp(T_S) + 1 \). In view of the above, to show that \( \chi(p_S) + \chi(u_S) = \text{rk } g \), it suffices to check that \( \sharp(Q_S \cap R^+_{S_1}) = \sharp(S_1) \), which is a straightforward verification.

6) Type \( D_{2n+1} \).

In this case, \( \sharp K(\Pi) = \text{rk } g - 1 \). Let us use the numbering of simple roots in [14, Chapter 18]. We check easily that \( \alpha_1, \ldots, \alpha_{\ell-2} \in V_{\Pi} \).
Table 2

<table>
<thead>
<tr>
<th>$S_1$</th>
<th>$\dim V_{S_1}$</th>
<th>$T_S \cap S_1$</th>
<th>$\sharp \mathcal{K}(S_1)$</th>
<th>$\sharp \mathcal{K}(T_S \cap S_1)$</th>
<th>$K_1$</th>
<th>$\sharp(\Gamma^{K_1} \cap \Gamma^{S_1})$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$A_1$</td>
<td>5</td>
<td>$\emptyset$</td>
<td>1</td>
<td>0</td>
<td>$A_3$ or $A_5$</td>
<td>1</td>
</tr>
<tr>
<td>$A_2$</td>
<td>5</td>
<td>$\emptyset$</td>
<td>1</td>
<td>0</td>
<td>$A_5$</td>
<td>2</td>
</tr>
<tr>
<td>$A_3$</td>
<td>6</td>
<td>$A_1$</td>
<td>1</td>
<td>1</td>
<td>$A_3$</td>
<td>2</td>
</tr>
<tr>
<td>$A_4$</td>
<td>5</td>
<td>$A_1$</td>
<td>2</td>
<td>1</td>
<td>$A_5$</td>
<td>3</td>
</tr>
<tr>
<td>$A_5$</td>
<td>6</td>
<td>$A_1$</td>
<td>2</td>
<td>1</td>
<td>$E_6$</td>
<td>3</td>
</tr>
<tr>
<td>$A_6$</td>
<td>5</td>
<td>$A_3$</td>
<td>4</td>
<td>2</td>
<td>$E_6$</td>
<td>4</td>
</tr>
<tr>
<td>$D_4$</td>
<td>5</td>
<td>$A_3$</td>
<td>4</td>
<td>2</td>
<td>$E_6$</td>
<td>4</td>
</tr>
<tr>
<td>$D_5$</td>
<td>6</td>
<td>$A_3$</td>
<td>4</td>
<td>2</td>
<td>$E_6$</td>
<td>9</td>
</tr>
</tbody>
</table>

If $\dim V_S = \sharp \mathcal{K}(\Pi)$, then the inequality follows from (3) and (6), and the condition for equality follows from Point 2).

Suppose now that $\dim V_S = \text{rk } g$ and $\alpha_{\ell-1} \in S$ (the case $\alpha_\ell \in S$ being analogous). Then

$$\chi(p_S) + \chi(u_S) = \text{rk } g + \sharp \mathcal{K}(S) - \sharp \mathcal{K}(T_S) - 2 + \sharp(Q_S),$$

and the connected component $S_1$ of $S$ containing $\alpha_{\ell-1}$ is not in $\mathcal{K}(\Pi)$; otherwise, we would have $\dim V_S = \sharp \mathcal{K}(\Pi)$.

By Point 1), $e_{S_1} \in Q_S$. By examining the possibilities for $S_1$ and $K_1$ (Examples 1.2), we verify that

$$\sharp \mathcal{K}(S_1) - \sharp \mathcal{K}(T_S \cap S_1) + \sharp(\Gamma^{K_1} \cap \Gamma^{S_1}) \geq 2$$

with equality if and only if $S_1$ is of type $A_1$ or $A_2$. Therefore, we have obtained the inequality.

In fact, we showed in the previous paragraph that already we have

$$\text{rk } g + \sharp \mathcal{K}(S_1) - \sharp \mathcal{K}(T_S \cap S_1) - 2 + \sharp(\Gamma^{K_1} \cap \Gamma^{S_1}) \geq \text{rk } g.$$ 

So if $\chi(p_S) + \chi(u_S) = \text{rk } g$, then from (9) and the above inequality, we must have $\mathcal{K}(S \setminus S_1) \subset \mathcal{K}(\Pi)$, and also the equality in (10). Hence conditions (i) and (ii). Conversely, suppose that conditions (i) and (ii) are verified; then the fact that $\alpha_1, \ldots, \alpha_{\ell-2} \in V_\Pi$ implies that $\mathcal{K}(S \setminus S_1) \subset \mathcal{K}(\Pi)$ and $\sharp \mathcal{K}(S_1) = 1$. Hence $S_1$ is of type $A_1$, $A_2$. It is then easy to check that $\chi(p_S) + \chi(u_S) = \text{rk } g$.

7) Type $E_6$.

Here, we have $\sharp \mathcal{K}(\Pi) = 4$ and $\alpha_2, \alpha_4 \in V_\Pi$. Let $S_1$ be a connected component of $S$ such that $\dim V_{S_1} > 4$. Under these conditions, the possibilities are shown in Table 2. Thus,

$$\sharp \mathcal{K}(S_1) - \sharp \mathcal{K}(T_S \cap S_1) + \sharp(\Gamma^{S_1} \cap \Gamma^{K_1}) \geq 2(\dim V_S - \sharp \mathcal{K}(\Pi)).$$

A direct verification gives the result. Note that as in the case of type $A_\ell$, $\mathcal{K}(\Pi)$ is totally ordered by inclusion, so $T_S$ is connected.

**Remark 2.3.** Theorem 2.2 says that if $\mathcal{K}(S) \subset \mathcal{K}(\Pi)$ or, equivalently, $S' \in \mathcal{K}(\Pi)$ for any connected component $S'$ of $S$, then $\chi(p_S) + \chi(u_S) = \text{rk } g$. 

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Example 2.4. Let us consider the case of minimal parabolic subalgebras. So $S = \{\alpha\}$ and it follows that

$$\chi(p) + \chi(u) = \begin{cases} 
\text{rk} \, g & \text{if } \{\alpha\} \in K(\Pi), \\
\text{rk} \, g & \text{if } \{\alpha\} \not\subset K(\Pi) \text{ and } \dim V_S = \sharp K(\Pi) + 1, \\
\text{rk} \, g + 2 & \text{if } \{\alpha\} \not\subset K(\Pi) \text{ and } \dim V_S = \sharp K(\Pi).
\end{cases}$$

Thus the minimal parabolic subalgebras $p_S$ verifying $\chi(p_S) + \chi(u_S) = \text{rk} \, g$ are (in the simple roots numbering of [14, Chapter 18]) as shown in Table 3.

Example 2.5. In the other extreme, it is easy to check that maximal parabolic subalgebras of $g = \mathfrak{sl}_{\ell+1}$ verifying $\chi(p) + \chi(u) = \text{rk} \, g$ are exactly the ones associated to simple roots at the extremities of the Dynkin diagram.

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References


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