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ALL DIHEDRAL DIVISION ALGEBRAS OF DEGREE FIVE ARE CYCLIC

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Abstract. In 1982 Rowen and Saltman proved that every division algebra which is split by a dihedral extension of degree $2n$ of the center, n odd, is in fact cyclic. The proof requires roots of unity of order n in the center. We show that for $n = 5$, this assumption can be removed. It then follows that ${}_5\mathrm{Br}(F)$, the 5-torsion part of the Brauer group, is generated by cyclic algebras, generalizing a result of Merkurjev (1983) on the 2 and 3 torsion parts.

1. Mathematical background

We begin with basic notions needed for this work and refer the reader to [8] or [5] for more details. Let R be a ring and let $\mathrm{C}(R) = \{r \in R \mid rx = xr \ \ \forall x \in R\}$ denote the center of R.

Definition 1.1. A ring R will be called a simple ring if R has no non-trivial two-sided ideals. In particular R is a division ring if every non-zero element is invertible.

Remark 1.2. Notice that if R is simple, its center is naturally a field.

Definition 1.3. An F-algebra R is called an F-central simple algebra if R is simple with $\mathrm{C}(R) = F$ and $\dim_F(R) < \infty$.

Remark 1.4. Every F-central simple algebra A has $\dim_F(A) = n^2$, and we define the degree of A, denoted $\deg(A)$, to be n.

By Wedderburn's Theorem every F-central simple algebra is of the form $M_n(D)$, where D is a division algebra with center F.

The Brauer group of a field F, denoted $\mathrm{Br}(F)$, is the set of isomorphism classes of F-central simple algebras modulo the following relation: two central simple algebras A, B are equivalent if and only if there exist natural numbers n, m such that $M_n(A) \cong M_m(B)$.

Proposition 1.5. *Let D be an F-central division algebra of degree n, and K a subfield of D; then K is a maximal subfield if and only if $[K : F] = n$.*

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Definition 1.6. A crossed product is an F-central simple algebra A of degree n containing a commutative F-subalgebra C Galois over F, with $[C : F] = n$. Note that if A is a division algebra, then C is a maximal subfield of A.

Definition 1.7. Let D be an F-central division algebra of degree n. We will say that D is split by a group G if D contains a maximal subfield K with Galois closure E such that $\mathrm{Gal}(E/F) = G$.

Theorem 1.8. *Let A be a crossed product where $K \subset A$ is a maximal subfield with Galois group $\mathrm{Gal}(K/F) = G$. Then A has the following description: $A = \bigoplus_{\sigma\in G} Kx_\sigma$ as a left K-vector space, and multiplication in A is according to the rules:*

$$x_\sigma k = \sigma(k)x_\sigma \quad \forall k \in K$$

and

$$x_\sigma x_\tau = c(\sigma, \tau)x_\tau x_\sigma$$

where $c \in \mathrm{H}^2(G, K^\times)$ is a 2-cocycle. In this case A is denoted $A = (K, G, c)$.

Remark 1.9. If $G = \langle\sigma\rangle$ we can give a simpler representation of A as follows: $A = \bigoplus_{i=0}^{n-1} Kx^i$ as a left K-vector space, where $n = \deg(A) = |G|$ and the multiplication is according to the rules

$$xk = \sigma(k)x \quad \forall k \in K$$

and

$$x^i x^j = \begin{cases} x^{i+j}, & i + j < n, \\ \beta x^{i+j-n}, & i + j \geq n. \end{cases}$$

In this case, A is denoted as $A = (K, \sigma, \beta)$.

Remark 1.10. If F contains a primitive n-th root of unity ρ, we can give an even simpler description of A (since then $K = F[x \mid x^n = \alpha \in F]$) as follows:

$$A = F[x, y \mid x^n = \alpha; y^n = \beta; xy = \rho_n yx] \quad \alpha, \beta \in F.$$

2. Some preliminary results

In this section we briefly repeat the arguments of Rowen and Saltman in [9] but we do not assume F contains roots of unity.

The situation we will be handling is the following: D/F is a central simple algebra of odd degree n having a maximal subfield $K \subset D$ with Galois closure $E \supset K \supset F$, such that

$$\mathrm{Gal}(E/F) = D_n = \left\langle \sigma, \tau : \sigma^n = \tau^2 = 1 \ , \tau\sigma\tau = \sigma^{-1} \right\rangle$$

and $K = E^{\langle\tau\rangle}$.

Extending scalars to $E^{\langle\sigma\rangle}$, we may view $E \subset D' = D \otimes E^{\langle\sigma\rangle}$. Now $\mathrm{Gal}(E/E^{\langle\sigma\rangle}) = \langle\sigma\rangle$, i.e. D' is cyclic, so we have an element $\beta \in D'$ such that

$$\beta^{-1}x\beta = \sigma(x) \quad \forall x \in E. \tag{1}$$

In particular $\beta^n \in E^{\langle\sigma\rangle}$. Notice that τ can be extended to $D' = D \otimes E^{\langle\sigma\rangle}$ by its action on $E^{\langle\sigma\rangle}$, that is, we write τ instead of $1 \otimes \tau$.

Lemma 2.1. *We may assume that $\tau(\beta) = \beta^{-1}$.*

Proof. Applying τ to (1) yields

$$\tau(\beta)^{-1}\tau(x)\tau(\beta) = \sigma^{-1}(\tau(x)) \quad \forall x \in E.$$

Now since τ is an automorphism of E, $\tau(x)$ runs over all elements of E, and thus

$$\tau(\beta)^{-1}y\tau(\beta) = \sigma^{-1}(y) \quad \forall y \in E;$$

that is $\tau(\beta)$ acts on E as σ^{-1}. Now define $\beta' = \beta^r\tau(\beta)^{-r}$, where $r = (n+1)/2$, and compute that $\tau(\beta') = \beta'^{-1}$, and β' acts on E as σ. □

Let $P_t(X) = X^n + \sum_{i=1}^n c_i(t)X^{n-i}$ denote the characteristic polynomial of $t \in D'$. Note that $c_1(t) = -\operatorname{tr}(t)$ and $c_n(t) = (-1)^n \operatorname{N}(t)$ where $tr(t)$ and $N(t)$ are the reduced trace and norm of t.

Lemma 2.2. *Let $t = \beta^i e$, for $e \in E$ and $0 < i < n$, $i \neq 0$. Then $tr(t) = 0$.*

Proof. Let $d = \gcd(i, n)$. Clearly we have $t^{n/d} = \beta^{ni/d}\operatorname{N}_{\sigma^i}(e) \in E^{\langle\sigma^i\rangle}$ where $\operatorname{N}_{\sigma^i}$ is the norm from E to $E^{\langle\sigma^i\rangle}$. Now $[E : E^{\langle\sigma^i\rangle}] = n/d$, implying $P(X) = X^{n/d} - \beta^{ni/d}\operatorname{N}_{\sigma^i}(e)$ is the characteristic polynomial of t, hence $\operatorname{tr}_{E/E^{\langle\sigma^i\rangle}}(t) = 0$ which implies $\operatorname{tr}_{E/F}(t) = 0$. □

Lemma 2.3. *Let $t = (\beta + \beta^{-1})e$ for $e \in E$. Then the coefficients of $P_t(X)$ satisfy $c_i(t) = 0$ for every odd $0 < i < n$.*

Proof. Notice that for i odd, t^i is a sum of elements of the form $a\beta^s$ where $a \in E$ and s odd, $-n < s < n$, so by Lemma 2.2 and Newton's identities we are done in the characteristic zero case. For the general case, we refer the reader to [9] where the main idea is that you can form a model for this situation in the form of an Azumaya algebra and then use a specialization argument. □

Corollary 2.4. *There is an element $t \in D$ such that for every $e \in E$ (and so also for $k \in K \subset E$), $c_i = 0$ for every odd $0 < i < n$ in $P_{te}(X)$.*

Proof. Since $D = D'^{\langle\tau\rangle}$ we have $t = \beta + \beta^{-1}$ is the desired element. □

Remark 2.5. Notice that if $n = p$ is prime $\operatorname{Char}(F) = p$, the element $t = \beta + \beta^{-1} \in D$ we found satisfies $t^p \in F$ and $t \notin F$, and so by a theorem of Albert in the "special results" chapter of his seminal book [1], which is known as Albert's cyclicity criterion, D is cyclic (this is not a new result, as J.P Tignol and P. Mammone did this for any field F with $\operatorname{Char}(F) \mid n$ in [6] using the corestriction, but it shows that the proof of Rowen and Saltman also applies to this case).

3. The case $n = 5$

Now we would like to focus on the particular case where $n = 5$. The main tool we will be using is the following proposition taken from [3, Proposition 2.2].

Proposition 3.1. *Let $G(x_1, ..., x_n)$ be a homogeneous form of degree 3 defined over a field F. If G has a solution, $\alpha \in K^{(n)}$, defined over a quadratic extension K of F, then G has a solution defined over F.*

Proof. The proof in [2] uses basic intersection theory which we will not use; instead we will give an algebraic proof (which is actually a translation of the proof in [2]) which will enable us to find an explicit solution in section 3. Since $[K : F] = 2$ the solution α has the following form: $\alpha = (\alpha_1 + \beta_1 t, ..., \alpha_n + \beta_n t)$ where $\alpha_i, \beta_i \in k$, and

$t \in K$ such that $K = F[t]$. Now specialize $G(x_1, ..., x_n)$ to $G(\alpha_1+\beta_1 Z, ..., \alpha_n+\beta_n Z)$, denoting it by $g(Z)$. Notice that the coefficient of Z^3 in $g(Z)$ is $G(\beta_1, ..., \beta_n)$; hence if $G(\beta_1, ..., \beta_n) = 0$ we have a solution defined over F, otherwise $g(Z)$ is a degree 3 polynomial defined over F. Since $g(t) = 0$ we get that $g(Z) = cm_t(Z)(Z-w)$, where $c = G(\beta_1, ..., \beta_n)$ and $m_t(Z)$ is the minimal polynomial of t over F. Now c, $g(Z)$ and $m_t(Z)$ are defined over F; hence w is in F and clearly $G(\alpha_1 + \beta_1 w, ..., \alpha_n + \beta_n w) = g(w) = 0$, so we have found a solution $\gamma = (\alpha_1 + \beta_1 w, ..., \alpha_n + \beta_n w) \in F^n$. □

Theorem 3.2. *Let D be a division algebra of degree 5 split by the group D_5; then D is cyclic.*

Proof. In view of Remark 2.5, we may assume $\mathrm{Char}(F) \neq 5$. First we remark that by Albert's cyclicity criterion it is enough to find an element $t \in D - F$ such that $t^5 \in F$, that is, $c_i = 0$ for every $0 < i < n$. Now by Corollary 2.4 we have $t \in D$ with the property $c_i(te) = 0$ for every odd $0 < i < n$ and $\forall e \in E$. Now since $P_{t^{-1}}(x) = -N(t)^{-1}P_t(x^{-1})x^5$ we have $c_i(et^{-1}) = 0$ for every even $0 < i < n$ and $\forall e \in E$. Hence we are left with finding a solution for $c_1(et^{-1}) = 0$ (which is linear) and $c_3(et^{-1}) = 0$ (which is cubic) in the five dimensional vector space Et^{-1}. Define $V := \{et^{-1} \in Et^{-1} \mid c_1(et^{-1}) = 0\}$, which is a four dimensional subspace of Et^{-1}. We have to find a solution for $c_3(v) = 0$ in V. Let us add a fifth root of unity to F, which is either a quadratic extension or a chain of two quadratic extensions. After this extension we are in the case of Rowen and Saltman where they gave an explicit element whose fifth power is in F which was $(v + v^{-1})t^{-1}$, where $v \in E$. This element is clearly in $V \otimes_F F[\rho_5]$. Now by Proposition 3.1 since $c_3(v)$ is homogeneous of degree 3, we have a solution after either one or two quadratic extensions. Thus, we have a solution before the extension, and we are done. □

Remark 3.3. If the fifth root of unity is in a quadratic extension of F, we know D is cyclic by a theorem of Vishne [10, Theorem 13.6] and D. Haile, M. A. Knus, M. Rost, J. P. Tignol [4], so that what actually is new is the last case of $[F[\rho] : F] = 4$.

4. A generic example

Fixing p let $K = F[\rho_p]$ and denote $\mathrm{Gal}(K/F) = \langle\tau\rangle$. In [7, Theorem 2] Merkurjev proves that ${}_p\mathrm{Br}(F)$ is generated by F-central simple algebras, A, of degree p such that $A \otimes K \simeq (\alpha, \beta)$ where $K[\sqrt[p]{\alpha}]$ is cyclic over K Galois over F.

In [10] Vishne calls these algebras quasi-symbols and gives more details about them, including generic examples. We will show that for $p = 5$ these algebras are cyclic and conclude that ${}_5\mathrm{Br}(F)$ is generated by cyclic algebras.

4.1. **A generic quasi-symbol of degree 5.** For $p = 5$ we have two possibilities for $[K : F]$. The first is $[K : F] = 2$; in this case Vishne shows that every quasi-symbol is cyclic. The second case is $[K : F] = 4$; in this case every quasi-symbol A has one of the following forms (after extending scalars to K):

(1) $A \otimes K = (\alpha, \beta)$, where $\alpha \in F$ and $\tau(\beta) \equiv \beta^2 \pmod{K^{\times 5}}$.
(2) $A \otimes K = (\alpha, \beta)$, where $\tau(\alpha) = \alpha^{-1}$ and $\tau(\beta) \equiv \beta^{-2} \pmod{K^{\times 5}}$.

The first kind is known to be cyclic by [10, Theorem 10.3]. So we are left with the second kind for which Vishne gives the following generic construction which we will show is cyclic. Thus every quasi-symbol of degree 5 is cyclic and hence, by [7, Theorem 2] we conclude that ${}_5\mathrm{Br}(F)$ is generated by cyclic algebras.

Let k_0 be a field of characteristic $\neq 5$ and $k = k_0[\rho]$ where ρ is a fixed primitive fifth root of unity, and $\mathrm{Gal}(k/k_0) = \langle\tau\rangle$ where $\tau(\rho) = \rho^2$. Set $K = k(a, b, \eta)$ a transcendental extension and extend τ to K by

$$\tau(a) = a^{-1}, \qquad \tau(b) = \eta^5 b^{-2}, \qquad \tau(\eta) = \eta^2 b^{-1}.$$

Notice that we still have $\tau^5 = 1$. Define $F = K^{\langle\tau\rangle}$ and

$$D = (a, b)_K = K[x, y \mid x^5 = a, \quad y^5 = b, \quad yxy^{-1} = \rho x],$$

and extend τ to D by $\tau(x) = x^{-1}, \quad \tau(y) = \eta y^{-2}$. Notice that $\tau^2(\eta) = \eta^{-1}$ and $\tau^2(y) = y^{-1}$. Now define $D_0 = D^{\langle\tau\rangle}$; D_0/F is the generic quasi-symbol of degree 5 of the second type.

Remark 4.1. Vishne's construction is much more general and we specialized it to the above case; for the general construction we refer the reader to [10].

Proposition 4.2. *D_0 is split by D_5.*

Proof. Notice that $\mathrm{Gal}(K[y]/F) = C_5 \rtimes C_4 = \langle\sigma\rangle \rtimes \langle\tau\rangle$ and now we will see how τ acts on σ. Applying τ to $x^{-1}tx = \sigma(t)$, which holds for every $t \in K[y]$, yields $\tau(\sigma(t)) = \tau(x^{-1})\tau(t)\tau(x) = x\tau(t)x^{-1} = \sigma^{-1}(\tau(t))$ and so we get $\tau\sigma\tau^{-1} = \sigma^{-1}$. Hence τ^2 is a central element in $\mathrm{Gal}(K[y]/F)$ and it is clear that $E = K[y]^{\langle\tau^2\rangle} \subset K[y]$ is Galois over F with $\mathrm{Gal}(E/F) = D_5 = \langle\sigma\rangle \rtimes \langle\tau\rangle$, and we are done. □

Corollary 4.3. *D_0 is cyclic.*

In [7] Merkurjev proves the following theorem:

Theorem 4.4. *Let F be a field. ${}_n\mathrm{Br}(F)$ is generated by cyclic algebras, for $n = 2, 3$.*

Now as a result of the above we can extend Merkurjev's theorem to $n = 5$ and get

Theorem 4.5. *${}_5\mathrm{Br}(F)$ is generated by cyclic algebras.*

Proof. By section 8 of [10] ${}_5\mathrm{Br}(F)$ is generated by quasi-symbols of degree 5, and so we are done. □

4.2. **Finding an explicit solution.** Since the above example is a generic one, it would be nice to give an explicit element with fifth power in F, which is what we do now by going over the general proof.

Let $P_t(X) = X^n + \sum_{i=1}^n c_i X^{n-i}$ denote the characteristic polynomial of $t \in D_0$. $V = (x + x^{-1})^{-1}K[y]^{\langle\tau\rangle}$ is a 5-dimensional F-subspace of D_0, satisfying $c_2(v) = c_4(v) = 0$ for all $v \in V$. Also, we want to find a solution in V for $tr(Z) = c_1((x+x^{-1})^{-1}Z) = 0$ and $G(Z) = c_3((x+x^{-1})^{-1}Z) = 0$. Extending scalars from F to $F[\rho+\rho^{-1}]$, we have the solutions $Z_1 = y+y^{-1} = \alpha+\beta(\rho+\rho^{-1})$ and $Z_2 = \tau(Z_1) = \alpha+\beta\tau(\rho+\rho^{-1}) = \alpha+\beta\tau(\rho^2+\rho^{-2})$ where $\alpha = (\alpha_1, ..., \alpha_5), \beta = (\beta_1, ..., \beta_5) \in K[y]^{\langle\tau\rangle}$ so $\alpha_i, \beta_i \in F$. Now define the following line: $L = \{\alpha+\beta t\} = \{(\alpha_1+\beta_1 t, ..., \alpha_5+\beta_5 t)\}$ defined over F.

Proposition 4.6. *For every $l \in L$ we have $tr(l) = 0$.*

Proof. By standard linear algebra, $L \cap \{tr(Z) = 0\}$ is either one point or the whole line L. Since $Z_1, Z_2 \in L \cap \{tr(Z) = 0\}$, we get $L \cap \{tr(Z) = 0\} = L$ and we are done. □

Now let us study the variety $\{G(Z)=0\}\cap L$. First we need to compute $G(Z)$. In order to do that we use the representation of D induced by right multiplication on $D=K[y]+K[y]x+K[y]x^2+K[y]x^3+K[y]x^4$, namely

$$x \longrightarrow \begin{pmatrix} 0 & 0 & 0 & 0 & a \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix},$$

$$m \in K[y] \longrightarrow \mathrm{Diag}(m, \sigma(m), \sigma^2(m), \sigma^3(m), \sigma^4(m)).$$

Now the minimal polynomial of $x+x^{-1}$ is

$$\lambda^5 - 5\lambda^3 + 5\lambda - (a+a^{-1});$$

hence

$$\begin{aligned}(x+x^{-1})^{-1} &= ((x+x^{-1})^4 - 5(x+x^{-1})^2 + 5)(a+a^{-1})^{-1} \\ &= (a+a^{-1})^{-1}(x^4+x^{-4}-x^2-x^{-2}+1)\end{aligned}$$

implying

$$(x+x^{-1})^{-1} \longrightarrow (a+a^{-1})^{-1}\begin{pmatrix} 1 & a & -1 & -a & 1 \\ a^{-1} & 1 & a & -1 & -a \\ -1 & a^{-1} & 1 & a & -1 \\ -a^{-1} & -1 & a^{-1} & 1 & a \\ 1 & -a^{-1} & -1 & a^{-1} & 1 \end{pmatrix}.$$

Now when we compute the characteristic polynomial of $(x+x^{-1})^{-1}m$ we get that

$$\begin{aligned}c_3((x+x^{-1})^{-1}m) &= (a+a^{-1})^{-1}(m\sigma(m)\sigma^2(m) + \sigma(m)\sigma^2(m)\sigma^3(m) \\ &\quad + \sigma^2(m)\sigma^3(m)\sigma^4(m) + \sigma^3(m)\sigma^4(m)m + \sigma^4(m)m\sigma(m)) \\ &= (a+a^{-1})^{-1}\,\mathrm{tr}_\sigma(m\sigma(m)\sigma^2(m)),\end{aligned}$$

yielding $F(Z) = (a+a^{-1})^{-1}\,\mathrm{tr}_\sigma(Z\sigma(Z)\sigma^2(Z))$. Now clearly $\{F(Z)=0\}\cap L$ is defined over F by the polynomial

$$\begin{aligned}f(t) = F(\alpha+\beta t) &= (a+a^{-1})^{-1}\,\mathrm{tr}_\sigma(\alpha+\beta t)\sigma(\alpha+\beta t)\sigma^2(\alpha+\beta t)) \\ &= (a+a^{-1})^{-1}\,\mathrm{tr}_\sigma(\beta\sigma(\beta)\sigma^2(\beta)t^3 + \ldots) = F(\beta)t^3 + \ldots.\end{aligned}$$

But we know two solutions for $f(t)$, namely $t_1 = \rho+\rho^{-1}$ and $t_2 = \rho^2+\rho^{-2}$, so we get $f(t) = F(\beta)(t-t_1)(t-t_2)(t-t_3)$. Now since $f(t)$ and $F(\beta)(t-t_1)(t-t_2)$ are defined over F, we get $t_3 \in F$. Explicitly, $f(0) = -t_1t_2t_3F(\beta)$ implies $t_3 = \frac{-f(0)}{t_1t_2F(\beta)} = \frac{f(0)}{F(\beta)} = \frac{F(\alpha)}{F(\beta)}$ is in F. Hence we get:

Theorem 4.7. *The element* $w = (x+x^{-1})^{-1}(\alpha+\beta\frac{F(\alpha)}{F(\beta)}) \in D_0 - F$ *satisfies* $w^5 \in F$.

Now we are left with solving for α, β from the two equations

$$\begin{gathered} y + y^{-1} = \alpha + \beta(\rho+\rho^{-1}), \\ \eta y^{-2} + \eta^{-1}y^2 = \tau(y+y^{-1}) = \alpha + \beta(\rho^2+\rho^{-2}). \end{gathered}$$

Hence

$$\beta = \frac{y + y^{-1} - \eta y^{-2} - \eta^{-1} y^2}{\rho + \rho^{-1} - \rho^2 - \rho^{-2}},$$
$$\alpha = y + y^{-1} - \beta(\rho + \rho^{-1}).$$

4.3. **The general case.** We will now show that the above solution for the case of quasi-symbols, where we descend from $F[\rho + \rho^{-1}]$ to F, is valid for the general case of $D_5 = \left\langle \sigma, \tau : \sigma^5 = \tau^2 = 1, \tau\sigma\tau = \sigma^{-1} \right\rangle$ division algebras, where we need to descend from $F[\rho] \otimes E^{\langle\sigma\rangle}$ to F. The situation is the following: we look for a solution to $c_3(t) = c_1(t) = 0$ where $c_i(t)$ are as in Section 3 and $t \in (\beta + \beta^{-1})^{-1} E^{\langle\tau\rangle}$. Let $\mathrm{Gal}(F[\rho]/F) = \langle\pi\rangle$; hence $\mathrm{Gal}(E \otimes F[\rho]/F) = D_5 \times \langle\pi\rangle$ and so after extending scalars to $F[\rho]$ we want a solution in $(\beta + \beta^{-1})^{-1}(E \otimes F[\rho])^{\langle\tau\rangle \times \langle\pi\rangle}$, which will then be defined over F.

Proposition 4.8. *We may assume $v + v^{-1} \in (E \otimes F[\rho])^{\langle\tau\rangle \times \langle\pi^2\rangle}$, for v as in the proof of Theorem* 3.2.

Proof. Since $v = x^r \tau(x)^{-r}$, where x is any eigenvector of σ with eigenvalue ρ, we may write $x = \sum_{i=0}^{4} \rho^{-i} \sigma^i(k)$ for $k \in E^{\langle\tau\rangle \times \langle\pi\rangle}$. Now $\tau(x) = \pi^2(x)$ and so $\tau(v) = \tau(x)^r x^{-r} = \pi^2(x)^r x^{-r} = \pi^2(x^r \pi^2(x)^{-r}) = \pi^2(x^r \tau(x)^{-r}) = \pi^2(v)$ implying $\tau(v + v^{-1}) = v + v^{-1}$, hence $v + v^{-1}$ is in $(E \otimes F[\rho])^{\langle\tau\rangle \times \langle\pi^2\rangle}$, as desired. □

Now it is clear that after extending scalars to $F[\rho + \rho^{-1}]$ we have the solution $(\beta + \beta^{-1})^{-1}(v + v^{-1})$, and so we are in the same situation as in the quasi-symbol case; hence the above solution is valid for the general case, too.

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