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THE PROJECTIVE π-CHARACTER BOUNDS THE ORDER OF A π-BASE

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Abstract. All spaces below are Tychonov. We define the projective π-character $p\,\pi\chi(X)$ of a space X as the supremum of the values $\pi\chi(Y)$ where Y ranges over all (Tychonov) continuous images of X. Our main result says that every space X has a π-base whose order is $\leq p\,\pi\chi(X)$; that is, every point in X is contained in at most $p\,\pi\chi(X)$-many members of the π-base. Since $p\,\pi\chi(X) \leq t(X)$ for compact X, this is a significant generalization of a celebrated result of Shapirovskii.

Let us start by recalling a few definitions and basic facts. A π-base $\mathcal{B}$ of a space X (resp. a local π-base at a point $x \in X$) is a family of non-empty open sets such that every non-empty open set (resp. every neighbourhood of x) includes some member of $\mathcal{B}$. The π-weight $\pi(X)$ of X is the smallest infinite cardinal such that X has a π-base of at most that cardinality. The π-character $\pi\chi(x, X)$ of x in X is the smallest cardinality of a local π-base at $x \in X$, and

$$\pi\chi(X) = \sup\{\pi\chi(x, X) : x \in X\}$$

is the π-character of the space X. Finally, the local tightness at $x \in X$ is the smallest cardinal κ such that if x belongs to the closure $\overline{A}$ of a set A, then there is $B \subset A$ with $|B| \leq \kappa$ and $x \in \overline{B}$; moreover

$$t(X) = \sup\{t(x, X) : x \in X\}$$

is the tightness of the space X.

Shapirovskii proved the following two important results concerning these cardinal functions for compacta: If X is compact, then $\pi\chi(X) \leq t(X)$. Moreover X has a π-base $\mathcal{B}$ of order $\leq t(X)$; i.e. every point of X is contained in at most $t(X)$-many members of $\mathcal{B}$. (A trivial consequence is that if $t(X)^+$ is a caliber of X—i.e. among $t(X)^+$-many open sets there are always $t(X)^+$-many with non-empty intersection—then X has a π-base of cardinality at most $t(X)$.) The first result was proved in [5], and alternative proofs were given in [1] and [3]. The second result first appeared in [6] and then in [8]. A very short and elegant new proof (using a variant of Shapirovskii's "algebraic" approach to free sequences) was presented in [9].

Arhangel'skii has recently introduced in [2] the concept of a space of *countable projective π-character* and noticed that any compact space of countable tightness

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has countable projective π-character. Then he showed that a compact space of countable projective π-character that has ω_1 as a caliber is separable (or equivalently, has a countable π-base), thereby strengthening the above consequence of Shapirovskii's result for countably tight compacta.

In this paper we introduce the general concept of projective π-character and give the following significant generalization of Shapirovskii's full result: Any Tychonov space has a π-base of order at most the projective π-character of the space. Not only is this result stronger for compacta, because it replaces tightness with projective π-character that is smaller, but somewhat surprisingly it extends to all Tychonov spaces.

Let φ be any cardinal function defined on a class $\mathcal{C}$ of topological spaces. We define the projective version $p\,\varphi$ of φ on $\mathcal{C}$ as follows. For any $X \in \mathcal{C}$ we let $p\,\varphi(X)$ be the the supremum of the values $\varphi(Y)$ where Y ranges over all continuous images of X belonging to $\mathcal{C}$. In particular, we shall consider the case in which $\varphi = \pi\chi$, the π-character defined on the class of Tychonov spaces. It is easy to show then that a Tychonov space X has countable projective π-character in the sense of [2] iff $p\,\pi\chi(X) \leq \omega$.

Also, as was already mentioned, if X is compact Hausdorff, then we have $p\,\pi\chi(X) \leq t(X)$. In fact, this follows because $t(Y) \leq t(X)$ for any continuous image of X and, by Shapirovskii's first result above, $\pi\chi(Y) \leq t(Y)$ for every compact Y. But are $p\,\pi\chi(X)$ and $t(X)$ really different? Arhangel'skii asked more specifically if there is a compactum of countable projective π-character that is not countably tight; see [2], problem 7. Our next example yields such a compactum.

Example 1. Let X be a compactification of ω whose remainder is (homeomorphic to) the ordinal $\omega_1 + 1$. Then $p\,\pi\chi(X) \leq \omega < t(X)$.

Proof. It is obvious that $t(\omega_1, X) = t(X) = \omega_1$. To see $p\,\pi\chi(X) \leq \omega$, consider any continuous surjection $f : X \to Y$. If $f(\omega_1) = p$ is an isolated point in Y, then there is an $\alpha < \omega_1$ such that f is constant on the interval $[\alpha, \omega_1]$; hence Y is countable and compact and so, trivially, $\pi\chi(Y) \leq w(Y) = \omega$.

If, however, p is not isolated, then Y has a countable dense subset S with $p \notin S$. So there is a closed G_δ set F such that $p \in F \subset Y \backslash S$, and again we can find an $\alpha < \omega_1$ such that $f[\alpha, \omega_1] \subset F$. But then $G = Y \backslash F$ is countable and dense open in Y; moreover $w(G) = \omega$ because every countable and locally compact space is second countable. So we have $\pi\chi(Y) \leq \pi(Y) = w(G) = \omega$. $\square$

We recall from [3] that $\pi sw(X)$ denotes the *π-separating weight* of a space X, that is, the minimum order of a π-base of X; see p. 74 of [3].

With this we may now formulate our main result as follows.

Theorem 2. *For any Tychonov space X we have $\pi sw(X) \leq p\,\pi\chi(X)$. In particular, any Tychonov space of countable projective π-character has a point-countable π-base.*

Our proof of Theorem 2 will go along similar lines as Shapirovskii's proof of the weaker result $\pi sw(X) \leq t(X)$ for compact spaces. The main idea of that was to show that the compactum X admits an irreducible map onto a subspace of the $\Sigma_{t(X)}$-power of the unit interval. The role of irreducible maps in our proof will be played by a new, more general, type of maps that we shall call π-irreducible. So we shall first define and deal with these maps. (The referee has pointed out to us that [8] is an excellent source concerning Shapirovskii's original method.)

Definition 3. Let f be a continuous map of X *onto* Y. We say that the map f is π-irreducible if for every proper closed subset $F \subset X$ its image $f[F]$ is not dense in Y.

Clearly, an onto map f is π-irreducible iff the f-image of a non-dense set is non-dense. Also, it is obvious that a closed map is π-irreducible iff it is irreducible; consequently the two concepts coincide for maps between compact Hausdorff spaces.

The following proposition will be used in the proof of Theorem 2 and explains our terminology.

Proposition 4. *Let f be a continuous map of X onto Y. Then the following five statements are equivalent:*

(1) f is π-irreducible;

(2) for every π-base $\mathcal{B}$ of X and for every $B \in \mathcal{B}$ the f-image of its complement, $f[X \backslash B]$, is not dense in Y;

(3) there is a π-base $\mathcal{B}$ of X such that for every $B \in \mathcal{B}$ the f-image $f[X \backslash B]$ is not dense in Y;

(4) for every π-base $\mathcal{C}$ of Y the family $\{f^{-1}(C) : C \in \mathcal{C}\}$ is a π-base of X;

(5) there is a π-base $\mathcal{C}$ of Y such that $\{f^{-1}(C) : C \in \mathcal{C}\}$ is a π-base of X.

Proof. We shall show (3)$\Rightarrow$(4) and (5)$\Rightarrow$(1) only because the other three implications of the cycle are trivial.

So, let $\mathcal{B}$ be as in (3) and $\mathcal{C}$ be any π-base of Y. For every non-empty open set U in X choose $B \in \mathcal{B}$ with $B \subset U$. Then there is a $C \in \mathcal{C}$ such that $C \cap f[X \backslash B] = \emptyset$, and hence $f^{-1}(C) \subset B \subset U$.

Now, let $\mathcal{C}$ be as in (5) and F be a proper closed subset of X. Then there is a $C \in \mathcal{C}$ with $F \cap f^{-1}(C) = \emptyset$; consequently we have $f[F] \cap C = \emptyset$ and so $f[F]$ is not dense in Y. $\square$

Corollary 5. *If $f : X \to Y$ is π-irreducible, then* $\pi(X) = \pi(Y)$.

Proof. $\pi(X) \leq \pi(Y)$ is immediate from part (4) of Proposition 4. To see $\pi(X) \geq \pi(Y)$ first note that for any non-empty open $U \subset X$ the interior of $f[U]$ in Y is non-empty. So for any π-base $\mathcal{B}$ of X the family $\{\mathrm{Int}_Y(f[B]) : B \in \mathcal{B}\}$ is a π-base of Y. Indeed, this is because if V is non-empty open in Y and $B \in \mathcal{B}$ with $B \subset f^{-1}(V)$, then $f[B] \subset V$. $\square$

We now consider another key ingredient of the proof of our main result: certain specially embedded subspaces of Tychonov cubes. As usual, we shall denote the unit interval $[0, 1]$ by I. The members of the Tychonov cube I^κ will be construed as functions from κ to I. So if $x \in I^\kappa$ and $\alpha < \kappa$, then $x \restriction \alpha$ is the projection of x to the subproduct I^α.

Definition 6. We say that $Y \subset I^\kappa$ is 0-embedded in the Tychonov cube I^κ if

$$\{y \restriction \alpha : y \in Y \text{ and } y(\alpha) = 0\}$$

is dense in the projection $Y \restriction \alpha = \{y \restriction \alpha : y \in Y\}$ for every $\alpha < \kappa$.

We now present two results concerning 0-embedded subspaces of Tychonov cubes which will be crucial in the proof of our main theorem and are also interesting in themselves.

Theorem 7. *Assume that Y is 0-embedded in the Tychonov cube I^κ where κ is a regular cardinal and $y \in Y$ is such that $y(\alpha) > 0$ for all $\alpha < \kappa$. Then $\pi\chi(y, Y) = \kappa$.*

Proof. Of course, only $\pi\chi(y, Y) \geq \kappa$ needs to be proven. To see this, let $\mathcal{U}$ be any family of elementary open sets in I^κ such that $|\mathcal{U}| < \kappa$ and $U \cap Y \neq \emptyset$ for all $U \in \mathcal{U}$. Every elementary open set $U \in \mathcal{U}$ is supported by a finite subset of κ; hence the regularity of κ implies the existence of an ordinal $\alpha < \kappa$ such that the support of each $U \in \mathcal{U}$ is included in α.

Since Y is 0-embedded in I^κ, this implies that for every $U \in \mathcal{U}$ we may pick a point $y_U \in U \cap Y$ such that $y_U(\alpha) = 0$. But then $y(\alpha) > 0$ clearly implies that the point y is not in the closure of the set $\{y_U : U \in \mathcal{U}\}$; consequently $\mathcal{U}$ cannot be a local π-base at y in Y, completing the proof. □

From Theorem 7 we can immediately obtain the following useful corollary about the projective π-character of 0-embedded subspaces of Tychonov cubes.

Corollary 8. *If Y is 0-embedded in the Tychonov cube I^κ, then for every non-isolated point $y \in Y$ we have*

$$p\,\pi\chi(Y) \geq \big|\{\alpha : y(\alpha) > 0\}\big|,$$

and if $y \in Y$ is isolated, then $\{\alpha : y(\alpha) > 0\}$ is finite.

Our next result shows that every Tychonov space admits a π-irrreducible map onto a suitable 0-embedded subspace of a Tychonov cube.

Theorem 9. *Let X be any Tychonov space of π-weight $\pi(X) = \kappa$. Then there is a π-irreducible map f of X onto a 0-embedded subspace Y of the Tychonov cube I^κ.*

Proof. To begin with, let us choose a π-base $\mathcal{B}$ of X with $|\mathcal{B}| = \kappa$ and fix a well-ordering $\prec$ of $\mathcal{B}$ of order-type κ.

We shall define by transfinite induction on $\alpha < \kappa$ the co-ordinate maps $g_\alpha = p_\alpha \circ f : X \to I$, where $p_\alpha(y) = y(\alpha)$ is the αth co-ordinate projection, and sets $B_\alpha \in \mathcal{B}$. So assume that $\alpha < \kappa$ and for all $\beta < \alpha$ the maps $g_\beta : X \to I$ and the sets $B_\beta \in \mathcal{B}$ have been defined.

Let $f_\alpha : X \to I^\alpha$ be the map whose βth co-ordinate map is g_β for all $\beta < \alpha$ and set $Y_\alpha = f_\alpha[X]$. Then, in view of Corollary 5, the map $f_\alpha : X \to Y_\alpha$ cannot be π-irreducible because $\pi(Y_\alpha) < \kappa = \pi(X)$; hence using part (2) of Proposition 4 there is a member $B \in \mathcal{B}$ for which $f_\alpha[X \backslash B]$ is dense in Y_α. Let B_α be the $\prec$-first such member of $\mathcal{B}$. We then define $g_\alpha : X \to I$ as any continuous function that is identically 0 on $X \backslash B_\alpha$ and takes the value 1 at some point in B_α. As was intended, with the induction completed we let $f : X \to I^\kappa$ be the unique map having the g_α for $\alpha < \kappa$ as its co-ordinate functions and we also set $Y = f[X]$.

Note first that if $\beta < \alpha$, then $B_\beta \prec B_\alpha$. Indeed, since we have $Y_\beta = Y_\alpha \restriction \beta$, the density of $f_\alpha[X \backslash B_\alpha]$ in Y_α implies that $f_\beta[X \backslash B_\alpha]$ is dense in Y_β; hence $B_\alpha \prec B_\beta$ would contradict the choice of B_β. Moreover, by our construction, $f_{\beta+1}[X \backslash B_\beta]$ is not dense in $Y_{\beta+1}$ and consequently $f_\alpha[X \backslash B_\beta]$ is not dense in Y_α, which implies $B_\alpha \neq B_\beta$.

Since $\mathcal{B}$ is of order type κ under $\prec$, it follows from this that for every $B \in \mathcal{B}$ there is an $\alpha < \kappa$ with $B \prec B_\alpha$. But then, by the choice of B_α, we have that $f_\alpha[X \backslash B]$ is not dense in $Y_\alpha = Y \restriction \alpha$ and hence $f[X \backslash B]$ cannot be dense in Y. Using part (3) of Proposition 4 implies that f is indeed a π-irreducible map of X onto Y.

Finally, by our construction, for every $\alpha < \kappa$ the image $f_\alpha[X \backslash B_\alpha]$ is dense in $Y_\alpha = Y \restriction \alpha$; moreover we have

$$f_\alpha[X \backslash B_\alpha] \subset \{y \restriction \alpha : y \in Y \text{ and } y(\alpha) = 0\},$$

and consequently Y is indeed 0-embedded in I^κ. □

Let us now recall that the κth Σ_λ-power of I, denoted by $\Sigma_\lambda(I, \kappa)$, is the subspace of I^κ consisting of all points whose support is of size at most λ. The support of a point $y \in I^\kappa$ is the set $\{\alpha < \kappa : y(\alpha) > 0\}$. Thus, from Theorem 9 and from Corollary 8—moreover from the trivial fact that $p\,\pi\chi(Y) \leq p\,\pi\chi(X)$ if Y is any continuous image of X—we immediately obtain the following result.

Corollary 10. *If X is a Tychonov space such that $\pi(X) = \kappa$ and $p\,\pi\chi(X) = \lambda$, then some π-irreducible image Y of X embeds into $\Sigma_\lambda(I, \kappa)$.*

This corollary is clearly a strengthening of the following result of Shapirovskii from [6] (see also 3.22 of [3]): If X is compact Hausdorff, then some irreducible image of X embeds into a $\Sigma_{t(X)}$-power of I.

The proof of our main theorem, Theorem 2, can now be easily established by recalling the following result of Shapirovskii from [6] (see also [3], 3.24).

Theorem (Shapirovskii)**.** *If the space Y embeds into a Σ_λ-power of I, then $\pi sw(Y) \leq \lambda$.*

Proof of Theorem 2. Now, to prove Theorem 2, consider any non-discrete Tychonov space X. By Corollary 10 then X has a π-irreducible image Y that embeds into a Σ_λ-power of I, where $\lambda = p\,\pi\chi(X)$. By the previous theorem of Shapirovskii the space Y has a π-base $\mathcal{C}$ of order at most λ. But by part (4) of Proposition 4, $\{f^{-1}(C) : C \in \mathcal{C}\}$ is a π-base of X that clearly has the same order as $\mathcal{C}$. □

The following result is then an immediate consequence of Theorem 2.

Corollary 11. *Let X be any Tychonov space and $\kappa > p\,\pi\chi(X)$ be a cardinal such that κ is a caliber of X. Then $\pi(X) < \kappa$.*

Since $t(X) \geq p\,\pi\chi(X)$ for a compact Hausdorff space X, this corollary implies Shapirovskii's theorem saying that if $t(X)^+$ is a caliber of such a space X, then $\pi(X) \leq t(X)$. Moreover, it also extends from compacta to all Tychonov spaces Arhangel'skii's result from [2] saying that spaces of countable projective π-character and having ω_1 as a caliber are separable.

Let us conclude by pointing out that neither Theorem 2 nor Corollary 11 remain valid if the projective π-character $p\,\pi\chi$ is replaced by the simple π-character $\pi\chi$ in them. In fact, it has recently been shown in [4] that there are even first countable spaces whose π-separating weight is as large as you wish. Moreover, in the same paper it was also shown that it is consistent to have first countable spaces with caliber ω_1 which have uncountable π-weight (or equivalently, density). However, since first countability implies countable tightness, none of these examples are (or could be) compact, so the following intriguing questions remain open.

Problem 12. Let X be a compact Hausdorff space of countable π-character. Does X have a point-countable π-base? If, in addition, ω_1 is a caliber of X, is X then separable?

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