DYNAMICS OF THE $w$ FUNCTION  
AND THE GREEN-TAO THEOREM  
ON ARITHMETIC PROGRESSIONS IN THE PRIMES

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Abstract. Let $A_3$ be the set of all positive integers $pqr$, where $p, q, r$ are primes and possibly two, but not all three of them are equal. For any $n = pqr \in A_3$, define a function $w$ by $w(n) = P(p + q)P(p + r)P(q + r)$, where $P(m)$ is the largest prime factor of $m$. It is clear that if $n = pqr \in A_3$, then $w(n) \in A_3$. For any $n \in A_3$, define $w^i(n) = n$, $w'(n) = w(w^{i-1}(n))$ for $i = 1, 2, \ldots$. An element $n \in A_3$ is semi-periodic if there exists a nonnegative integer $s$ and a positive integer $t$ such that $w^{s+t}(n) = w^s(n)$. We use $\text{ind}(n)$ to denote the least such nonnegative integer $s$. Wushi Goldring [Dynamics of the $w$ function and primes, J. Number Theory 119(2006), 86-98] proved that any element $n \in A_3$ is semi-periodic. He showed that there exists $i$ such that $w^i(n) \in \{20, 98, 63, 75\}$, $\text{ind}(n) \leq 4(\pi(P(n)) - 3)$, and conjectured that $\text{ind}(n)$ can be arbitrarily large.

In this paper, it is proved that for any $n \in A_3$ we have $\text{ind}(n) = O(\log P(n))^2$, and the Green-Tao Theorem on arithmetic progressions in the primes is employed to confirm Goldring’s above conjecture.

1. Introduction

Let $A_3 = \{n = pqr \mid p, q, r \text{ are all primes}\} \setminus \{n = p^3 \mid p \text{ is prime}\}$. For any $n \in A_3$, define a function $w$ by

$$w(n) = P(p + q)P(p + r)P(q + r),$$

where $P(m)$ is the largest prime factor of $m$. It is clear that $w(A_3) \subseteq A_3$ (see [1 Lemma 2.1]). For any $n \in A_3$, define $w^0(n) = n$, $w^i(n) = w(w^{i-1}(n))$ for $i = 1, 2, \ldots$, and define the $w$-orbit of $n$ to be the sequence $W(n) = [n, w(n), \ldots, w^i(n), \ldots]$.

Wushi Goldring [1] proved that for any $n \in A_3$, there exists $i$ such that $W(n) = [n, w(n), \ldots, w^{i-1}(n), 20, 98, 63, 75]$. For $n \in A_3$, the periodicity index of $n$ is defined to be the least nonnegative integer $i$ such that $w^i(n) \in \{20, 98, 63, 75\}$. Denote the periodicity index of $n$ by $\text{ind}(n)$. Wushi Goldring[1] proved that $\text{ind}(n) \leq 4(\pi(P(n)) - 3)$, and posed several conjectures related to $w(n)$. Two of them are

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Conjecture 2.9. \( \text{ind}(n) = O(\log \pi(P(n))). \)

Conjecture 2.10. There are sets in \( A_3 \) of arbitrarily large periodicity index.

In this paper we improve the upper bound of \( \text{ind}(n) \) and employ the Green-Tao Theorem on arithmetic progressions in the primes to confirm Goldring’s Conjecture 2.10.

Theorem 1. For any \( n \in A_3 \), we have \( \text{ind}(n) = O((\log \pi(P(n)))^2). \)

Theorem 2. \( \text{ind}(n) \) can be arbitrarily large.

Remark. By the Prime Number Theorem, it is clear that \( \log(\pi(P(n))) \sim \log P(n). \) By the proofs of Theorems 1 and 2, we believe that Conjecture 2.9 is not true.

2. Proofs of theorems

We will make repeated use of the following trivial facts.

Lemma 1. Let \( p, q \) be odd primes. Then

(a) \( P(p + q) \leq \frac{1}{2}(p + q); \)
(b) if \( p + 2 \) is composite, then \( P(p + 2) \leq \frac{1}{3}(p + 2); \)
(c) if \( p + 2 \) is prime and \( p > 3 \), then \( p + 4 \) must be composite and \( P(p + 4) \leq \frac{1}{4}(p + 4). \)

Lemma 2. Let \( X \) be an integer with \( X \geq 3 \) and \( \alpha \) be a real number with \( 0 < \alpha < 1 \). For any \( n = pqr \in A_3 \), where \( p, q, r \) are all primes with \( p \leq X, q \leq \alpha X \) and \( r \leq \alpha X \), there exists \( 1 \leq i \leq 3 \) such that

\[
P(w^i(n)) \leq \frac{1}{4}(\alpha + 3)X + 4.
\]

Proof. By the definition of \( w \) we have

\[
w(n) = P(p + q)P(p + r)P(q + r).
\]

If \( p \leq \alpha X \), then by Lemma 1,

\[
P(w(n)) \leq \alpha X + 2 \leq \frac{1}{4}(\alpha + 3)X + 4.
\]

Now we may assume that \( p > \alpha X \). If \( p = 3 \), then by \( q \leq \alpha X \) and \( r \leq \alpha X \) we have \( q = r = 2 \). Then

\[
P(w(n)) = 5 \leq \frac{1}{4}(\alpha + 3)X + 4.
\]

Thus we may assume that \( p \geq 5 \) and \( p > \alpha X \).

Case 1. \( p + 2 \) is composite.

By Lemma 1 we have

\[
P(p + q) \leq \begin{cases} \frac{p + 2}{3}, & \text{if } q = 2, \\ \frac{1 + \alpha X}{1 + 2}, & \text{if } q \geq 3, \end{cases},
\]

\[
P(p + r) \leq \begin{cases} \frac{p + 2}{4}, & \text{if } r = 2, \\ \frac{1 + \alpha X}{2 + 1}, & \text{if } r \geq 3, \end{cases}
\]

and

\[
P(q + r) \leq \begin{cases} \alpha X, & \text{if } q \geq 3 \text{ and } r \geq 3, \\ \alpha X + 2, & \text{if } q = 2 \text{ or } r = 2. \end{cases}
\]

Hence

\[
P(w(n)) \leq \frac{1}{4}(\alpha + 3)X + 4.
\]
Case 2. $p + 2$ is prime. Since $p > 3$ and $p$, $p + 2$ are both primes, we have $3 | p + 4$.

Subcase 2.1. $q$, $r$ are both odd primes. By Lemma 1 we have

$$P(w(n)) \leq \max\{\frac{1 + \alpha}{2} X, \alpha X\} \leq \frac{1}{4}(\alpha + 3)X + 4.$$  

Subcase 2.2. $q > r = 2$. Then $w(n) = (p + 2)P(p + q)P(q + 2).$ Let $p_1 = p + 2$, $q_1 = P(p + q)$, $r_1 = P(q + 2).$ Since $p_1 \leq p + 2$, $q_1 \leq \frac{1}{2}(1 + \alpha)X$, $r_1 \leq \alpha X + 2$, and $r_1$ is odd, by Lemma 1 we have

$$P(p_1 + q_1) \leq \begin{cases} \frac{p+4}{3}, & \text{if } q_1 = 2, \\ \frac{3+\alpha}{4}X + 1, & \text{if } q_1 \text{ is odd}, \end{cases}$$

$$P(q_1 + r_1) \leq \begin{cases} \alpha X + 4, & \text{if } q_1 = 2, \\ \frac{1+3\alpha}{4}X + 1, & \text{if } q_1 \text{ is odd}, \end{cases}$$

and

$$P(p_1 + r_1) \leq \frac{1+\alpha}{2}X + 2.$$  

Hence

$$P(w^2(n)) \leq \frac{1}{4}(\alpha + 3)X + 4.$$  

Similarly, when $r > q = 2$, we can get the same conclusion.

Subcase 2.3. $q = r = 2$. Then

$$w^3(n) = P(p + 4) \cdot P^2(P(p + 4) + p + 2).$$

By Lemma 1 we have

$$P(p + 4) \leq \frac{p+4}{3}, P(P(p + 4) + p + 2) \leq \frac{p+2 + (p+4)/3}{2} = \frac{2p+5}{3}.$$  

Hence

$$P(w^3(n)) \leq \frac{1}{4}(\alpha + 3)X + 4.$$  

This completes the proof of Lemma 2. \hfill \Box

Corollary 3. Let $X$ be an integer with $X \geq 3$ and $\alpha$ be a real number with $0 < \alpha < 1$. For $n = pqr \in A_3$ with $p \geq q \geq r \geq 3$, $p \leq X$ and $r \leq \alpha X$, there exists $2 \leq i \leq 4$ such that

$$P(w^i(n)) \leq \frac{1}{8}(\alpha + 7)X + 4.$$  

Proof. Let

$$p_1 = P(p + q), q_1 = P(p + r), r_1 = P(q + r).$$

By Lemma 1 we have

$$p_1 \leq X, q_1 \leq \frac{1+\alpha}{2}X, r_1 \leq \frac{1+\alpha}{2}X.$$  

Since $w(n) = P(p + q)P(p + r)P(q + r)$, by Lemma 2, there exists $1 \leq j \leq 3$ such that

$$P(w^j(w(n))) \leq \frac{3 + (1 + \alpha)/2}{4}X + 4.$$  

Therefore, there exists $2 \leq i \leq 4$ such that

$$P(w^i(n)) \leq \frac{1}{8}(\alpha + 7)X + 4.$$
This completes the proof of Corollary 3. □

**Lemma 4.** For any $n \in A_3$ there exists $1 \leq i \leq 2\log(P(n) + 2) + 6$ such that

$$P(w^i(n)) \leq \frac{15}{16} P(n) + 6.$$ 

**Proof.** For any $n = pqr \in A_3$ with $p \geq q \geq r \geq 2$, we have

$$w(n) = P(p+q)P(p+r)P(q+r).$$

**Case 1.** $p$, $q$, $r$ are all odd primes.

**Subcase 1.1.** At most one of $\frac{1}{2}(p+q)$, $\frac{1}{2}(p+r)$, and $\frac{1}{2}(q+r)$ is an odd prime.

If there is an odd prime, then this odd prime is not larger than $p$, but the other two prime factors of $w(n)$ are not larger than $\frac{1}{2}p$. Otherwise all of $P(p+q)$, $P(p+r)$ and $P(q+r)$ are not larger than $\frac{1}{2}p$. Hence, by Lemma 2, there exists $1 \leq i \leq 4$ such that

$$P(w^i(n)) \leq \frac{1}{4} (3 + \frac{1}{2}) p + 4 = \frac{7}{8} p + 4.$$ 

**Subcase 1.2.** Precisely two of $\frac{1}{2}(p+q)$, $\frac{1}{2}(p+r)$, and $\frac{1}{2}(q+r)$ are odd primes.

Firstly, we suppose that $\frac{1}{2}(p+q)$ and $\frac{1}{2}(p+r)$ are odd primes. Let

$$p_1 = P(p+q) = \frac{p+q}{2}, \quad q_1 = P(p+r) = \frac{p+r}{2}, \quad r_1 = P(q+r).$$

Since $p_1 + q_1 = p + \frac{q+r}{2}$ is even, $1 \neq \frac{q+r}{2}$ is odd. Thus $3 \leq r_1 = P(q+r) \leq \frac{1}{2}p$. By Corollary 3, there exists $1 \leq i \leq 5$ such that

$$P(w^i(n)) \leq \frac{1}{8} (7 + \frac{1}{2}) p + 4 = \frac{15}{16} p + 4.$$ 

Similarly, we have the same conclusion for the other two cases: (a) $\frac{1}{2}(p+q)$ and $\frac{1}{2}(q+r)$ are odd primes; (b) $\frac{1}{2}(p+r)$ and $\frac{1}{2}(q+r)$ are odd primes.

**Subcase 1.3.** All of $\frac{1}{2}(p+q)$, $\frac{1}{2}(p+r)$, and $\frac{1}{2}(q+r)$ are odd primes.

Let

$$p_1 = \frac{p+q}{2}, \quad q_1 = \frac{p+r}{2}, \quad r_1 = \frac{q+r}{2},$$

$$p_i = \frac{p_{i-1} + q_{i-1}}{2}, \quad q_i = \frac{p_{i-1} + r_{i-1}}{2}, \quad r_i = \frac{q_{i-1} + r_{i-1}}{2} \quad (i = 2, 3, ...).$$

Assume that for $i = 1, 2, ..., k$, all of $p_i$, $q_i$, $r_i$ are odd primes, and at least one of $p_{k+1}$, $q_{k+1}$, $r_{k+1}$ is not an odd prime. Since

$$1 \leq p_k - r_k = \frac{p_{k-1} - r_{k-1}}{2} = \cdots = \frac{p - r}{2^k},$$

we have $k \leq 2 \log p$. By Subcases 1.1 and 1.2, there exists $1 \leq j \leq 5$ such that

$$P(w^j(p_kq_kr_k)) \leq \frac{15}{16} p_k + 4 \leq \frac{15}{16} p + 4.$$ 

Thus

$$P(w^{k+j}(n)) \leq \frac{15}{16} p + 4.$$ 

Hence, if $p$, $q$, $r$ are all odd primes, there exists $1 \leq i \leq 2 \log p + 5$ such that

$$P(w^i(n)) \leq \frac{15}{16} p + 4.$$
Case 2. \( p \geq q > r = 2 \).

Now we note that \( P(p + 2) \) and \( P(q + 2) \) are odd primes.

**Subcase 2.1.** \( P(p + q) \) is an odd prime.

As in Case 1, there exists \( 1 \leq i \leq 2 \log(p + 2) + 6 \) such that
\[
P(w^i(n)) \leq \frac{15}{16}(p + 2) + 4 \leq \frac{15}{16}p + 6.
\]

**Subcase 2.2.** \( P(p + q) = 2 \).

- If at least one of \( p + 2 \) and \( q + 2 \) is prime, then by Lemma 1 we have
  \[
P(p + 2) \leq \frac{p + 2}{3} \quad \text{or} \quad P(q + 2) \leq \frac{p + 2}{3}.
\]

\(\text{Noting that } 2 \leq \frac{1}{2}(p + 2), \text{by Lemma 2 there exists } 1 \leq i \leq 4 \text{ such that} \)
\[
P(w^i(n)) \leq \frac{1}{4}(3 + \frac{1}{2}(p + 2) + 4 = \frac{7}{8}(p + 2) + 4.
\]

- If \( p + 2, q + 2 \) are both primes, then \( p + 4 \) is odd composite and \( q + 4 \) is odd prime or 7, and

\( w^2(n) = P(p + 4)P(q + 4)P(p + q + 4). \)

Since \( P(p + q) = 2 \), we have \( 4 \mid p + q + 4 \). So

\[
P(p + q + 4) \leq \frac{p + q + 4}{4}.
\]

By Lemma 1 we have
\[
P(p + 4) \leq \frac{p + 4}{3}, \quad P(q + 4) \leq \max\{7, \frac{p + 4}{3}\}.
\]

Hence
\[
P(w^2(n)) \leq \frac{15}{16}p + 6.
\]

**Case 3.** \( q = r = 2. \) Then \( n = 2^2p. \) By Lemma 2 with \( \alpha = \frac{2}{3} \) and \( X = p \), there exists \( 1 \leq i \leq 3 \) such that
\[
P(w^i(n)) \leq \frac{11}{12}p + 4.
\]

This completes the proof of Lemma 4. \(\square\)

**Proof of Theorem 1.** For any \( n = pqr \in A_3 \) let \( i_0 = 0 \) and \( P(n) = p. \) By Lemma 4 there exist positive integers \( i_1 < i_2 < \cdots \) such that
\[
(1) \quad P(w^{i_k}(n)) \leq \frac{15}{16}P(w^{i_{k-1}}(n)) + 6, \quad k = 1, 2, \ldots
\]

and
\[
(2) \quad i_k - i_{k-1} \leq 2\log(P(w^{i_{k-1}}(n)) + 2) + 6, \quad k = 1, 2, \ldots.
\]

By (1) we have
\[
P(w^{i_k}(n)) \leq \frac{15}{16}P(w^{i_{k-1}}(n)) + 6 \leq \left(\frac{15}{16}\right)^2P(w^{i_{k-2}}(n)) + \frac{15}{16} + 6 \leq \cdots
\]
\[
\leq \left(\frac{15}{16}\right)^kP(w^{i_0}(n)) + \frac{15}{16}(k-1) + 6\left(\frac{15}{16}\right)^{k-2} + \cdots + 6\left(\frac{15}{16}\right) + 6
\]
\[
< \left(\frac{15}{16}\right)^kp + 96.
\]
If 

\[ k \geq \frac{\log p}{\log 16 - \log 15} \]

then

\[ P(w^{ik}(n)) \leq (\frac{15}{16})^k p + 96 \leq 97. \]

For each \( m \in A_3 \) with \( P(m) \leq 97 \), by [1] Theorem 1.1 there exists \( j_m \) such that

\[ w^{j_m}(n) \in \{20, 98, 63, 75\}. \]

Let \( c = \max\{j_m \mid m \in A_3, P(m) \leq 97\} \). Then there exists \( j \) with \( 1 \leq j \leq c \) such that

\[ w^{ik+j}(n) = w^{j}(w^{ik}(n)) \in \{20, 98, 63, 75\}. \]

By [1, Corollary 2.6] we have

\[ P(w^{ik}(n)) \leq p + 4, \quad k = 1, 2, \ldots. \]

By (2) and (3) we have

\[ i_k - i_{k-1} \leq 2 \log(p + 6) + 6, \quad k = 1, 2, \ldots. \]

Thus

\[ i_k \leq 2k \log(p + 6) + 6k, \quad k = 1, 2, \ldots. \]

Take an integer \( k \) with

\[ k \geq \frac{\log p}{\log 16 - \log 15} > k - 1. \]

Then

\[ \text{ind}(n) \leq i_k + j \leq 2k \log(p + 6) + 6k + c \ll (\log p)^2. \]

That is,

\[ \text{ind}(n) = O((\log p)^2). \]

This completes the proof of Theorem 1. \( \square \)

Finally, we employ the Green-Tao Theorem on arithmetic progressions in the primes to confirm Goldring’s Conjecture 2.10 [1]; i.e., \( \text{ind}(n) \) can be arbitrarily large.

**Proof of Theorem 2.** By the Green-Tao Theorem on arithmetic progressions in the primes (see [2]), for any positive integer \( k \geq 7 \), there exist two positive integers \( a, d \) such that \( a + id \) \((0 \leq i \leq 2^k)\) are all primes. Let

\[ p_0 = a + 2^kd, \quad q_0 = a + 2^{k-1}d, \quad r_0 = a, \]

and

\[ p_i = \frac{p_{i-1} + q_{i-1}}{2}, \quad q_i = \frac{p_{i-1} + r_{i-1}}{2}, \quad r_i = \frac{q_{i-1} + r_{i-1}}{2}, \quad i = 1, 2, \ldots, k - 1. \]

Then \( p_i, q_i \) and \( r_i \) \((i = 0, 1, 2, \ldots, k-1)\) are in the arithmetic progression \( a + id \) \((0 \leq i \leq 2^k)\). Hence \( p_i \), \( q_i \) and \( r_i \) \((i = 0, 1, 2, \ldots, k-1)\) are all primes. Let \( n = p_0 q_0 r_0 \). Then \( w^i(n) = p_i q_i r_i \) \((i = 1, 2, \ldots, k-1)\). Noting that

\[ p_i + q_i + r_i = p_0 + q_0 + r_0 > 2^k, \]

we have \( w^i(n) \notin \{20, 63, 75, 98\} \) \((i = 1, 2, \ldots, k-1)\). So \( \text{ind}(n) \geq k \). This completes the proof of Theorem 2. \( \square \)
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