ON DRAZIN INVERTIBILITY

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(Communicated by Joseph A. Ball)

Abstract. The left Drazin spectrum and the Drazin spectrum coincide with the upper semi-$B$-Browder spectrum and the $B$-Browder spectrum, respectively. We also prove that some spectra coincide whenever $T$ or $T^*$ satisfies the single-valued extension property.

1. Introduction and preliminaries

Throughout this note $L(X)$ will denote the algebra of all bounded linear operators acting on an infinite-dimensional complex Banach space $X$. The operator $T \in L(X)$ is said to be upper semi-Fredholm if its kernel $\ker T$ is finite-dimensional and the range $T(X)$ is closed, while $T \in L(X)$ is said to be lower semi-Fredholm if $T(X)$ is finite-codimensional. If either $T$ is upper or lower semi-Fredholm, then $T$ is said to be a semi-Fredholm operator, while $T$ is said to be a Fredholm operator if it is both upper and lower semi-Fredholm. If $T \in L(X)$ is semi-Fredholm, the classical index of $T$ is defined by $\text{ind}(T) := \dim \ker T - \text{codim} T(X)$.

The concept of semi-Fredholm operators has been generalized by Berkani ([9], [13] and [11]) in the following way: for every $T \in L(X)$ and a nonnegative integer $n$ let us denote by $T[n]$ the restriction of $T$ to $T^n(X)$ viewed as a map from the space $T^n(X)$ into itself (we set $T[0] = T$). $T \in L(X)$ is said to be semi-$B$-Fredholm, (resp. $B$-Fredholm, upper semi-$B$-Fredholm, lower semi-$B$-Fredholm,) if for some integer $n \geq 0$ the range $T^n(X)$ is closed and $T[n]$ is a semi-Fredholm operator (resp. Fredholm, upper semi-Fredholm, lower semi-Fredholm). In this case $T[m]$ is a semi-Fredholm operator for all $m \geq n$ ([13]). This enables one to define the index of a semi-$B$-Fredholm operator as $\text{ind } T = \text{ind } T[n]$.

A bounded operator $T \in L(X)$ is said to be a Weyl operator if $T$ is a Fredholm operator having index 0. A bounded operator $T \in L(X)$ is said to be $B$-Weyl if for some integer $n \geq 0$ the range $T^n(X)$ is closed and $T[n]$ is Weyl. The Weyl spectrum and the $B$-Weyl spectrum are defined, respectively, by

$$\sigma_w(T) := \{ \lambda \in \mathbb{C} : \lambda I - T \text{ is not Weyl} \}$$

and

$$\sigma_{bw}(T) := \{ \lambda \in \mathbb{C} : \lambda I - T \text{ is not } B\text{-Weyl} \}.$$ 

Recall that the ascent of an operator $T \in L(X)$ is defined as the smallest nonnegative integer $p := p(T)$ such that $\ker T^p = \ker T^{p+1}$. If such an integer does not
exist, we put \( p(T) = \infty \). Analogously, the \textit{descent} of \( T \) is defined as the smallest nonnegative integer \( q := q(T) \) such that \( T^q(X) = T^{q+1}(X) \), and if such an integer does not exist, we put \( q(T) = \infty \). It is well known that if \( p(T) \) and \( q(T) \) are both finite, then \( p(T) = q(T) \); see [1] Theorem 3.3. Moreover, if \( \lambda \in \mathbb{C} \), the condition \( 0 < p(\lambda I - T) = q(\lambda I - T) < \infty \) is equivalent to saying that \( \lambda \) is a pole of the resolvent. In this case \( \lambda \) is an eigenvalue of \( T \) and an isolated point of the spectrum \( \sigma(T) \); see [17] Prop. 50.2.

The concept of Drazin invertibility [14] has been introduced in a more abstract setting than operator theory [14]. In the case of the Banach algebra \( L(X) \), \( T \in L(X) \) is said to be \textit{Drazin invertible} (with a finite index) precisely when \( p(T) = q(T) < \infty \) and this is equivalent to saying that \( T = T_0 \oplus T_1 \), where \( T_0 \) is invertible and \( T_1 \) is nilpotent; see [19] Corollary 2.2 and [18] Prop. A. Every \( B \)-Fredholm operator \( T \) admits the representation \( T = T_0 \oplus T_1 \), where \( T_0 \) is Fredholm and \( T_1 \) is nilpotent [11], so every Drazin invertible operator is \( B \)-Fredholm.

The concept of Drazin invertibility for bounded operators may be extended as follows.

**Definition 1.1.** \( T \in L(X) \) is said to be \textit{left Drazin invertible} if \( p := p(T) < \infty \) and \( T^{p+1}(X) \) is closed; while \( T \in L(X) \) is said to be \textit{right Drazin invertible} if \( q := q(T) < \infty \) and \( T^q(X) \) is closed.

It should be noted that the condition \( q = q(T) < \infty \) does not entail that \( T^q(X) \) is closed; see Example 5 of [21]. Clearly, \( T \in L(X) \) is both right and left Drazin invertible if and only if \( T \) is Drazin invertible. In fact, if \( 0 < p := p(T) = q(T) \), then \( T^p(X) = T^{p+1}(X) \) is the kernel of the spectral projection associated with the spectral set \( \{0\} \); see [17] Prop. 50.2.

The \textit{left Drazin spectrum} is then defined as

\[
\sigma_{ld}(T) := \{ \lambda \in \mathbb{C} : \lambda I - T \text{ is not left Drazin invertible} \},
\]

the \textit{right Drazin spectrum} is defined as

\[
\sigma_{rd}(T) := \{ \lambda \in \mathbb{C} : \lambda I - T \text{ is not right Drazin invertible} \},
\]

and the \textit{Drazin spectrum} is defined as

\[
\sigma_d(T) := \{ \lambda \in \mathbb{C} : \lambda I - T \text{ is not Drazin invertible} \}.
\]

Obviously, \( \sigma_d(T) = \sigma_{ld}(T) \cup \sigma_{rd}(T) \).

A bounded operator \( T \in L(X) \) is said to be \textit{Browder} (resp. upper semi-Browder, lower semi-Browder) if \( T \) is Fredholm and \( p(T) = q(T) < \infty \) (resp. \( T \) is upper semi-Fredholm and \( p(T) < \infty \), \( T \) is lower semi-Fredholm and \( q(T) < \infty \)). Every Browder operator is Weyl and hence, if

\[
\sigma_b(T) := \{ \lambda \in \mathbb{C} : \lambda I - T \text{ is not Browder} \}
\]

denotes the Browder spectrum of \( T \), then \( \sigma_w(T) \subseteq \sigma_b(T) \). In the sequel by \( \sigma_{usb}(T) \) we shall denote the \textit{upper semi-Browder spectrum} of \( T \) defined by

\[
\sigma_{usb}(T) := \{ \lambda \in \mathbb{C} : \lambda I - T \text{ is not upper semi-Browder} \}.
\]

Clearly, every bounded below operator \( T \in L(X) \) (\( T \) injective with closed range) is upper semi-Browder, while every surjective operator is lower semi-Browder. The classical \textit{approximate point spectrum} of \( T \) will be denoted by \( \sigma_a(T) \) while by \( \sigma_s(T) \) we shall denote the \textit{surjectivity spectrum} of \( T \).
It is natural to extend the concept of semi-Browder operators as follows: A bounded operator $T \in L(X)$ is said to be $B$-Browder (resp. upper semi-$B$-Browder, lower semi-$B$-Browder) if for some integer $n \geq 0$ the range $T^n(X)$ is closed and $T_{[n]}$ is Browder (resp. upper semi-Browder, lower semi-Browder). The respective $B$-Browder spectra are denoted by $\sigma_{bb}(T)$, $\sigma_{usbb}(T)$ and $\sigma_{lbb}(T)$.

The main result of this paper establishes that $T \in L(X)$ is $B$-Browder (respectively, upper semi-$B$-Browder, lower semi-$B$-Browder) if and only if $T$ is Drazin invertible (respectively, left Drazin invertible, right Drazin invertible); consequently $\sigma_{bb}(T) = \sigma_d(T)$, $\sigma_{usbb}(T) = \sigma_{ld}(T)$ and $\sigma_{lbb}(T) = \sigma_{rd}(T)$. We also prove that many of the spectra before introduced coincide whenever $T$, or its dual $T^*$, satisfies the single-valued extension property.

2. SVEP and semi-$B$-Browder spectra

A useful tool in the Fredholm theory is given by the localized single-valued extension property. This property has an important role in local spectral theory; see the recent monographs by Laursen and Neumann [20] and Aiena [1].

Definition 2.1. Let $X$ be a complex Banach space and $T \in L(X)$. The operator $T$ is said to have the single-valued extension property at $\lambda_0 \in \mathbb{C}$ (abbreviated SVEP at $\lambda_0$) if for every open disc $D$ of $\lambda_0$, the only analytic function $f : U \to X$ that satisfies the equation $(\lambda I - T)f(\lambda) = 0$ for all $\lambda \in D$ is the function $f \equiv 0$. An operator $T \in L(X)$ is said to have SVEP if $T$ has SVEP at every point $\lambda \in \mathbb{C}$.

Evidently, $T \in L(X)$ has SVEP at every point of the resolvent $\rho(T) := \mathbb{C} \setminus \sigma(T)$. Moreover, from the identity theorem for analytic functions it is easily seen that $T$ has SVEP at every point of the boundary $\partial \sigma(T)$ of the spectrum. In particular, $T$ has SVEP at every isolated point of the spectrum. Note that the localized SVEP is inherited by the restriction to closed invariant subspaces; i.e., if $T$ has SVEP at $\lambda_0$ and $M$ is a closed $T$-invariant subspace of $X$, then $T|M$ has SVEP at $\lambda_0$. Moreover, the set $\Sigma(T)$ of all points $\lambda \in \mathbb{C}$ such that $T$ does not have SVEP at $\lambda$ is an open set contained in the interior of the spectrum of $T$. Consequently, if $T$ has SVEP at each point $\lambda$ of an open punctured disc $D \setminus \{\lambda_0\}$ centered at $\lambda_0$, then $T$ also has SVEP at $\lambda_0$.

We have

\begin{align*}
(1) & \quad p(\lambda I - T) < \infty \Rightarrow T \text{ has SVEP at } \lambda, \\
(2) & \quad q(\lambda I - T) < \infty \Rightarrow T^* \text{ has SVEP at } \lambda;
\end{align*}

see [1 Theorem 3.8]. Furthermore, from the definition of localized SVEP it is easily seen that

\begin{align*}
(3) & \quad \sigma_a(T) \text{ does not cluster at } \lambda \Rightarrow T \text{ has SVEP at } \lambda, \\
(4) & \quad \sigma_s(T) \text{ does not cluster at } \lambda \Rightarrow T^* \text{ has SVEP at } \lambda.
\end{align*}

Remark 2.2. The implications (1), (2), (3) and (4) are actually equivalences if $T$ is a semi-Fredholm operator; see [5] or [1 Chapter 3].
Lemma 2.3. If $T \in L(X)$ and $p = p(T) < \infty$, then the following statements are equivalent:

(i) there exists $n \geq p + 1$ such that $T^n(X)$ is closed;

(ii) $T^n(X)$ is closed for all $n \geq p$.

Proof. Define $c'_i(T) := \dim(\ker T^i/\ker T^{i+1})$. Clearly, $p = p(T) < \infty$ entails that $c'_i(T) = 0$ for all $i \geq p$, so $k_i(T) := c'_i(T) - c'_{i+1}(T) = 0$ for all $i \geq p$. The equivalence then easily follows from \cite[Lemma 12]{[10]}. Define

$$\Delta(T) := \{ n \in \mathbb{N} : m \geq n, m \in \mathbb{N} \Rightarrow T^n(X) \cap \ker T \subseteq T^m(X) \cap \ker T \}.$$ 

The degree of stable iteration is defined as $\text{dis}(T) := \inf \Delta(T)$ if $\Delta(T) \neq \emptyset$, while $\text{dis}(T) = \infty$ if $\Delta(T) = \emptyset$.

Definition 2.4. $T \in L(X)$ is said to be quasi-Fredholm of degree $d$ if there exists $d \in \mathbb{N}$ such that:

(a) $\text{dis}(T) = d$,

(b) $T^n(X)$ is a closed subspace of $X$ for each $n \geq d$,

(c) $T(X) + \ker T^d$ is a closed subspace of $X$.

It should be noted that by Proposition 2.5 of \cite{[13]} every semi-$B$-Fredholm operator is quasi-Fredholm.

Theorem 2.5. For every $T \in L(X)$ the following statements are equivalent:

(i) $T$ is left Drazin invertible;

(ii) There exists $n \in \mathbb{N}$ such that $T^n(X)$ is closed and $T_{[n]}$ is bounded below;

(iii) $T$ is semi-$B$-Fredholm and $T$ has SVEP at 0.

Dually, if $T \in L(X)$ the following statements are equivalent:

(iv) $T$ is right Drazin invertible;

(v) there exists $n \in \mathbb{N}$ such that $T^n(X)$ is closed and $T_{[n]}$ is onto;

(vi) $T$ is semi-$B$-Fredholm and $T^*$ has SVEP at 0.

Proof. (i) $\iff$ (ii) Suppose that $T$ is left Drazin invertible. Then $p = p(T) < \infty$ and $T^{p+1}(X)$ is closed. From Lemma 2.3 it follows that $T^p(X)$ is closed. By \cite[Lemma 3.2]{[10]} we have $\ker T \cap T^p(X) = \ker T_{[p]} = \{0\}$, so $T_{[p]}$ is injective. The range of $T_{[p]}$ is closed, since it coincides with $T^{p+1}(X)$; hence $T_{[p]}$ is bounded below, so the condition (ii) is satisfied.

Conversely, suppose that there exists $n \in \mathbb{N}$ such that $T^n(X)$ is closed and $T_{[n]}$ is bounded below. Let us consider an element $x \in \ker T^{n+1}$. Clearly, $T(T^n x) = 0$ so $T^n x \in \ker T$. Since $T^n x \in T^n(X)$ it then follows that $T^n x \in \ker T \cap T^n(X) = \ker T_{[n]} = \{0\}$; thus $x \in \ker T^n$. Therefore, $\ker T^{n+1} = \ker T^n$, so $T$ has finite ascent $p := p(T) \leq n$. The range of $T_{[n]}$ is the closed subspace $T^{n+1}(X)$, with $p + 1 \leq n + 1$. Therefore $T^{p+1}(X)$ is closed; thus $T$ is left Drazin invertible.

(ii) $\iff$ (iii) Assume (i) or equivalently (ii). Then $T$ has SVEP at 0, since $p(T) < \infty$ and $T_{[n]}$ is upper semi-Fredholm, so $T$ is upper semi-$B$-Fredholm.

Conversely, suppose that $T$ is semi-$B$-Fredholm and $T$ has SVEP at 0. By Proposition 3.2 of \cite{[10]} if $T$ quasi-Fredholm, in particular if $T$ is semi-$B$-Fredholm, then there exists $n \in \mathbb{N}$ such that $T^n(X)$ is closed and $T_{[n]}$ is semi-regular (i.e., it has closed range and its kernel is contained in the range of each iterate of $T_{[n]}$). Since the restriction $T_{[n]}$ has SVEP at 0, from Theorem 2.49 of \cite{[11]} it then follows that $T_{[n]}$ is bounded below.
(iv) \( \Leftrightarrow \) (v) If \( q := q(T) < \infty \), then \( T(T^n(X)) = T^{n+1}(X) = T^n(X) \), so \( T^n \) is onto. Moreover, \( T^n(X) \) is closed by assumption. Conversely, if (v) holds, then \( T^{n+1}(X) = T^n(X) \) so \( q := q(T) \leq n \). Obviously, \( T^n(X) = T^n(X) \) is closed.

(v) \( \Leftrightarrow \) (vi). Assume (v), or equivalently (iv). Since \( q := q(T) < \infty \), then \( T^* \) has SVEP at 0 and, clearly, \( T^n \) is lower semi-Fredholm, so (vi) holds. The opposite implication has been proved in [2, Theorem 2.7].

**Corollary 2.6.** \( T \in L(X) \) is Drazin invertible if and only if \( T \) is semi-B-Fredholm and both \( T \) and \( T^* \) have SVEP at 0.

The condition that \( T \), or \( T^* \), has SVEP at 0 for semi-B-Fredholm operators, more generally for quasi-Fredholm operators, may be characterized as follows:

**Theorem 2.7.** [2] Suppose that \( T \in L(X) \) is quasi-Fredholm. Then the following statements are equivalent:

(i) \( T \) has SVEP at 0;

(ii) \( \sigma_a(T) \) does not cluster at 0.

Dually, if \( T \in L(X) \) is quasi-Fredholm, then the following statements are equivalent:

(iii) \( T^* \) has SVEP at 0;

(iv) \( \sigma_a(T) \) does not cluster at 0.

Given \( n \in \mathbb{N} \) let us denote by \( \hat{T}_n : X/\ker T^n \to X/\ker T^n \) the quotient map defined canonically by \( \hat{T}_n \hat{x} := \hat{T}x \) for each \( \hat{x} \in \hat{X} := X/\ker T^n \), where \( x \in \hat{x} \).

**Lemma 2.8.** Suppose that \( T \in L(X) \) and \( T^n(X) \) is closed for some \( n \in \mathbb{N} \). If \( T^n \) is upper semi-Fredholm, then \( \hat{T}_n \) is upper semi-Fredholm and \( \text{ind} \hat{T}_n = \text{ind} T^n \). Analogous statements hold if \( T^n \) is assumed to be lower semi-Fredholm, Weyl, upper or lower semi-Browder, respectively.

**Proof.** The operator \([T^n] : X/\ker T^n \to T^n(X)\) defined by

\[ [T^n]\hat{x} = T^n x, \quad \text{where } x \in \hat{x}, \]

is a bijection, and it easy to check that \([T^n] \hat{T}_n = T^n \hat{T} \), from which the statements follow.

**Theorem 2.9.** Suppose that \( T \in L(X) \). Then the following equivalences hold:

(i) \( T \) is upper semi-B-Browder if and only if \( T \) is left Drazin invertible.

(ii) \( T \) is lower semi-B-Browder if and only if \( T \) is right Drazin invertible.

(iii) \( T \) is B-Browder if and only if \( T \) is Drazin invertible.

**Proof.** (i) Trivially, every bounded below operator is upper semi-Browder. By Theorem 2.5 if \( T \) is left Drazin invertible, then \( T \) is upper semi-B-Browder.

Conversely, suppose that \( T \) is upper semi-B-Browder. By Lemma 2.8 then \( \hat{T}_n \) is upper semi-Browder for some \( n \in \mathbb{N} \) and hence by Remark 2.2 the condition \( p(\hat{T}_n) < \infty \) is equivalent to saying that \( \sigma_a(\hat{T}_n) \) does not cluster at 0. Let \( D(0, \varepsilon) \) be an open ball centered at 0 such that \( D(0, \varepsilon) \setminus \{0\} \) does not contain points of \( \sigma_a(\hat{T}_n) \), so

\[ \ker (\lambda I - \hat{T}_n) = \{0\} \quad \text{for all } 0 < |\lambda| < \varepsilon. \]

Since the restriction \( T|\ker T^n \) is nilpotent we also have that \( D(0, \varepsilon) \setminus \{0\} \subseteq \rho(T|\ker T^n), \rho(T|\ker T^n) \) the resolvent of \( T|\ker T^n \), so

\[ (\lambda I - T)(\ker T^n) = \ker T^n \quad \text{for all } 0 < |\lambda| < \varepsilon. \]
We have proved that \( \sigma \) points of \( \hat{T} \) is lower semi-Browder. By Lemma 2.8, then \( \hat{T} \) is right Drazin invertible.

From (6) we know that there exists \( \{w_n\} \subset X \) such that \( (\lambda I - \hat{T})w_n \to \hat{w} \) as \( n \to +\infty \); thus
\[
(\lambda I - T)w_n - \hat{w} \to z_n \in \ker T^n.
\]

From (4) we know that there exists \( y_n \in \ker T^n \) such that \( z_n = (\lambda I - T)y_n \), and hence
\[
(\lambda I - T)w_n - (\lambda I - T)y_n = (\lambda I - T)(w_n - y_n) \to w,
\]
so that \( (\lambda I - T)(X) \) is closed. We have shown that \( \lambda I - T \) is bounded below for all \( 0 < |\lambda| < \varepsilon \) and, consequently, 0 is an isolated point of \( \sigma_n(T) \). This implies that \( T \) has SVEP at 0 and since by assumption \( T \) is upper semi-Browder according to Theorem 2.5, we then conclude that \( T \) is Drazin invertible.

(ii) By Theorem 2.5 if \( T \) is right Drazin invertible, then there exists \( n \in \mathbb{N} \) such that \( T_{[n]} \) is onto and hence lower semi-Browder.

Conversely, suppose that \( T \) is lower semi-Browder and let \( n \in \mathbb{N} \) such that \( T_{[n]} \) is lower semi-Browder. By Lemma 2.8, then \( T_n \) is lower semi-Browder and hence the condition \( q(T_n) < \infty \) is equivalent to saying that \( \sigma(T_n) \) does not cluster at 0. Let \( \mathbb{B}(0, \varepsilon) \) be an open ball centered at 0 such that \( \mathbb{B}(0, \varepsilon) \setminus \{0\} \) does not contain points of \( \sigma(T_n) \). As in the proof of part (i) we have \( (\lambda I - T)(\ker T^n) = \ker T^n \) for all \( 0 < |\lambda| < \varepsilon \). We show that \( (\lambda I - T)(X) = X \) for all \( 0 < |\lambda| < \varepsilon \). Since \( \lambda I - T_n \) is onto, for each \( x \in X \) there exists \( y \in X \) such that \( (\lambda I - T_n)y = \hat{x} \) and hence
\[
x - (\lambda I - T)y \in \ker T^n = (\lambda I - T)(\ker T^n).
\]
Consequently, there exists \( z \in \ker T^n \) such that \( x - (\lambda I - T)y = (\lambda I - T)z \), from which it follows that
\[
x = (\lambda I - T)(z + y) \in (\lambda I - T)(X).
\]
We have proved that \( \lambda I - T \) is onto for all \( 0 < |\lambda| < \varepsilon \); thus \( \sigma_n(T) \) does not cluster at 0 and consequently \( T^* \) has SVEP at 0. By Theorem 2.5 we then conclude that \( T \) is right Drazin invertible.

(iii) Clear.

Corollary 2.10. For every \( T \in L(X) \) we have
\[
\sigma_{usbb}(T) = \sigma_{ld}(T), \quad \sigma_{lsbb}(T) = \sigma_{rd}(T), \quad \sigma_{bb}(T) = \sigma_{d}(T).
\]

3. BROWDER TYPE THEOREMS

Let us denote by \( USBF^-(X) \) the class of all upper semi-B-Fredholm operators with index less than or equal to 0, while by \( LSBF^+(X) \) we denote the class of all lower semi-B-Fredholm operators with index greater than or equal to 0. Set
\[
\sigma_{usbf^-}(T) := \{ \lambda \in \mathbb{C} : \lambda I - T \notin USBF^-(X) \}
\]
and
\[
\sigma_{lsbf^+}(T) := \{ \lambda \in \mathbb{C} : \lambda I - T \notin LSBF^+(X) \}.
\]
Theorem 3.1. If \( T \in L(X) \), then the following equalities hold:

(i) \( \sigma_{\text{usbf}}(T) = \sigma_{\text{usbf}}(T) \cup \text{acc } \sigma_a(T) \).
(ii) \( \sigma_{\text{shb}}(T) = \sigma_{\text{shb}}(T) \cup \text{acc } \sigma_a(T) \).
(iii) \( \sigma_{\text{shb}}(T) = \sigma_{\text{shw}}(T) \cup \text{acc } \sigma_a(T) \).

Proof. The proof of the equalities (i), (iii) may be found in [3] and [7]. To show the equality (ii), we observe first that

\( \sigma_{\text{shf}}(T) \subseteq \sigma_{\text{rd}}(T) \).

Indeed, if \( \lambda \notin \sigma_{\text{rd}}(T) \), then, by Theorem 2.25 \( \lambda I - T \[n \] \) is onto some \( n \in \mathbb{N} \), hence lower semi-Fredholm and

\[
\text{ind}(\lambda I - T) = \text{ind}(\lambda I - T \[n \]) = \alpha(\lambda I - T \[n \]) \geq 0;
\]

thus \( \lambda \notin \sigma_{\text{shf}}(T) \).

By Corollary 2.10 in order to show the inclusion \( \sigma_{\text{shb}}(T) \supseteq \sigma_{\text{shf}}(T) \cup \text{acc } \sigma_a(T) \) we need only to prove that \( \text{acc } \sigma_a(T) \subseteq \sigma_{\text{shb}}(T) \). If \( \lambda \notin \sigma_{\text{shb}}(T) = \sigma_{\text{rd}}(T) \), then \( \lambda I - T \) is right Drazin invertible, and hence by Theorem 2.25 \( \lambda I - T \) is semi-B-Fredholm with \( q(\lambda I - T) < \infty \). By Corollary 4.8 of [16] it then follows that \( \lambda I - T \) is onto in a punctured disc centered at \( \lambda \); thus \( \lambda \notin \text{acc } \sigma_a(T) \).

To show the opposite inclusion \( \sigma_{\text{shb}}(T) \subseteq \sigma_{\text{shf}}(T) \cup \text{acc } \sigma_a(T) \), suppose that \( \lambda \notin \sigma_{\text{shf}}(T) \cup \text{acc } \sigma_a(T) \). Since \( \lambda \notin \text{acc } \sigma_a(T) \), then \( T^* \) has SVEP at \( \lambda \). Since \( \lambda I - T \) is lower semi-B-Fredholm by Theorem 2.25 \( \lambda I - T \) is right Drazin invertible. By Corollary 2.10 then \( \lambda \notin \sigma_{\text{rd}}(T) = \sigma_{\text{shb}}(T) \), so the equality (ii) is proved.

A bounded operator \( T \in L(X) \) is said to satisfy Browder’s theorem if \( \sigma_w(T) = \sigma_b(T) \). Denote by \( \sigma_{\text{usf}^+}(T) \) the essential approximate point spectrum of \( T \), defined as the complement in \( \mathbb{C} \) of the set of all \( \lambda \) such that \( \lambda I - T \) is upper semi-Fredholm with \( \text{ind } T \leq 0 \). The operator \( T \in L(X) \) is said to satisfy a-Browder’s theorem if \( \sigma_{\text{usf}^+}(T) = \sigma_{\text{usf}^+}(T) \); see for instance [4].

According to [12], a bounded operator \( T \in L(X) \) is said to satisfy the generalized Browder’s theorem if \( \sigma(T) \setminus \sigma_{\text{usbf}}(T) = \sigma_0(T) \), while \( T \in L(X) \) is said to satisfy the generalized a-Browder’s theorem if \( \sigma_0(T) \setminus \sigma_{\text{usbf}}(T) = \sigma_{\text{rd}}(T) \).

Note that in all the papers concerning generalized Browder’s theorems (see for instance [7], [15], [12], [8]), there is no trace of the role of B-Browder spectra. Our Corollary 2.10 shows that this is only apparent. In fact, by Corollary 2.10 we have:

generalized Browder’s theorem holds for \( T \iff \sigma_{\text{usbf}}(T) = \sigma_{\text{shb}}(T) \),

while

generalized a-Browder’s theorem holds for \( T \iff \sigma_{\text{usbf}^+}(T) = \sigma_{\text{usbf}}(T) \).

Browder’s theorem may be characterized by localized SVEP: Browder’s theorem (resp. generalized Browder’s theorem) holds for \( T \) if and only if \( T \) has SVEP at every \( \lambda \notin \sigma_w(T) \) (resp. \( T \) has SVEP at every \( \lambda \notin \sigma_{\text{usbf}}(T) \), see [7]), while a-Browder’s theorem (resp. generalized a-Browder’s theorem) holds for \( T \) if and only if \( T \) has SVEP at every \( \lambda \notin \sigma_{\text{usf}^+}(T) \) (resp. \( T \) has SVEP at every \( \lambda \notin \sigma_{\text{usbf}^+}(T) \), see [6]). The inclusions \( \sigma_{\text{usbf}}(T) \subseteq \sigma_w(T) \) and \( \sigma_{\text{usbf}^+}(T) \subseteq \sigma_{\text{usbf}}(T) \) immediately entail that the generalized Browder’s theorem implies Browder’s theorem, and, analogously, the generalized a-Browder’s theorem implies a-Browder’s theorem. The main result of a very recent paper [8] proves that Browder’s theorem and the generalized Browder’s theorem (respectively, a-Browder’s theorem and the
generalized $a$-Browder’s theorem) are equivalent. These results may be shown in a few lines as follows:

**Theorem 3.2.** For every $T \in L(X)$ the following equivalences hold:

(i) $\sigma_w(T) = \sigma_{bb}(T) \Leftrightarrow \sigma_{bw}(T) = \sigma_{bb}(T)$.

(ii) $\sigma_{usbf}^{-}(T) = \sigma_{ub}^{-}(T) \Leftrightarrow \sigma_{usbb}^{-}(T) = \sigma_{usbb}(T)$.

**Proof.** (i) We have only to show the implication $\Rightarrow$. Assume that $\sigma_w(T) = \sigma_{bb}(T)$. Clearly, $\sigma_{bw}(T) \subseteq \sigma_{bb}(T)$ for all $T \in L(X)$. To show the opposite inclusion, assume that $\lambda_0 \notin \sigma_{bw}(T)$, i.e. that $\lambda_0 I - T$ is $W$-Weyl. By [13 Corollary 3.2], then there exists an open disc $D$ such that $\lambda I - T$ is Weyl and hence Browder for all $\lambda \in D \setminus \{\lambda_0\}$. Since $p(\lambda I - T) = q(\lambda I - T) < \infty$, then both $T$ and $T^*$ have SVEP at every $\lambda \in D \setminus \{\lambda_0\}$, and hence both $T$ and $T^*$ have SVEP at $\lambda_0$. By Theorem 2.5 then $\lambda_0 I - T$ is Drazin invertible, or equivalently $\lambda_0 \notin \sigma_{bb}^{-}(T)$. Hence $\sigma_{bw}(T) = \sigma_{bb}^{-}(T)$.

(ii) Also here it suffices to prove the implication $\Rightarrow$. Assume that $\sigma_{usbf}^{-}(T) = \sigma_{ub}^{-}(T)$. Clearly, $\sigma_{usbb}^{-}(T) \subseteq \sigma_{usf}(T)$ for all $T \in L(X)$. Suppose that $\lambda_0 \notin \sigma_{usbb}^{-}(T)$. Then $\lambda_0 I - T \in USBF^{-}(X)$ and by [13 Corollary 3.2] there exists an open disc $D$ such that $\lambda I - T$ is upper semi-Fredholm with index less than or equal to $0$ for all $\lambda \in D \setminus \{\lambda_0\}$. From assumption then $\lambda I - T$ is upper semi-Browder; hence $p(\lambda I - T) < \infty$. Thus, $T$ has SVEP at every $\lambda \in D \setminus \{\lambda_0\}$ and hence $T$ also has SVEP at $\lambda_0$. By Theorem 2.5 we then conclude that $\lambda_0 \notin \sigma_{bb}^{-}(T) = \sigma_{usbb}(T)$, so the equality $\sigma_{usbf}^{-}(T) = \sigma_{usbb}(T)$ is proved.

The following result shows that many of the spectra considered before coincide whenever $T$ or $T^*$ has SVEP.

**Theorem 3.3.** Suppose that $T \in L(X)$. Then the following statements hold:

(i) If $T$ has SVEP, then

\begin{equation}
\sigma_{bsh}^d(T) = \sigma_{sbb}(T) = \sigma_{d}(T) = \sigma_{bw}(T).
\end{equation}

(ii) If $T^*$ has SVEP, then

\begin{equation}
\sigma_{usbf}^{-}(T) = \sigma_{usbb}(T) = \sigma_{bw}(T) = \sigma_{d}(T).
\end{equation}

(iii) If both $T$ and $T^*$ have SVEP, then

\begin{equation}
\sigma_{usbf}^{-}(T) = \sigma_{bsh}^d(T) = \sigma_{bw}(T) = \sigma_{d}(T).
\end{equation}

**Proof.** (i) By Theorem 3.1 and Corollary 2.10 we have

\begin{equation}
\sigma_{bsh}^d(T) \subseteq \sigma_{sbb}(T) = \sigma_{d}(T) \subseteq \sigma_{bw}(T).
\end{equation}

We show now that $\sigma_{d}(T) \subseteq \sigma_{bsh}^d(T)$. Assume that $\lambda \notin \sigma_{bsh}^d(T)$. We may assume $\lambda = 0$. Since $T$ is lower semi-$B$-Fredholm and since $T^*$ has SVEP, in particular $T^*$ has SVEP at $0$, by Theorem 2.5 then $T$ is right Drazin invertible or, equivalently, lower semi-$B$-Browder. Therefore there exists $n \in \mathbb{N}$ such that $T_{[n]}$ is lower semi-Fredholm and $q(T_{[n]}) < \infty$. By Theorem 3.4 of [1] it then follows that $\text{ind } T_{[n]} \leq 0$. On the other hand, since $\lambda \notin \sigma_{bsh}^d(T)$, we also have $\text{ind } T_{[n]} \geq 0$ from which we obtain $\text{ind } T_{[n]} = 0$. This implies, again by Theorem 3.4 of [1], that also $p(T_{[n]}) < \infty$, so that $T_{[n]}$ is Browder and hence $T$ is $B$-Browder. By part (iii) of Theorem 2.4 then $T$ is Drazin invertible, so $0 \notin \sigma_d(T)$, as desired. Finally, since $T$ has SVEP by which the $T$ satisfies the generalized Browder’s theorem, we have $\sigma_{bw}(T) = \sigma_{d}(T)$ and the equalities (8) are proved.
(ii) The inclusion $\sigma_{\text{lsbf}}(T) \subseteq \sigma_{\text{usbf}}(T) = \sigma_{\text{id}}(T) \subseteq \sigma_{d}(T)$ holds for every $T \in L(X)$ by Theorem 3.1 and Corollary 2.10.

We show that $\sigma_{d}(T) \subseteq \sigma_{\text{usbf}}(T)$. Suppose that $\lambda \notin \sigma_{\text{usbf}}(T)$ and assume that $\lambda = 0$. Since $T$ is upper semi-$B$-Fredholm, then there exists $n \in \mathbb{N}$ such that $T_{[n]}$ is upper semi-Fredholm. The restriction $T_{[n]} := T|T^n(X)$ has SVEP, in particular SVP at 0 and hence, see Remark 2.2 $p(T_{[n]}) < \infty$. By Theorem 3.4 of [1] it then follows that $\text{ind } T_{[n]} \leq 0$. On the other hand, since $\lambda \notin \sigma_{\text{lsbf}}(T)$, we also have $\text{ind } T_{[n]} \geq 0$ from which we obtain $\text{ind } T_{[n]} = 0$. This implies, again by Theorem 3.4 of [1], that also $q(T_{[n]}) < \infty$, so that $T_{[n]}$ is Browder and hence $T$ is $B$-Browder. By part (iii) of Theorem 2.9 then $T$ is Drazin invertible, so $0 \notin \sigma_{d}(T)$, as desired. Finally, since $T$ has SVEP, then $T$ satisfies the generalized Browder’s theorem, so $\sigma_{\text{bw}}(T) = \sigma_{d}(T)$.

(iii) Clear from parts (i), (ii). □

References


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