HUREWICZ SETS OF REALS WITHOUT PERFECT SUBSETS

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Abstract. We show that even for subsets $X$ of the real line that do not contain perfect sets, the Hurewicz property does not imply the property $S_1(\Gamma, \Gamma)$, asserting that for each countable family of open $\gamma$-covers of $X$, there is a choice function whose image is a $\gamma$-cover of $X$. This settles a problem of Just, Miller, Scheepers, and Szeptycki. Our main result also answers a question of Bartoszyński and the second author, and implies that for $C_p(X)$, the conjunction of Sakai’s strong countable fan tightness and the Reznichenko property does not imply Arhangel’skii’s property $\alpha_2$.

1. Introduction

By a set of reals we mean a separable, zero-dimensional, and metrizable space (such spaces are homeomorphic to subsets of the real line $\mathbb{R}$). Fix a set of reals $X$. Let $\mathcal{O}$ denote the collection of all open covers of $X$. An open cover $\mathcal{U}$ of $X$ is a $\gamma$-cover of $X$ if it is infinite and for each $x \in X$, $x$ is a member of all but finitely many members of $\mathcal{U}$. Let $\Gamma$ denote the collection of all open $\gamma$-covers of $X$. Motivated by Menger’s work, Hurewicz [6] introduced the Hurewicz property $U_{\text{fin}}(\mathcal{O}, \Gamma)$:

For each sequence $\{U_n\}_{n \in \mathbb{N}}$ of members of $\mathcal{O}$ that do not contain a finite subcover, there exist finite sets $\mathcal{F}_n \subseteq U_n$, $n \in \mathbb{N}$, such that \(\bigcup \mathcal{F}_n : n \in \mathbb{N}\) $\in \Gamma$.

Every $\sigma$-compact space satisfies $U_{\text{fin}}(\mathcal{O}, \Gamma)$, but the converse fails [7, 2].

Let $\mathcal{A}$ and $\mathcal{B}$ be any two families. Motivated by works of Rothberger, Scheepers introduced the following prototype of properties [12]:

$S_1(\mathcal{A}, \mathcal{B})$: For each sequence $\{U_n\}_{n \in \mathbb{N}}$ of members of $\mathcal{A}$, there exist members $U_n \in U_n$, $n \in \mathbb{N}$, such that $\{U_n : n \in \mathbb{N}\} \in \mathcal{B}$.

It is easy to see that $U_{\text{fin}}(\mathcal{O}, \Gamma) = U_{\text{fin}}(\Gamma, \Gamma)$, and therefore $S_1(\Gamma, \Gamma)$ implies $U_{\text{fin}}(\mathcal{O}, \Gamma)$ [12]. However, a set of reals satisfying $S_1(\Gamma, \Gamma)$ cannot contain perfect subsets [7]. It follows that, for example, $\mathbb{R}$ satisfies $U_{\text{fin}}(\mathcal{O}, \Gamma)$ but not $S_1(\Gamma, \Gamma)$. In the fundamental paper [7], we are asked whether there are nontrivial examples showing that $U_{\text{fin}}(\mathcal{O}, \Gamma)$ does not imply $S_1(\Gamma, \Gamma)$.

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Problem 1.1 (Just, Miller, Scheepers, Szeptycki [7]). Let $X$ be a set of reals that does not contain a perfect set, but that does have the Hurewicz property. Does $X$ then satisfy $S_1(\Gamma, \Gamma)$?

We give a negative answer that also yields a new result concerning function spaces.

2. The main theorem

We prove a stronger assertion than what is needed to settle Problem 1.1 this will be useful for the next section. Let $C_T$ denote the collection of all clopen $\gamma$-covers of $X$. Clearly, $S_1(\Gamma, \Gamma)$ implies $S_1(C_T, C_T)$. The hypothesis in the following theorem is a consequence of the Continuum Hypothesis. See [3] for a survey of the involved cardinals.

Theorem 2.1. Assume that $b = c$. There exists a set of reals $X$ such that:

1. $X$ does not contain a perfect set;
2. all finite powers of $X$ have the Hurewicz property $U_{fin}(O, \Gamma)$;
3. no set of reals containing $X$ satisfies $S_1(C_T, C_T)$.

Theorem 2.1 is proved in three steps. The first step is analogous to Theorem 4.2 of [5] and will be used to show that the constructed set is not contained in a set of reals satisfying $S_1(C_T, C_T)$. We say that a convergent sequence $\{x_n\}_{n \in \mathbb{N}}$ is nontrivial if $\lim_n x_n \notin \{x_n : n \in \mathbb{N}\}$.

Lemma 2.2. Let $X$ be a subspace of a zero-dimensional metrizable space $Y$ satisfying $S_1(C_T, C_T)$, and let $\{x^n_m\}_{n \in \mathbb{N}}$, $m \in \mathbb{N}$, be nontrivial convergent sequences in $X$. Then there are a countable closed cover $\{F_k : k \in \mathbb{N}\}$ of $X$ and an infinite $A \subseteq \mathbb{N}$, such that $F_k \cap \{x^n_m : n \in A\}$ is finite for all $k, m$.

Proof. Let $d$ be a metric on $Y$ that generates its topology. For each $m$, do the following. Let $x_m = \lim_n x^m_n$, and for each $n$ take a clopen neighborhood $C^m_n$ of $x^m_n$ in $Y$, whose diameter is smaller than $d(x^n_m, x_m)/2$. For each $m, n$, set

$$\mathcal{U}^m_n = Y \setminus (C^m_n \cup C^m_{n+1} \cup \cdots \cup C^m_m) .$$

For each $m$, $\{\mathcal{U}^m_n : n \in \mathbb{N}\}$ is a clopen $\gamma$-cover of $Y$. Apply $S_1(C_T, C_T)$ to get $f : \mathbb{N} \rightarrow \mathbb{N}$ such that $\{\mathcal{U}^m_{f(m)} : m \in \mathbb{N}\}$ is a (clopen) $\gamma$-cover of $Y$. As $\mathcal{U}^m_{f(m)} \subseteq Y \setminus C^m_{f(m)}$ for each $m$, we have that the image $A$ of $f$ is infinite.

For each $k$, let $F_k = \bigcap_{i \geq k} \mathcal{U}^i_{f(i)}$. $\{F_k : k \in \mathbb{N}\}$ is a closed ($\gamma$-)cover of $Y$. Fix $k$ and $m$. If $n$ is large enough and $n \in A$, then $n = f(i)$ with $i \geq m, k$. As $x^n_m = x^n_{f(i)} \in C^m_{f(i)}$ and $i \geq m$, $x^n_m \notin U^i_{f(i)}$. As $i \geq k$, $U^i_{f(i)} \supseteq F_k$, and therefore $x^n_m \notin F_k$. □

To make sure that our constructed set does not contain a perfect set and that it satisfies the Hurewicz property in all finite powers, we will use the following. Let $\overline{\mathbb{N}} = \mathbb{N} \cup \{\infty\}$ be the one-point compactification of $\mathbb{N}$, and $\overline{\mathbb{N}}^{\mathbb{N}}$ be the collection of all nondecreasing elements $f$ of $\overline{\mathbb{N}}^\mathbb{N}$ (endowed with the Tukey product topology) such that $f(n) < f(n + 1)$ whenever $f(n) < \infty$. $\overline{\mathbb{N}}^{\mathbb{N}}$ is homeomorphic to the Cantor space (see [13] for an explicit homeomorphism) and can therefore be viewed as a set of reals.

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1. It is an open problem whether the converse implication holds [3][11].
Let $S$ be the family of all nondecreasing finite sequences in $\mathbb{N}$. For $s \in S$, $|s|$ denotes its length. For each $s \in S$, define $q_s \in \mathbb{N}^\mathbb{N}$ by $q_s(n) = s(n)$ if $n < |s|$, and $q_s(n) = \infty$ otherwise. Let $Q$ be the collection of all these elements $q_s$. $Q$ is dense in $\mathbb{N}^\mathbb{N}$.

For a set $D$ and $f, g \in \mathbb{N}^D$, $f \leq^* g$ means: $f(d) \leq g(d)$ for all but finitely many $d \in D$. A $\mathcal{b}$-scale is an unbounded (with respect to $\leq^*$) set $\{f_\alpha : \alpha < b\} \subseteq \mathbb{N}^\mathbb{N}$ of increasing functions, such that $f_\alpha \leq^* f_\beta$ whenever $\alpha < \beta$.

**Theorem 2.3** (Bartoszyński-Tsaban [2]). Let $X \subseteq \mathbb{N}^\mathbb{N}$ be a union of a $\mathcal{b}$-scale and $Q$. Then $X$ contains no perfect subset, and all finite powers of $H$ satisfy the Hurewicz property $U_{\text{fin}}(O, \Gamma)$.

For each $s \in S$, $\{q_{s^*}n\}_{n \in \mathbb{N}}$ (where $^*$ denotes a concatenation of sequences) is a nontrivial convergent sequence in $\mathbb{N}^\mathbb{N}$ and

$$\lim_{n \to \infty} q_{s^*}n = q_s.$$

The following will be used in our construction.

**Lemma 2.4.** Let $X$ be a closed subspace of $\mathbb{N}^\mathbb{N}$. If $X \cap \{q_{s^*}n : n \in \mathbb{N}\}$ is finite for each $s \in S$, then there exists $\phi : S \to \mathbb{N}$ such that for all $x \in X$ and all $n \geq 2$, $x(n) \geq \phi(x \upharpoonright n) \leq \phi(x \upharpoonright (n+1))$.

**Proof.** For each $s \in S$, let $k(s)$ be such that $q_{s^*}k \in \mathbb{N}^\mathbb{N} \setminus X$ for all $k \geq k(s)$. As $X$ is closed in $\mathbb{N}^\mathbb{N}$, for each $k \geq k(s)$ there is $m(s, k)$ such that

$$\{z \in \mathbb{N}^\mathbb{N} : z \upharpoonright (|s|+1) = s^*k, z(|s|+1) > m(s, k)\} \cap X = \emptyset.$$

(Note that $\{z \in \mathbb{N}^\mathbb{N} : z \upharpoonright (|s|+1) = s^*k, z(|s|+1) > m\}$, $m \in \mathbb{N}$, is a neighborhood base at $q_{s^*}k$.) Define $\phi : S \to \mathbb{N}$ by

$$\phi(s) = \max\{k(s), m(s \upharpoonright (|s|-1)), s(|s|-1)\}$$

when $|s| \geq 2$, and by $\phi(s) = 0$ when $|s| < 2$. Let $x \in X$ and $n \geq 2$. If $x(n) \geq \phi(x \upharpoonright n)$, then $x(n) \geq k(x \upharpoonright n)$; hence $x(n+1) \leq m(x \upharpoonright n, x(n)) \leq \phi(x \upharpoonright (n+1))$. $\square$

It remains to prove the following.

**Proposition 2.5.** Assume that $b = \mathfrak{c}$. There exists a $\mathcal{b}$-scale $B = \{b_\alpha : \alpha < b\}$ such that for each closed cover $\{F_\alpha : n \in \mathbb{N}\}$ of $B \cup Q$ and each infinite set $A \subseteq \mathbb{N}$, there are $n$ and $s \in S$ such that $F_\alpha \cap \{q_{s^*}k : k \in A\}$ is infinite.

**Proof.** Let $\{A_\alpha : \alpha < \mathfrak{c}\}$ be an enumeration of all infinite subsets of $\mathbb{N}$, such that for each infinite $A \subseteq \mathbb{N}$, there are $\mathfrak{c}$ many $\alpha < \mathfrak{c}$ with $A_\alpha = A$.

As $b = \mathfrak{d} = \mathfrak{c}$, there is a (standard) scale in $\mathbb{N}^\mathbb{N}$, that is, a family $\{\phi_\alpha : \alpha < \mathfrak{c}\} \subseteq \mathbb{N}^\mathbb{N}$ such that:

1. For each $\phi \in \mathbb{N}^\mathbb{N}$, there is $\beta < \mathfrak{c}$ such that $\phi \leq^* \phi_\beta$;
2. For all $\alpha < \beta < \mathfrak{c}$, $\phi_\alpha \leq^* \phi_\beta$.

\(\text{2}^\text{Strictly speaking, } q_{s^*}n \notin \mathbb{N}^\mathbb{N} \text{ when } n < s(|s| - 1), \text{ but since we are dealing with convergent sequences, we can ignore the first few elements.}
For an infinite $A \subseteq \mathbb{N}$, let $\overline{A} = A \cup \{\infty\}$ and

$$\overline{A}^\mathbb{N} = \{ x \in \mathbb{N}^\mathbb{N} : x(n) \in \overline{A} \text{ for all } n \}. $$

The order isomorphism between $A \cup \{\infty\}$ and $\mathbb{N} \cup \{\infty\}$ induces an order isomorphism $\Psi_A : \overline{A}^\mathbb{N} \to \mathbb{N}^\mathbb{N}$. By induction on $\alpha < b = c$, construct a $b$-scale $B = \{b_\alpha : \alpha < c\}$ such that for each $\alpha < c$, $b_\alpha \in (A_\alpha)^\mathbb{N}$, and

$$\Psi_{A_\alpha}(b_\alpha)(n) > \phi_\alpha(\Psi_{A_\alpha}(b_\alpha) \upharpoonright n)$$

for all $n \geq 2$.

We claim that $X = B \cup Q$ is as required. Indeed, let $A$ be an infinite subset of $\mathbb{N}$. Take an increasing enumeration $\{\beta_\alpha : \alpha < c\}$ of $\{\alpha < c : A_\alpha = A\}$. For each $\alpha < c$, $b_\beta_\alpha \in \overline{A}^\mathbb{N}$. Set $c_\alpha = \Psi_A(b_\beta_\alpha)$, and $C = \{c_\alpha : \alpha < c\}$. By the construction of the functions $b_\alpha$,

$$c_\alpha(n) > \phi_\beta_\alpha(c_\alpha \upharpoonright n) \geq \phi_\alpha(c_\alpha \upharpoonright n)$$

for all but finitely many $n$.

Let $\{K_m : m \in \mathbb{N}\}$ be a closed cover of $C \cup Q$. Then there are $m$ and $s \in S$ such that $K_m \cap \{q_{s,k} : k \in \mathbb{N}\}$ is infinite: Otherwise, by Lemma 2.4, for each $m$ there is $\psi_m \in S^\mathbb{N}$ such that for all $x \in K_m$ and $n \geq 2$, $x(n) \geq \psi_m(x \upharpoonright n)$ implies $x(n+1) \leq \psi_m(x \upharpoonright (n+1))$. Let $\alpha < c$ be such that for each $m$, $\phi_\alpha(s) \geq \psi_m(s)$ for all but finitely many $s \in S$. It is easy to verify that $c_\alpha \not\in K_m$ for all $m$, a contradiction.

Now consider any closed cover $\{F_m : m \in \mathbb{N}\}$ of $B \cup Q$ and set $K_m = \Psi_A(F_m \cap \overline{A}^\mathbb{N})$. Let $s \in S$ and $m$ be such that $K_m \cap \{q_{s,k} : k \in \mathbb{N}\}$ is infinite. Then for $\tilde{s} \in S$ such that $\tilde{s}(i)$ is the $s(i)$-th element of $A$ for each $i < |s|$, we have that $F_m \cap \{q_{s,k} : k \in A\}$ is infinite.

This completes the proof of Theorem 2.1. The following corollary of Theorem 2.1 answers in the negative Problem 15(1) of Bartoszyński and the second author [1].

**Corollary 2.6.** The union of a $b$-scale and $Q$ need not satisfy $S_1(G, \Gamma)$. \hfill \Box

### 3. Reformulation for spaces of continuous functions

Let $Y$ be a (not necessarily metrizable) topological space. For $y \in Y$ and $A \subseteq Y$, write $\lim A = y$ if $A$ is countable and an (any) enumeration of $A$ converges nontrivially to $y$. Let $\Gamma_y = \{A \subseteq Y : \lim A = y\}$. $Y$ has the Arhangel’skii property $\alpha_2$ if $S_1(\Gamma_y, \Gamma_y)$ holds for all $y \in Y$.

Fix a set of reals $X$. $C_p(X)$ is the subspace of the Tychonoff product $\mathbb{R}^X$ consisting of the continuous functions. It was recently discovered, independently by Bukovský and Hales [3] and by Sakai [11], that $C_p(X)$ has the property $\alpha_2$ if, and only if, $X$ satisfies $S_1(C_\Gamma, C_\Gamma)$.

Many additional connections of this type are studied in the literature. For families $\mathscr{A}$ and $\mathscr{B}$, consider the following prototype [12].

$S_{\text{fin}}(\mathscr{A}, \mathscr{B})$: For each sequence $\{U_n\}_{n \in \mathbb{N}}$ of members of $\mathscr{A}$, there exist finite subsets $\mathcal{F}_n \subseteq U_n$, $n \in \mathbb{N}$, such that $\bigcup_n \mathcal{F}_n \in \mathscr{B}$. 
For a topological space $Y$ and $y \in Y$, let $\Omega_y = \{A \subseteq Y : y \in \overline{A} \setminus A\}$. $Y$ has the Arhangel’skiĭ countable fan tightness if $S_1(\Omega_y,\Omega_y)$ holds for each $y \in Y$. $Y$ has the Reznichenko property if for each $y \in Y$ and each $A \in \Omega_y$, there are pairwise disjoint finite sets $F_n \subseteq A$, $n \in \mathbb{N}$, such that each neighborhood $U$ of $y$ intersects $F_n$ for all but finitely many $n$.

For sets of reals $X$, $C_p(X)$ has countable fan tightness and the Reznichenko property if, and only if, all finite powers of $X$ have the Hurewicz property $U_{\text{fin}}(\mathcal{O},\Gamma)$ [9]. Thus, Theorem 2.1 can be reformulated as follows.

**Theorem 3.1.** Assume that $b = c$. There exists a set of reals $X$ without perfect subsets such that $C_p(X)$ has countable fan tightness and the Reznichenko property, but does not have the Arhangel’skiĭ property $\alpha_2$.

A topological space $Y$ has the Sakai strong countable fan tightness if $S_1(\Omega_y,\Omega_y)$ holds for each $y \in Y$. Sakai proved that for sets of reals, $C_p(X)$ has strong countable fan tightness if, and only if, all finite powers of $X$ satisfy $S_1(\mathcal{O},\mathcal{O})$ [10]. For sets of reals $X$, $C_p(X)$ has strong countable fan tightness and the Reznichenko property if, and only if, all finite powers of $X$ satisfy $U_{\text{fin}}(\mathcal{O},\Gamma)$ as well as $S_1(\mathcal{O},\mathcal{O})$ [8].

If $b \leq \text{cov}(\mathcal{M})$ and $X$ is a union of a $b$-scale and $Q$, then all finite powers of $X$ satisfy $U_{\text{fin}}(\mathcal{O},\Gamma)$ as well as $S_1(\mathcal{O},\mathcal{O})$ [2]. As the Continuum Hypothesis (or just Martin’s Axiom) implies that $b = \text{cov}(\mathcal{M}) = c$, we have the following.

**Corollary 3.2.** Even for $C_p(X)$ where $X$ is a set of reals, the conjunction of strong countable fan tightness and the Reznichenko property does not imply the Arhangel’skiĭ property $\alpha_2$. □

4. **Concluding remarks and open problems**

Our results are consistency results. What is not settled is whether the answers to the problems addressed in this paper are undecidable.

**Problem 4.1.** Is it consistent that all sets of reals that have the Hurewicz property $U_{\text{fin}}(\mathcal{O},\Gamma)$ but have no perfect subsets satisfy $S_1(\Gamma,\Gamma)$?

**Problem 4.2.** Is it consistent that each union of a $b$-scale and $Q$ satisfies:

1. $S_1(\Gamma,\Gamma)$?
2. $S_1(\Gamma,\Gamma)$ in all finite powers?

**Problem 4.3.** Is it consistent that for each set of reals $X$, if $C_p(X)$ has both strong countable fan tightness and the Reznichenko property, then $C_p(X)$ has the Arhangel’skiĭ property $\alpha_2$?

**References**


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