HYPONORMAL TOEPLITZ OPERATORS
AND ZEROS OF POLYNOMIALS

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Abstract. The problem of hyponormality for Toeplitz operators with (trigonometric) polynomial symbols is studied. We give a necessary and sufficient condition using the zeros of the analytic polynomial induced by the Fourier coefficients of the symbol.

Let \( L^p \) be the Lebesgue space on the unit circle \( T \) and let \( H^p \) be the corresponding Hardy space for \( 1 \leq p \leq \infty \). The Toeplitz operator \( T_\phi \) with symbol \( \phi \) in \( L^\infty \) is the operator on \( H^2 \) defined by \( T_\phi f = P(\phi f) \) for \( f \) in \( H^2 \), where \( P \) is the orthogonal projection from \( L^2 \) onto \( H^2 \). In this paper, we are interested in when \( T_\phi \) is hyponormal.

Two characterizations of the hyponormality of \( T_\phi \) are known as the following:

(I) Suppose \( \phi_1 \) and \( \phi_2 \) are functions in \( H^2 \) with \( \phi = \phi_1 + \overline{\phi}_2 \) in \( L^\infty \). Then \( T_\phi \) is hyponormal if and only if there exists a constant \( c \) and a function \( k \) in \( H^\infty \) with \( \|k\|_\infty \leq 1 \) such that \( \phi_2 = c + T_\phi \overline{k} \phi_1 \).

(II) \( T_\phi \) is hyponormal if and only if there exist two functions \( k \) and \( g \) in \( H^\infty \) such that \( \phi = k\overline{\phi} + g \) and \( \|k\|_\infty \leq 1 \).

The characterization (I) is due to Cowen \[1\]. Cowen \[1\] and Zhu \[6\] used this characterization. (II) is due to Nakazi-Takahashi \[4\] Lemma 1. It is easy to prove (II) if we use (I). Nakazi-Takahashi \[4\] and Hwang-Lee \[3\] used this one. Hwang-Lee \[3\] established an explicit and useful criterion using (II) when the symbol \( \phi \) is a trigonometric polynomial. Their criterion involves the zeros of an analytic polynomial induced by the Fourier coefficients of \( \phi \). On the other hand, Zhu \[6\] gave a characterization which is related to the coefficients of the analytic polynomial induced by the Fourier coefficients of \( \phi \), using (I) and a theorem of Schur \[5\]. In this paper, we give a necessary and sufficient condition which is related to the zeros of an analytic polynomial induced by the Fourier coefficient of \( \phi \), using (II) and the Carathéodory-Schur interpolation theorem (cf. \[3\]).
Theorem 1. Suppose \( \phi \) is a trigonometric polynomial such that
\[
\phi = z^\ell \prod_{j=1}^{t} (z - \alpha_j) \prod_{j=1}^{s} (z - \beta_j),
\]
where \( \ell \geq 1, \ |\alpha_j| < 1 \) and \( |\beta_j| \geq 1 \). When \( t = 0 \) or \( s = 0 \), we assume that
\[
\prod_{j=1}^{t} (z - \alpha_j) = 1 \quad \text{or} \quad \prod_{j=1}^{s} (1 - \beta_j) = 1.
\]
Let
\[
f = \prod_{j=1}^{t} \frac{z - \alpha_j}{1 - \alpha_j z} \quad \text{and} \quad h = \prod_{j=1}^{s} \frac{1 - \beta_j z}{z - \beta_j}.
\]
Then \( T_\phi \) is hyponormal if and only if \( 2\ell \leq t + s \) and there exists a solution \( a_0, \ldots, a_{\ell-1} \) of the linear system of equations
\[
f^{(i)}(0) = \sum_{j=0}^{i} (i-1)(i-2) \cdots (i-j+1)a_j h^{(i-j)}(0) \quad (0 \leq i \leq \ell - 1)
\]
for which the associated lower triangular Toeplitz matrix
\[
T(a_0, \ldots, a_{\ell-1}) = \begin{bmatrix}
a_0 & \cdots & 0 \\
a_1 & a_0 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
a_{\ell-1} & a_{\ell-2} & \cdots & a_0
\end{bmatrix}
\]
has \( ||T(a_0, \ldots, a_{\ell-1})|| \leq 1 \).

Proof. By the characterization (II), \( T_\phi \) is hyponormal if and only if there exists a function \( K \) in \( H^\infty \) with \( ||K||_\infty \leq 1 \) and a function \( g \) in \( H^\infty \) such that \( \phi = K\phi + g \). Hence \( T_\phi \) is hyponormal if and only if \( 2\ell \leq t + s \) by (1) of Corollary 5 in [4] and there exists a function \( K \) in \( H^\infty \) with \( ||K||_\infty \leq 1 \) and a function \( g \) in \( H^\infty \) such that
\[
z^\ell \prod_{j=1}^{t} (z - \alpha_j) \prod_{j=1}^{s} (z - \beta_j) = K z^\ell \prod_{j=1}^{t} (\bar{z} - \bar{\alpha}_j) \prod_{j=1}^{s} (\bar{z} - \bar{\beta}_j) + g.
\]
The above equality can be written as follows:
\[
f = K z^{2\ell(t+s)} h + z^\ell G,
\]
where \( G = g / \prod_{j=1}^{t} (1 - \bar{\alpha}_j z) \prod_{j=1}^{s} (1 - \bar{\beta}_j z) \). Since \( z^{(t+s)-2\ell} (f - z^\ell G) = Kh \),
\[
z^{(t+s)-2\ell} (f - z^\ell G) \prod_{j=1}^{s} (z - \beta_j) = K \prod_{j=1}^{s} (1 - \bar{\beta}_j z).
\]
This implies that \( K \) is divisible in \( H^\infty \) by \( z^{(t+s)-2\ell} \) because \( |\beta_j| \leq 1 \). Hence, if \( k = z^{2\ell(t+s)} K \), then \( k \) belongs to \( H^\infty \) and \( f = kh + z^\ell G \). Hence \( T_\phi \) is hyponormal if and only if \( 2\ell \leq t + s \) and there exists a function \( k \in H^\infty \) with \( ||k||_\infty \leq 1 \) such that
\[
f^{(i)}(0) = \sum_{j=0}^{i} s C_j k^{(j)}(0) h^{(i-j)}(0) \quad (0 \leq i \leq \ell - 1),
\]
Let \( C_j = i!/j!(i-j)! \). Put \( k = \sum_{j=0}^{\infty} a_j z^j \). Then \( k^{(j)}(0) = j! a_j \) and so
\[
f^{(i)}(0) = \sum_{j=0}^{i} (i-1)(i-2) \cdots (i-j+1) a_j h^{(i-j)}(0)
\]
for \( 0 \leq i \leq \ell - 1 \). Now the theorem follows from the Carathéodory-Schur interpolation theorem (cf. [2]).

In the characterization (II) of hyponormality, put \( \mathcal{E}(\phi) = \{k \in H^\infty : \phi = k\bar{\phi} + g, g \in H^\infty, \text{ and } ||k||_\infty \leq 1\}. \mathcal{E}(\phi) \) has been studied and it may contain more than two elements (see [5]). Hence the \( k \) in the proof of Theorem 1 may not be unique in general, and so \( (a_j)_{j=0}^{\infty} \) may not be unique.

By a result in the previous paper [4 Corollary 5], if \( \{1/\beta_j\}_{j=1}^{s} \subseteq \{\alpha_j\}_{j=1}^{t} \) (see Theorem 1), then \( T_\phi \) is hyponormal. Here we give a necessary and sufficient condition for hyponormality of \( T_\phi \) in terms of a relation between \( \{\alpha_j\}_{j=1}^{t} \) and \( \{\beta_j\}_{j=1}^{s} \) when \( \ell = 1 \) or 2.

**Corollary 1.** Let \( \ell = 1 \) in Theorem 1. Then \( T_\phi \) is hyponormal if and only if
\[
\prod_{j=1}^{t} |\alpha_j| \times \prod_{j=1}^{s} |\beta_j| \leq 1.
\]
When \( t = 0 \) or \( s = 0 \), we assume
\[
\prod_{j=1}^{t} |\alpha_j| = 1 \quad \text{or} \quad \prod_{j=1}^{s} |\beta_j| = 1.
\]

**Proof.** By Theorem 1 \( T_\phi \) is hyponormal if and only if \( f(0) = a_0 h(0) \) and \( |a_0| \leq 1 \).

**Corollary 2.** Let \( \ell = 2 \) in Theorem 1. Then \( T_\phi \) is hyponormal if and only if there exist constants \( a_0, a_1 \) such that \( |a_1| \leq 1 - |a_0|^2 \) and
\[
\prod_{j=1}^{t} |\alpha_j| \times \prod_{j=1}^{s} |\beta_j| \leq 1
\]
\[
\sum_{k=1}^{t} \left\{ (1 - |\alpha_k|^2) \prod_{j \neq k} (-\alpha_j) \right\} = a_0 \sum_{k=1}^{t} \left( \frac{|\beta_k|^2 - 1}{|\beta_k|^2} \right) \prod_{j \neq k} \left( -\frac{1}{\beta_j} \right) + a_1 \prod_{j=1}^{s} \left( -\frac{1}{\beta_j} \right).
\]
If \( s = 0 \), then
\[
\sum_{k=1}^{t} \left\{ (1 - |\alpha_k|^2) \prod_{j \neq k} (-\alpha_j) \right\} = 1
\]
and if \( t = 0 \), then there exist constants \( a_0, a_1 \) such that \( |a_1| \leq 1 - |a_0|^2 \) and
\[
a_0 \sum_{k=1}^{s} \left( \frac{|\beta_k|^2 - 1}{|\beta_k|^2} \right) \prod_{j \neq k} \left( -\frac{1}{\beta_j} \right) + a_1 \prod_{j=1}^{s} \left( -\frac{1}{\beta_j} \right) = 0.
\]

**Proof.** By Theorem 1 \( T_\phi \) is hyponormal if and only if \( f(0) = a_0 h(0) \), \( f'(0) = a_0 h'(0) + a_1 h''(0) \), and \( |a_1| \leq 1 - |a_0|^2 \).

**Corollary 3.** Let \( s = 0 \) in Theorem 1. Then \( T_\phi \) is hyponormal if and only if \( f^{(i)}(0) = a_i (0 \leq i \leq \ell - 1) \) and \( \|T(a_0, a_1, \ldots, a_{\ell-1})\| \leq 1 \).
Proof. By Theorem \(1\), \(T_\phi\) is hyponormal if and only if \(f^{(i)}(0) = a_0\) for \(i = 0, 1, 2\) and \(|T(a_0, 0, 0)| = |a_0| \leq 1\). □

Corollary 4. Let \(t = 0\) in Theorem \(1\). Then \(T_\phi\) is hyponormal if and only if

\[
1 = a_0 h(0), \sum_{j=0}^{i} a_j h^{(i-j)}(0) = 0 \quad (1 \leq i \leq \ell - 1) \quad \text{and} \quad \|T(a_0, a_1, \ldots, a_{\ell-1})\| \leq 1.
\]

Our corollaries are new and different from Examples 6 and 7 in \([6]\). The author of \([6]\) proved them under some condition \(a_2 \neq 0\) in Example 6. Of course, his result is not for zeros of a polynomial.

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