ON ENDO MORPHISM RINGS OF LOCAL COHOMOLOGY MODULES

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ABSTRACT. Let $R$ be a local complete ring. For an $R$-module $M$ the canonical ring map $R \to \text{End}_R(M)$ is in general neither injective nor surjective; we show that it is bijective for every local cohomology module $M := H^h_I(R)$ if $H^l_I(R) = 0$ for every $l \neq h (= \text{height}(I))$ ($I$ an ideal of $R$); furthermore the same holds for the Matlis dual of such a module. As an application we prove new criteria for an ideal to be a set-theoretic complete intersection.

1. INTRODUCTION

For an ideal $I$ of a local ring $(R, m)$ we denote the $n$-th local cohomology functor with support in $I$ by $H^n_I$ and the (contravariant) Matlis dual functor by $D$; i.e., $D(M) = \text{Hom}_R(M, E)$ for any $R$-module $M$, where $E := E_R(R/M)$ is a fixed $R$-injective hull of the residue field $R/m$.

Let $R$ be a (always commutative, unitary) ring and $M$ an $R$-module. Consider the canonical map $\mu_M : R \to \text{End}_R(M)$ that maps $r \in R$ to multiplication by $r$ on $M$; it is a homomorphism of (associative) $R$-algebras. In general, $\mu_M$ is neither injective nor surjective. In section 2 we will show that, if $R$ is Noetherian local complete and $I$ an ideal of $R$ such that $H^l_I(R) = 0$ for every $l \neq h (= \text{height}(I))$, then $\mu_{H^h_I(R)}$ is bijective. In particular, the endomorphism ring of the $R$-module $H^h_I(R)$ is commutative and $\text{Ann}_R(H^h_I(R)) = 0$.

The proof of this result uses a generalization of Theorem 3.2 from [11], which says that, for a special class of Noetherian local complete rings $R$, it is true that $D(H^h_I(D(H^h_I(R))))$ is either zero or isomorphic to $R$ if $H^h_I(R) = 0$ for every $l > h = \text{height}(I)$; the generalization is due to Khashyarmanesh ([14 Corollary 2.6]) and says that $D(H^h_I(D(H^h_I(R)))) \cong R$ for every Noetherian local complete ring and every ideal $I \subseteq R$ such that $H^l_I(R) = 0$ for $l \neq h = \text{height}(I)$.

We also show in section 2 that $\mu_{D(H^h_I(R))}$ is an isomorphism if $H^l_I(R) = 0$ for every $l \neq h$.

Recently there was some work on Matlis duals of local cohomology modules (e.g. [6, 7, 10, 11, 12]). In [10 Corollary 1.1.4] the following was proved: If, for some $h \in \mathbb{N}$, $H^h_I(R) = 0$ for all $l > h$ and $\underline{x} = x_1, \ldots, x_h \in I$ is an $R$-regular sequence,
then one has the following equivalence:
\[ \sqrt{I} = \sqrt{\mathcal{I}R} \iff \mathcal{I} \text{ is a } D(H^b_I(R)) \text{ regular sequence.} \]

In section 3 we extend this equivalence:

**Theorem.** Let \((R, m)\) be a Noetherian local complete ring and \(I\) an ideal of \(R\) such that \(H^I_I(R) = 0\) for every \(l > h := \text{height}(I) \geq 1\); let \(\mathcal{I} = x_1, \ldots, x_h \in I\) be an \(R\)-regular sequence. Set \(D := D(H^b_I(R))\). The following statements are equivalent:

(i) \(\sqrt{I} = \sqrt{\mathcal{I}R}\); in particular, \(I\) is a set-theoretic complete intersection.

(ii) \(\mathcal{I}\) is an \(D\)-regular sequence.

(iii) The canonical map \(D/\mathcal{I}D \rightarrow H^b_{\mathcal{I}R}(D)\) (coming from \(H^b_{\mathcal{I}R}(D) = \lim_{\mathcal{I} \in \mathcal{N}} D/\mathcal{I}D\)) is injective.

(iv) The canonical map \(\{r \in R|\forall_{i \in \mathcal{N}} 3, s \in \mathcal{I} \cdot I^s \subseteq \mathcal{I}D\} \rightarrow \Gamma_I(R/\mathcal{I}R)\) is surjective.

The equivalence of (ii) and (iii) is inspired by a result of Marley and Rogers ([18, Prop. 2.3], which is a version of (ii) \(\iff\) (iii) for (arbitrary) finitely generated modules \(D\).

### 2. Endomorphism rings

**Definition 2.1.** (i) Let \(R\) be a ring and \(M\) an \(R\)-module. The map
\[ R \rightarrow \text{End}_R(M), \quad r \mapsto \text{multiplication by } r \text{ on } M \]
is an \(R\)-algebra homomorphism and will be denoted by \(\mu_M\).

(ii) Let \(R\) be a local ring and \(M\) an \(R\)-module. \(M\) has a canonical embedding
\[ M \rightarrow D(D(M)) = D^2(M), \quad m \mapsto (\varphi \mapsto \varphi(m)) \]
into its bidual; this map will be denoted by \(\iota_M\). We will consider \(M\) as a submodule of \(D^2(M)\) via \(\iota_M\).

(iii) Let \((R, m)\) be a Noetherian local ring and \(x = x_1, \ldots, x_h\) a sequence of elements of \(R\). For every \(R\)-module \(M\) there is a canonical map
\[ M/\mathcal{I}M \rightarrow \text{End}_R(M) \]
(coming from the description \(H^b_{\mathcal{I}R}(M) = \lim_{\mathcal{I} \in \mathcal{N}} M/(x_1^a, \ldots, x_h^a)M\), where the transition maps are induced by multiplication by \(x_1 \cdots x_h\).

**Theorem 2.2.** Let \((R, m)\) be a Noetherian local complete ring and \(I\) an ideal of \(R\) such that \(H^I_I(R) = 0\) for every \(l \neq h\) (\(h\) is then necessarily the height of \(I\)). Set \(H := H^b_I(R)\).

(i) \(\text{Hom}_R(H, \iota_H) : \text{End}_R(H) \rightarrow \text{Hom}_R(H, D^2(H))\) is an isomorphism.

(ii) There is a canonical isomorphism
\[ \gamma_H : \text{Hom}_R(H, D^2(H)) \rightarrow D(H^b_I(D(H))). \]

(iii) \(\mu_H : R \rightarrow \text{End}_R(H)\) is an isomorphism of \(R\)-algebras.

Consequently there is a canonical isomorphism
\[ \gamma_H \circ \text{Hom}_R(H, \iota_H) \circ \mu_H : R \rightarrow D(H^b_I(D(H))). \]

**Proof.** (i) It is clear that \(\text{Hom}_R(H, \iota_H)\) is injective. To show surjectivity, let \(\varphi \in \text{Hom}_R(H, D^2(H))\) be arbitrary; let \(x \in H\) be arbitrary and \(n \in \mathbb{N}\) such that \(I^n \cdot x = 0\). This implies \(I^n \cdot \varphi(x) = 0\); i.e.,
\[ \varphi(x) \in (0 : D^2(H) I^n) = D^2((0 : H I^n)) = (0 : H I^n) \subseteq H. \]
(the first equality follows from exactness of $D$; for the second equality we remark that $(0 :_H I^n)$ is finitely generated, as the spectral sequence
\[ E_2^{p,q} = \text{Ext}^p_R(R/I^n, H^q_I(R)) \Rightarrow \text{Ext}^{p+q}_R(R/I^n, R) \]
shows $(0 :_H I^n) = \text{Ext}^p_R(R/I^n, R)/I^n$). This means that the image of $\varphi$ is contained in $H \subseteq D^2(H)$, which was precisely what we had to show.

(ii) Hom-Tensor adjointness shows
\[ \text{Hom}_R(H, D^2(H)) = D(H \otimes_R D(H)). \]
On the other hand, our hypotheses imply $H^l_I = 0$ for every $l > h$; in particular, $H^h_I$ is right exact, we get
\[ H \otimes_R D(H) = H^h_I(R) \otimes_R D(H) = H^h_I(D(H)) \]
and statement (ii) is now clear.

(iii) [14 Corollary 2.6] implies that there exists an isomorphism of $R$-modules
\[ D(H^h_I(D(H))) \cong R. \]
Therefore, (i) and (ii) show that the $R$-module $\text{End}_R(H)$ is free of rank one. Fix any isomorphism $R \cong \text{End}_R(H)$ and let $\psi \in \text{End}_R(H)$ be the element corresponding to $1 \in R$. In particular there exists a (unique) $x \in R$ such that $\text{id}_H = x \circ \psi$, where $x$ is multiplication by $x$ on $H$. This implies $\varphi = x \circ \psi \circ x$ for every $\varphi \in \text{End}_R(H)$; in particular, multiplication by $x$ is surjective on $\text{End}_R(H) \cong R$, $x$ is a unit in $R$ and $\mu_H$ is bijective.

Remark 2.3. The hypothesis of the previous theorem is fulfilled if $I$ is perfect and $R$ is regular and has positive characteristic. This was shown in [14 Prop. 4.1].

Corollary 2.4. In the situation of Theorem 2.2 the endomorphism ring of $H^h_I(R)$ is canonically isomorphic to $R$; in particular, it is commutative and $\text{Ann}_R(H^h_I(R)) = 0$ holds.

Let $(R, m)$ be a Noetherian local ring and $M$ an $R$-module. Consider the sequence of $R$-modules
\[ R \to \text{End}_R(D(M)) \to \text{End}_R(D^2(M)) \to \text{Hom}_R(M, D^2(M)), \]
where the first map is $\mu_{D(M)}$, the second is given by $\alpha \mapsto D(\alpha)$ and the third is restriction to $M \subseteq D^2(M)$. The composition of the second and third is always injective:

Lemma 2.5. Let $(R, m)$ be a Noetherian local ring and $M$ an $R$-module. The $R$-linear map
\[ \text{End}_R(D(M)) \to \text{Hom}_R(M, D^2(M)), \quad \varphi \mapsto (m \mapsto (D(\varphi))(\iota_M(m))) \]
is injective.

Proof. This is straightforward: Let $\varphi$ be in the kernel of the above map; this means that for all $m \in M$ and for all $\psi \in D(M)$ one has $\varphi(\psi)(m) = 0$, i.e. $\varphi = 0$.

We apply this injectivity in the case where the local ring $(R, m)$ is complete, $M := H := H^h_I(R)$ and $I$ is an ideal of $R$ such that $H^l_I(R) = 0$ for every $l \neq h$; we
get $R$-linear maps

$$R \to \text{End}_R(D(H)) \to \text{End}_R(D^2(H)) \to \text{Hom}_R(H, D^2(H)) \equiv \text{End}_R(H) \equiv R.$$  

The composition of all these maps is clearly $\text{id}_R$. Thus, the injectivity statement from Lemma 2.5 shows:

**Theorem 2.6.** Let $(R, m)$ be a Noetherian local complete ring and $I$ an ideal of $R$ such that $H^l_I(R) = 0$ for every $l \neq h$. Then the canonical map

$$\mu_{D(H^h_I(R))} : R \to \text{End}_R(D(H^h_I(R)))$$

is an isomorphism of $R$-algebras.

3. Complete intersections and local cohomology

We need a couple of lemmata and remarks before we can prove Theorem 3.7, which is the main result of this section:

**Remark 3.1.** Let $(R, m)$ be a Noetherian local ring, $I$ an ideal of $R$ and $M$ an $R$-module such that

$$\text{Supp}_R(M) \subseteq V(I)$$

(where $V(I) = \{p \in \text{Spec}(R) | p \supseteq I\}$). Let $\hat{}$ denote $I$-adic completion. Then the natural map

$$D(M) \to \hat{D}(M)$$

is an isomorphism; in particular, $\bigcap_{l \in \mathbb{N}} I^l \cdot D(M) = 0$.

**Proof.** We have to show that the canonical map

$$D(M) \to \lim_{l \in \mathbb{N}} (D(M)/I^l D(M))$$

is bijective, but one has

$$D(M) = D(\Gamma_I(M)) = D(\lim_{l \in \mathbb{N}} \text{Hom}_R(R/I^l, M)) = \lim_{l \in \mathbb{N}} D(\text{Hom}_R(R/I^l, M)) = \lim_{l \in \mathbb{N}} D(M)/I^l D(M),$$

and it is easy to see that this is the canonical map $D(M) \to \hat{D}(M)$. \qed

Let $(R, m)$ be a Noetherian local ring and $\underline{x} = x_1, \ldots, x_h$ a sequence of elements of $R$. Marley and Rogers have shown ([18, Proposition 2.3]) that, for finitely generated $M$, $\iota_M \underline{x}$ is injective iff $\underline{x}$ is an $M$-regular sequence. In this context, note that the proof of the following lemma is strongly based on their proof; our additional ingredient is Remark 3.1.
Lemma 3.2. Let \((R, m)\) be a Noetherian local ring, \(I\) an ideal of \(R\), \(n, h \in \mathbb{N}\), \(x = x_1, \ldots, x_h \in I\) an arbitrary sequence and \(N\) an \(R\)-module. Set \(H := H^1_I(N)\) and \(D := D(H)\). The following two statements are equivalent:

(i) For every \(i = 1, \ldots, h\), multiplication by \(x_i\) on \(D/(x_1, \ldots, x_{i-1})D\) is injective (i.e., \(x_i\) is a \(D\)-quasiregular sequence).

(ii) \(D/\mathcal{I}_D D = H^1_{\mathcal{I}_D R}(D)\) is injective.

Proof. (i) \(\Rightarrow\) (ii): The finite case is well known; [20 Prop. 5.2.1] is a reference for the general case (note that (ii) holds trivially if \(D/\mathcal{I}_D D = 0\)).

(ii) \(\Rightarrow\) (i): By induction on \(h\): \(h = 1\): Set \(x = x_1\) and let \(\alpha \in D\) be such that \(x\alpha = 0\). We have to show \(\alpha = 0\). \(\alpha\) represents an element of \(\text{ker}(\iota_D \mathcal{I}_D)\); therefore, by assumption, \(\alpha \in xD\). Choose \(\alpha_1 \in D\) such that \(\alpha = x\alpha_1\). We conclude \(x^2\alpha_1 = 0\). Again, \(\alpha_1\) represents an element of \(\text{ker}(\iota_D \mathcal{I}_D)\) and so there exists \(\alpha_2 \in D\) such that \(\alpha_1 = x\alpha_2\). Continuing in this way, we get

\[
\alpha \in \bigcap_{k \in \mathbb{N}} x^kD
\]

and then \(\alpha = 0\), by Remark 3.1 \(h > 1\): First of all we prove injectivity of

\[
D/(x_1, \ldots, x_h-1)D \xrightarrow{\iota_D} x_{h-1}^H H^1_{(x_1, \ldots, x_{h-1})R}(D);
\]

to do so, let \(\alpha \in \text{ker}(\iota_D, x_1, \ldots, x_{h-1})\) be arbitrary. We show \(\alpha \in (x_1, \ldots, x_{h-1})D + x_h D\). For every \(k \in \mathbb{N}\) by induction on \(k\): \(k = 0\) is trivial, we assume \(k > 0\) and we write \(\alpha = \omega + x_h \beta\) for some \(\omega \in (x_1, \ldots, x_{h-1})D, \beta \in D\). By our choice of \(\alpha\) there exists \(t \in \mathbb{N}\) such that

\[
(x_1, \ldots, x_{h-1})^t x_h \beta \in (x_1^{t+1}, \ldots, x_{h-1}^{t+1})D
\]

and hence

\[
(x_1, \ldots, x_h)^{t+k} \beta \in (x_1^{t+k+1}, \ldots, x_{h-1}^{t+k+1})D.
\]

But \(\iota_D \mathcal{I}_D\) is injective; we conclude \(\beta \in (x_1, \ldots, x_h)D\) and our induction on \(k\) is finished:

\[
\alpha \in \bigcap_{k \in \mathbb{N}} ((x_1, \ldots, x_h-1)D + x_h^k D).
\]

The \(R\)-module

\[
D/(x_1, \ldots, x_h-1)D = D(\text{Hom}_R(R/(x_1, \ldots, x_h-1)R, H))
\]

is \(x_h R\)-adically separated by Remark 3.1. This means

\[
\bigcap_{k \in \mathbb{N}} ((x_1, \ldots, x_h-1)D + x_h^k D) = (x_1, \ldots, x_h-1)D
\]

and the stated injectivity of \(\iota_D, x_1, \ldots, x_{h-1}\) follows. The induction hypothesis shows that \(x_1, \ldots, x_{h-1}\) is \(D\)-quasiregular; we have to show that multiplication by \(x_h\) on \(D/(x_1, \ldots, x_{h-1})D\) is injective. Let \(\alpha \in D\) be such that \(x_h \alpha \in (x_1, \ldots, x_{h-1})D\). We state

\[
\forall k \in \mathbb{N} \alpha \in (x_1, \ldots, x_{h-1})D + x_h^k D
\]

and prove this statement by induction on \(k\). We may assume \(k > 0\) and write \(\alpha = \omega + x_h \beta\) for some \(\omega \in (x_1, \ldots, x_{h-1})D, \beta \in D\). From \(x_h \alpha = x_h \omega + x_h^{k+1} \beta\) we conclude \(x_h^{k+1} \beta \in (x_1, \ldots, x_{h-1})D\). Therefore,

\[
(x_1^{k+1}, \ldots, x_h^{k+1}) \beta \in (x_1^{k+2}, \ldots, x_{h-1}^{k+2})D.
\]
is injective and so $\beta \in (x_1, \ldots, x_h)D$, and induction on $k$ is finished:

$$\alpha \in \bigcap_{k \in \mathbb{N}} ((x_1, \ldots, x_{k-1})D + x_kD) \subseteq (x_1, \ldots, x_{k-1})D$$

(note that the last equality has been explained above in a similar situation). □

Let $(R, m)$ be a Noetherian local complete ring, $I$ an ideal of $R$, $h \in \mathbb{N}$; assume that $x = x_1, \ldots, x_h \in I$ is an $R$-regular sequence. It follows from the Grothendieck spectral sequence belonging to the composed functors $\Gamma_I \circ \Gamma_{\mathbb{Z}/R}$ that

$$H^1_R(I) = \Gamma_I(H^1_R(R)) \subseteq H^1_R(R).$$

By applying the functors $D$, $H^1_R$ and then $D$ again, we get a monomorphism (because $D$ is exact and $H^1_R$ is right exact)

$$D(H^1_R(D(H^1_R(R)))) \hookrightarrow D(H^1_R(D(H^1_R(R)))) = R,$$

Because of [14 Corollary 2.6], there is an isomorphism

$$D(H^1_R(D(H^1_R(R)))) \cong R.$$

Clearly, this isomorphism is unique up to a unit of $R$ and so we may consider $D(H^1_R(D(H^1_R(R))))$ as an ideal of $R$ (alternatively we use Theorem [2.2] and have a canonical isomorphism $D(H^1_R(D(H^1_R(R)))) = R$; the resulting ideal $J_{x, l}$ is the same in both cases).

**Definition 3.3.** In the above situation, set

$$J_{x, l} := D(H^1_R(D(H^1_R(R))))$$

and consider $J_{x, l}$ as an ideal of $R$.

**Remark 3.4.** Though the definition of $J_{x, l}$ is quite abstract, it also has the following concrete description: Because of the right exactness of $H^1_R$,

$$D(H^1_R(D(H^1_R(R)))) = D(H^1_R(R)) \otimes_R D(H^1_R(R))$$

and by Hom-Tensor adjointness, the latter module is

$$\text{Hom}_R(H^1_R(R), D^2(H^1_R(R))).$$

Now the arguments from the proof of Theorem [2.2] (i) show

$$\text{Hom}_R(H^1_R(R), D^2(H^1_R(R))) = \text{Hom}_R(D^1_R(R), H^1_R(R)).$$

But $H^1_R(R) = \Gamma_I(H^1_R(R))$ and we get

$$J_{x, l} = \{ \varphi \in \text{End}_R(H^1_R(R)) | \text{im}(\varphi) \subseteq \Gamma_I(H^1_R(R)) \}.$$
Therefore, if we restrict the canonical map \( R \to R/xR \) to \( J_{x,l} \), we get a canonical map from \( J_{x,l} \) to \( \Gamma_I(R/xR) \):

**Definition 3.5.** In the above situation, the canonical map

\[
J_{x,l} \to \Gamma_I(R/xR)
\]

is denoted by \( j_{x,l} \).

**Remark 3.6.** Let \( (R, m) \) be a Noetherian local complete ring. Let \( I \) be an ideal of \( R \), \( h \in \mathbb{N} \) and

\[
x = x_1, \ldots, x_h \subseteq I
\]
an \( R \)-regular sequence. Then

\[
I \subseteq \sqrt{xR + \text{Ann}_R(J_{x,l})}.
\]

In particular, if \( R \) is a domain and \( \sqrt{xR} \subseteq \sqrt{I} \), then \( J_{x,R} = 0 \).

**Proof.** We use the description of \( J_{x,R} \) from Remark 3.4. For the first statement we have to show

\[
V(\text{Ann}_R(J_{x,l}))) \cap V(xR) \subseteq V(I);
\]
i.e., for every \( r \in J_{x,R} \) we have to show

\[
V(\text{Ann}_R(r)) \cap V(xR) \subseteq V(I).
\]

Let \( r \in J_{x,l} \) be arbitrary; by Remark 3.4, \( J_{x,l} = \{ r \in R \mid \forall i \in \mathbb{N}_0 \exists s \in \mathbb{N}_0 \ r \cdot s \subseteq x^i R \} \).

For every \( p \in V(xR) \setminus V(I) \) we get

\[
r \cdot R_p \subseteq \bigcap_{i \in \mathbb{N}} x^i R_p \subseteq \bigcap_{i \in \mathbb{N}} p^i R_p = 0;
\]
i.e., \( \text{Ann}_R(r) \nsubseteq p \) and the first statement is proven. The second statement follows immediately from the first. \( \square \)

**Theorem 3.7.** Let \( (R, m) \) be a Noetherian local complete ring and \( I \) an ideal of \( R \) such that \( H^l_I(R) = 0 \) for every \( l > h := \text{height}(I) \geq 1 \); let \( x = x_1, \ldots, x_h \in I \) be an \( R \)-regular sequence (clearly, this implies \( H^l_I(R) = 0 \) for every \( l \neq h \)). Set \( D := D(H^h_I(R)) \). The following statements are equivalent:

(i) \( \sqrt{I} = \sqrt{xR} \); in particular, \( I \) is a set-theoretic complete intersection.

(ii) \( x \) is a \( D \)-regular sequence.

(iii) \( D/xD \overset{D^1} \longrightarrow \overset{i} \longrightarrow H^h_{xR}(D) \) is injective.

(iv) \( j_{x,l} \) is surjective.

(v) \( J_{x,l} = R \).

**Proof.** (i) \( \iff \) (ii) was shown (for more general \( R \)) in [10, Cor. 1.1.4].

(ii) \( \iff \) (iii) is a special case of Lemma 3.2 (note that \( D/xD = D(\text{Hom}_R(R/xR, H^h_{xR}(R))) \neq 0 \)).

(iii) \( \iff \) (iv): By definition, \( D(H^h_{xR}(D)) = J_{x,l} \). We have

\[
D(D/xD) = D(D(\text{Hom}_R(R/xR, H^h_{xR}(R)))).
\]

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But \( H^b_I(R) = \Gamma_I(H^b_{\underline{x}R}(R)) \) and, therefore,
\[
\text{Hom}_{R}(R/\underline{x}R, H^b_I(R)) = \Gamma_I(\text{Hom}_{R}(R/\underline{x}R, H^b_{\underline{x}R}(R))) = \Gamma_I(\text{Ext}^b_{R}(R/\underline{x}R, R)) = \Gamma_I(R/\underline{x}R)
\]
(for the second and the third equality use the fact that \( \underline{x} \) is an \( R \)-regular sequence).
The latter module is finitely generated. We get
\[
D(\text{Hom}_{R}(R/\underline{x}R, H^b_I(R))) = \Gamma_I(R/\underline{x}R).
\]
Thus \( D(\iota_{D,\underline{x}}) \) is a map \( J_{\underline{x},I} \to \Gamma_I(R/\underline{x}R) \); it is straightforward to see that it is in fact \( j_{\underline{x},I} \) (to do so one should start with the description \( H^b_{\underline{x}R}(D) = \lim_{\rightarrow} D/x^lD \)). □

**Question 3.8.** In the situation of Definition 3.3, when exactly is \( J_{\underline{x},I} = 0 \)?

**References**


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