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COMPUTATION OF THE MORDELL-TORNHEIM ZETA VALUES

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ABSTRACT. In this paper the authors present several algorithmic formulas which are potentially useful in computing the following Mordell-Tornheim zeta values:

$$\zeta_{MT,r}(s_1,\ \cdots,s_r;s):=\sum_{m_1,\ \cdots,m_r=1}^{\infty}\frac{1}{m_1^{s_1}\ \cdots m_r^{s_r}(m_1+\cdots+m_r)^s}$$

for the special cases

$$\zeta_{MT,r}(1,\ \cdots,1;s)\qquad\text{and}\qquad\zeta_{MT,r}(0,\ \cdots,0;s).$$

Some interesting (known or new) consequences and illustrative examples are also considered.

1. Introduction

The following double series:

$$\sum_{m_1,m_2=1}^{\infty}\frac{1}{m_1^{s_1}m_2^{s_2}(m_1+m_2)^s}\qquad(s_1,s_2,s\in\mathbb{C})\tag{1.1}$$

was defined and studied by Tornheim (see [14]). Subsequently, Mordell (see [10]) considered a multiple series in the form:

$$\sum_{m_1,\ \cdots,m_r=1}^{\infty}\frac{1}{m_1\cdots m_r\ \prod_{j=0}^{k}\{(m_1+\cdots+m_r+a+j)\}}\tag{1.2}$$

$$(r\in\mathbb{N}:=\{1,2,3,\ \cdots\};\ k\in\mathbb{N}_0:=\mathbb{N}\cup\{0\};\ a>-r).$$

In particular, he deduced that (see [10])

$$\sum_{m_1,m_2=1}^{\infty}\frac{1}{m_1m_2(m_1+m_2)}=2\zeta(3),\tag{1.3}$$

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where $\zeta(z)$ denotes the Riemann zeta function defined by (see, for details, [13, Section 2.3])

$$\zeta(z) := \sum_{k=1}^{\infty} \frac{1}{k^z} \qquad \big(\Re(z) > 1\big). \tag{1.4}$$

For

$$r \in \mathbb{N} \qquad \text{and} \qquad s_1, \cdots, s_r, s \in \mathbb{C},$$

Matsumoto (see [8] and [9]) defined the Mordell-Tornheim r-fold zeta function by

$$\zeta_{MT,r}(s_1, \cdots, s_r; s) := \sum_{m_1, \cdots, m_r=1}^{\infty} \frac{1}{m_1^{s_1} \cdots m_r^{s_r}(m_1 + \cdots + m_r)^s}. \tag{1.5}$$

He showed that the function $\zeta_{MT,r}(s_1, \cdots, s_r; s)$ can be continued meromorphically to the whole $(r+1)$-dimensional complex space and the multiple series in (1.5) is absolutely convergent when

$$\Re(s_j) > 1 \qquad (j = 1, \cdots, r) \qquad \text{and} \qquad \Re(s) > 0.$$

By using the following result of Mordell (see [10]):

$$\sum_{m_1, \cdots, m_r=1}^{\infty} \frac{1}{m_1 \cdots m_r(m_1 + \cdots + m_r + a)} = r! \sum_{\ell=0}^{\infty} \frac{(-1)^\ell}{\ell!\ (\ell+1)^{r+1}} \binom{a-1}{\ell}$$

$$(a > -r;\ r \in \mathbb{N}),$$

Hoffman proved that (see [4])

$$\zeta_{MT,r}(1, \cdots, 1; s) = r! \sum_{m_1 > m_2 > \cdots > m_s \geqq 1}^{\infty} \frac{1}{m_1^{r+1} m_2 \cdots m_s} \qquad (s \in \mathbb{N};\ r \in \mathbb{N}). \tag{1.6}$$

Analogously, Subbarao and Sitaramachandrarao (see [11]) established the following interesting result:

$$\zeta_{MT,r}(1, \cdots, 1; s) = (-1)^r\ r! \sum_{n=r}^{\infty} \frac{(-1)^n}{n!}\ \frac{\mathfrak{s}(n,r)}{n^s}, \tag{1.7}$$

where $\mathfrak{s}(n,k)$ are the Stirling numbers of the first kind defined by

$$z(z-1)\cdots(z-n+1) = \sum_{k=0}^{n} \mathfrak{s}(n,k) z^k \tag{1.8}$$

or, equivalently, by the generating function:

$$\{\log(1+z)\}^k = k! \sum_{n=k}^{\infty} \mathfrak{s}(n,k) \frac{z^n}{n!} \qquad (|z| < 1). \tag{1.9}$$

A detailed bibliography for the Mordell-Tornheim zeta values is given by Hoffman [5].

In this paper we derive algorithmic formulas for the evaluation of

$$\zeta_{MT,r}(1, \cdots, 1; s) \qquad \text{and} \qquad \zeta_{MT,r}(0, \cdots, 0; s),$$

which involve (for example) the Stirling numbers $\mathfrak{s}(n,k)$ and the Riemann zeta function $\zeta(z)$. Our proof of the algorithmic formula for $\zeta_{MT,r}(1, \cdots, 1; s)$ is based upon the results given earlier by Choi and Srivastava [3] and by Subbarao and Sitaramachandrarao [11].

2. Preliminary results

First of all, we find integral representations for the functions

$$\zeta_{MT,r}(0, \cdots, 0; s) \qquad \text{and} \qquad \zeta_{MT,r}(1, \cdots, 1; s).$$

Indeed, by applying the Eulerian integral:

$$z^{-s} = \frac{1}{\Gamma(s)} \int_0^\infty t^{s-1} e^{-zt}\, dt \qquad \big(\Re(s) > 0;\ \Re(z) > 0\big), \tag{2.1}$$

where $\Gamma(s)$ is the gamma function defined by

$$\Gamma(s) = \int_0^\infty t^{s-1} e^{-t}\, dt \qquad \big(\Re(s) > 0\big),$$

we obtain

$$\frac{1}{m_1^{s_1} \cdots m_r^{s_r} (m_1 + \cdots + m_r)^s} = \frac{1}{\Gamma(s)} \int_0^\infty \frac{t^{s-1} e^{-(m_1 + \cdots + m_r)t}}{m_1^{s_1} \cdots m_r^{s_r}}\, dt,$$

which, in view of the monotone convergence theorem, yields

$$\begin{aligned} \zeta_{MT,r}(s_1, \cdots, s_r; s) &:= \sum_{m_1, \cdots, m_r = 1}^{\infty} \frac{1}{m_1^{s_1} \cdots m_r^{s_r} (m_1 + \cdots + m_r)^s} \\ &= \frac{1}{\Gamma(s)} \int_0^\infty t^{s-1} \prod_{j=1}^{r} \left\{ \sum_{m_j = 1}^{\infty} \frac{e^{-m_j t}}{m_j^{s_j}} \right\} dt. \end{aligned} \tag{2.2}$$

Thus, for $s_1, \cdots, s_r \in \mathbb{C}$, we obtain

$$\zeta_{MT,r}(s_1, \cdots, s_r; s) = \frac{1}{\Gamma(s)} \int_0^\infty t^{s-1} \prod_{j=1}^{r} \left\{ \mathrm{Li}_{s_j}(e^{-t}) \right\} dt \qquad (s_j \in \mathbb{C};\ j = 1, \cdots, r), \tag{2.3}$$

where $\mathrm{Li}_\nu(z)$ is the polylogarithm function defined by [13, pp. 114 and 124]:

$$\mathrm{Li}_\nu(z) := \sum_{k=1}^{\infty} \frac{z^k}{k^\nu} \qquad \big(\nu \in \mathbb{C} \text{ when } |z| < 1;\ \Re(\nu) > 1 \text{ when } |z| = 1\big). \tag{2.4}$$

Hence we find that

$$\zeta_{MT,r}(0, \cdots, 0; s) = \frac{1}{\Gamma(s)} \int_0^\infty \frac{t^{s-1}}{(e^t - 1)^r}\, dt \qquad \big(\Re(s) > r;\ r \in \mathbb{N}\big) \tag{2.5}$$

and

$$\zeta_{MT,r}(1, \cdots, 1; s) = \frac{(-1)^r}{\Gamma(s)} \int_0^\infty t^{s-1} \left\{ \log(1 - e^{-t}) \right\}^r dt \qquad \big(\Re(s) > 1;\ r \in \mathbb{N}\big). \tag{2.6}$$

Next, for $m, r \in \mathbb{N}$, we introduce the following multiple sum:

$$P_m(s_1, \cdots, s_r) := \sum_{\substack{k_1 + \cdots + k_r = m \\ k_1, \cdots, k_r \geqq 1}} \frac{1}{k_1^{s_1} \cdots k_r^{s_r}} \qquad (m, r \in \mathbb{N}). \tag{2.7}$$

Using the Cauchy product of power series, we have

$$\begin{aligned}
&\prod_{j=1}^{r}\left\{\sum_{m_j=1}^{\infty}\frac{e^{-m_j t}}{m_j^{s_j}}\right\} \\
&= \sum_{m=1}^{\infty}\ \sum_{k_{r-1}=1}^{m}\ \sum_{k_{r-2}=1}^{m-k_{r-1}+1}\ \sum_{k_{r-3}=1}^{m-k_{r-1}-k_{r-2}+2} \\
&\qquad \cdots \sum_{k_1=1}^{m-k_{r-1}-\cdots-k_2+r-2}\frac{e^{-(m+r-1)t}}{k_1^{s_1}k_2^{s_2}\cdots k_{r-1}^{s_{r-1}}(m+r-1-k_1-k_2-\cdots-k_{r-1})^{s_r}} \\
&= \sum_{m=1}^{\infty}\ \sum_{\substack{k_1+\cdots+k_r=m+r-1\\ k_1,\cdots,k_r\geqq 1}}\frac{e^{-(m+r-1)t}}{k_1^{s_1}\cdots k_r^{s_r}},
\end{aligned} \tag{2.8}$$

so that

$$\prod_{j=1}^{r}\left\{\sum_{m_j=1}^{\infty}\frac{e^{-m_j t}}{m_j^{s_j}}\right\} = \sum_{m=r}^{\infty} e^{-mt}\, P_m(s_1,\ \cdots, s_r). \tag{2.9}$$

The relations (2.1), (2.2) and (2.9) yield the following result:

$$\zeta_{MT,r}(s_1,\ \cdots, s_r; s) = \sum_{m=r}^{\infty}\frac{P_m(s_1,\ \cdots, s_r)}{m^s}. \tag{2.10}$$

It is known that (*cf.* [6, p. 146, Equation (3)]; see also Equation (1.9) above)

$$\{\log(1-z)\}^r = r!\sum_{n=r}^{\infty}(-1)^n\ \mathfrak{s}(n,r)\frac{z^n}{n!}. \tag{2.11}$$

Since

$$\log(1-z) = -\sum_{n=1}^{\infty}\frac{z^n}{n},$$

we have

$$\{\log(1-z)\}^r = (-1)^r\sum_{n=r}^{\infty}P_n(\underbrace{1,\ \cdots, 1}_{r})z^n. \tag{2.12}$$

Thus, by comparing the coefficients of z^n in (2.11) and (2.12), we obtain the following relationship:

$$P_n(\underbrace{1,\ \cdots, 1}_{r}) = (-1)^{n+r}\ r!\ \frac{\mathfrak{s}(n,r)}{n!} \tag{2.13}$$

with the Stirling numbers $\mathfrak{s}(n,r)$ of the first kind.

The Stirling numbers $\mathfrak{s}(n,k)$ of the first kind satisfy a recurrence relation in the form (see [13, p. 56, Equation (3)]):

$$\mathfrak{s}(n+1,k) = \mathfrak{s}(n,k-1) - n\mathfrak{s}(n,k) \qquad (k=1,\ \cdots, n), \tag{2.14}$$

which, in conjunction with the relationship (2.12), yields

$$(n+1)P_{n+1}(\underbrace{1,\ \cdots,1}_{r}) = rP_n(\underbrace{1,\ \cdots,1}_{r-1}) + nP_n(\underbrace{1,\ \cdots,1}_{r}). \tag{2.15}$$

3. Main results

By examining the Eulerian beta integral:

$$\int_0^1 (1-t)^\lambda\, t^{\mu-1}\,dt = \frac{\Gamma(1+\lambda)\Gamma(1+\mu)}{\mu\,\Gamma(1+\lambda+\mu)} \qquad \big(\Re(\lambda) > -1;\ \Re(\mu) > 0\big), \tag{3.1}$$

in terms of the classical gamma function, Choi and Srivastava [3] introduced the following notation (see, for details, [3, p. 56]):

$$\frac{\Gamma(1+\lambda)\Gamma(1+\mu)}{\mu\,\Gamma(1+\lambda+\mu)} = \sum_{n=0}^{\infty} A_n(\mu)\lambda^n \qquad \big(A_0(\mu) = 1\big), \tag{3.2}$$

$$A_p(\mu) = \sum_{n=0}^{\infty} \Lambda_n^{(p)}\mu^n, \tag{3.3}$$

and

$$\psi(1+\lambda) - \psi(1+\lambda+\mu) = \sum_{n=0}^{\infty} B_n(\mu)\lambda^n \tag{3.4}$$

$$\big(B_0(\mu) = \psi(1) - \psi(1+\mu)\big),$$

where

$$B_n(\mu) = (-1)^{n+1}\,\zeta(n+1) - \frac{\psi^{(n)}(1+\mu)}{n!} \qquad (n \in \mathbb{N}) \tag{3.5}$$

and

$$(n+1)A_{n+1}(\mu) = \sum_{k=0}^{n} A_k(\mu)B_{n-k}(\mu) \qquad (n \in \mathbb{N}_0) \tag{3.6}$$

in terms of the polygamma functions $\psi^{(n)}(z)$ $(n \in \mathbb{N}_0)$ defined by

$$\psi^{(n)}(z) := \frac{d^{n+1}}{dz^{n+1}}\{\log\Gamma(z)\} = \frac{d^n}{dz^n}\{\psi(z)\} \qquad (n \in \mathbb{N}_0),$$

$$\psi^{(0)}(z) = \psi(z)$$

being the relatively more familiar psi (or digamma) function.

In the proof of one of our main results (see Theorem 1 below), we shall be using the following results of Choi and Srivastava [3, p. 56, Equation (2.3)] and Subbarao and Sitaramachandrarao [11, p. 250, Equation (3.4)]:

$$\int_0^1 \{\log(1-t)\}^p (\log t)^q\,\frac{dt}{t} = p!\ q!\ \Lambda_{q+1}^{(p)} \qquad (p, q \in \mathbb{N}_0) \tag{3.7}$$

and

$$\sum_{m_1,\ \cdots,m_r=1}^{\infty} \frac{1}{m_1\cdots m_r(m_1+\cdots+m_r+a)} = (-1)^r \int_0^1 t^{a-1}\{\log(1-t)\}^r\,dt, \tag{3.8}$$

respectively.

Remark 1. The above formulas (3.7) and (3.8) are the *corrected* versions of the corresponding results given in [3] and [11], respectively.

Theorem 1. *The following recursion formula holds true for* $\zeta_{MT,r}(1, \cdots, 1; s)$ *with respect to* r :

$$\zeta_{MT,r}(1, \cdots, 1; s) = \frac{(s+r-1)!}{s!}\zeta(s+r) - \sum_{n=1}^{s-1} \zeta(s-n+1)\zeta_{MT,r-1}(1, \cdots, 1; n)$$

$$+ \sum_{k=1}^{r-2}\left[\frac{(r-1)!}{k!}\zeta(r-k)\zeta_{MT,k}(1, \cdots, 1; s) - \binom{r-1}{k}\sum_{n=1}^{s}\frac{(s+r-n-k-1)!}{(s-n)!}\zeta(s+r-n-k)\zeta_{MT,k}(1, \cdots, 1; n)\right] \tag{3.9}$$

$$(r, s \in \mathbb{N}),$$

where an empty sum is interpreted to be nil.

Remark 2. It is easily observed from the assertion (3.9) of Theorem 1 that

$$\zeta_{MT,r}(1, \cdots, 1; 1) = r!\,\zeta(r+1) \qquad (r \in \mathbb{N}) \tag{3.10}$$

and

$$\zeta_{MT,1}(1; s) = \zeta(s+1) \qquad (s \in \mathbb{N}). \tag{3.11}$$

Proof of Theorem 1. Firstly, upon differentiating both members of (3.8) $s-1$ times with respect to a and then setting $a = 0$, we get

$$\zeta_{MT,r}(1, \cdots, 1; s) = \frac{(-1)^{s+r+1}}{(s-1)!}\int_0^1 \{\log(1-t)\}^r \,(\log t)^{s-1}\,\frac{dt}{t}. \tag{3.12}$$

Now, if we compare (3.12) with the case $p = r$ and $q = s-1$ of (3.7), we readily arrive at the following relationship:

$$\zeta_{MT,r}(1, \cdots, 1; s) = (-1)^{r+s+1}\,r!\,\Lambda_s^{(r)} \tag{3.13}$$

with the coefficients $\Lambda_n^{(p)}$ occurring in (3.3).

Secondly, in view of the following known expansion formula [1, p. 85, Equation (6.3.14)]:

$$\psi(z+1) = \psi(1) + \sum_{n=2}^{\infty}(-1)^n\,\zeta(n)z^{n-1} \qquad (|z| < 1),$$

we find from the equations (3.3) to (3.6) that

$$\begin{aligned}(r+1)\sum_{s=0}^{\infty}\Lambda_s^{(r+1)}\mu^s &= \sum_{k=0}^{r-1}A_k(\mu)B_{r-k}(\mu) + A_r(\mu)B_0(\mu)\\ &= \sum_{k=0}^{r-1}A_k(\mu)\left[(-1)^{r-k+1}\zeta(r-k+1) - \frac{\psi^{(r-k)}(1+\mu)}{(r-k)!}\right]\\ &\quad + A_r(\mu)\sum_{k=1}^{\infty}(-1)^k\,\zeta(k+1)\mu^k\end{aligned} \tag{3.14}$$

$$\big(\Re(\mu) > 0 \quad \text{and} \quad |\mu| < 1\big).$$

Now, by applying the known result [1, p. 86, Equation (6.4.9)]:

$$\psi^{(n)}(z+1) = (-1)^{n+1}\sum_{k=0}^{\infty}(-1)^k\,\frac{(n+k)!}{k!}\,\zeta(n+k+1)\,z^k \qquad (|z|<1)$$

in (3.14), we obtain

$$\begin{aligned}(r+1)\sum_{s=0}^{\infty}\Lambda_s^{(r+1)}\mu^s &= \sum_{s=0}^{\infty}\Lambda_s^{(r)}\mu^s\sum_{s=1}^{\infty}(-1)^s\zeta(s+1)\mu^s \\ &\quad + \sum_{k=0}^{r-1}\left(\sum_{s=0}^{\infty}\Lambda_s^{(k)}\mu^s\right)\cdot\Bigg[(-1)^{r-k+1}\,\zeta(r-k+1) \\ &\quad + \frac{1}{(r-k)!}\sum_{s=0}^{\infty}(-1)^{s+r-k}(s+1)_{r-k}\,\zeta(r-k+s+1)\mu^s\Bigg],\end{aligned} \tag{3.15}$$

where $(\kappa)_n$ represents the Pochhammer symbol (or the *shifted* factorial) given by

$$(\kappa)_0 = 1 \qquad \text{and} \qquad (\kappa)_n := \frac{\Gamma(\kappa+n)}{\Gamma(\kappa)} = \kappa(\kappa+1)\cdots(\kappa+n-1) \qquad (n\in\mathbb{N}).$$

Hence we have

$$\begin{aligned}(r+1)\sum_{s=0}^{\infty}\Lambda_s^{(r+1)}\mu^s &= \mu\left(\sum_{\ell=0}^{\infty}\Lambda_\ell^{(r)}\mu^\ell\right)\sum_{s=0}^{\infty}(-1)^{s+1}\zeta(s+2)\mu^s \\ &\quad + \sum_{\ell=0}^{\infty}\left(\sum_{k=0}^{r-1}(-1)^{r-k+1}\zeta(r-k+1)\Lambda_\ell^{(k)}\right)\mu^\ell \\ &\quad + \sum_{k=0}^{r-1}\frac{1}{(r-k)!}\left(\sum_{\ell=0}^{\infty}\Lambda_\ell^{(k)}\mu^\ell\right)\sum_{s=0}^{\infty}(-1)^{r-k+s}(s+1)_{r-k}\,\zeta(r-k+s+1)\mu^s,\end{aligned} \tag{3.16}$$

so that

$$\begin{aligned}(r+1)\sum_{s=0}^{\infty}\Lambda_s^{(r+1)}\mu^s &= \mu\sum_{s=0}^{\infty}\left(\sum_{n=0}^{s}\Lambda_n^{(r)}(-1)^{n-s+1}\zeta(s-n+2)\right)\mu^s \\ &\quad + \sum_{s=0}^{\infty}\left(\sum_{k=0}^{r-1}(-1)^{r-k+1}\zeta(r-k+1)\Lambda_s^{(k)}\right)\mu^s \\ &\quad + \sum_{k=0}^{r-1}\frac{1}{(r-k)!}\sum_{s=0}^{\infty}\Bigg(\sum_{n=0}^{s}\Lambda_n^{(k)}(-1)^{r-k+s-n}(s-n+1)_{r-k} \\ &\quad \cdot\zeta(r-k+s-n+1)\Bigg)\mu^s,\end{aligned}$$

that is, that

$$(r+1)\sum_{s=0}^{\infty}\Lambda_s^{(r+1)}\mu^s=\sum_{s=1}^{\infty}\left(\sum_{n=0}^{s-1}\Lambda_n^{(r)}(-1)^{n-s}\zeta(s-n+1)\right)\mu^s$$
$$+\sum_{s=1}^{\infty}\left(\sum_{k=0}^{r-1}(-1)^{r-k+1}\zeta(r-k+1)\Lambda_s^{(k)}\right)\mu^s$$
$$+\sum_{s=1}^{\infty}\left[\sum_{k=0}^{r-1}\frac{1}{(r-k)!}\left(\sum_{n=0}^{s}\Lambda_n^{(k)}(-1)^{r-k+s-n}(s-n+1)_{r-k}\right.\right.$$
$$\left.\left.\cdot\,\zeta(r-k+s-n+1)\right)\right]\mu^s. \tag{3.17}$$

We note here that the following recurrence relations hold true for $\Lambda_s^{(r)}$:

$$\Lambda_0^{(n)}=0,\quad \Lambda_0^{(0)}=1\quad\text{and}\quad \Lambda_n^{(0)}=0\qquad(n\in\mathbb{N}) \tag{3.18}$$

and

$$r\Lambda_s^{(r)}=\sum_{n=0}^{s-1}(-1)^{n-s}\zeta(s-n+1)\Lambda_n^{(r-1)}+\sum_{k=0}^{r-2}\left[(-1)^{r-k}\zeta(r-k)\Lambda_s^{(k)}\right.$$
$$\left.-\frac{1}{(r-k-1)!}\left(\sum_{n=0}^{s}(-1)^{r-k+s-n}\Lambda_n^{(k)}(s-n+1)_{r-k-1}\,\zeta(r-k+s-n)\right)\right]. \tag{3.19}$$

Formula (3.19) is obtained by comparing the coefficients of μ^s in (3.17).

Finally, using (3.18) and (3.19) in conjunction with (3.13), we have the assertion (3.9) of Theorem 1. The proof of Theorem 1 is thus completed. □

Remark 3. The equations in (3.18) follow from (3.2) and (3.17).

Example. For $s\in\mathbb{N}\setminus\{1\}$ and $r=2,3$, Theorem 1 would provide the following special cases:

$$\zeta_{MT,2}(1,1;s)=(s+1)\zeta(s+2)-\sum_{n=1}^{s-1}\zeta(s-n+1)\zeta(n+1) \tag{3.20}$$

and

$$\zeta_{MT,3}(1,1,1;s)=(s+1)(s+2)\zeta(s+3)+2\zeta(2)\zeta(s+1)$$
$$-\sum_{n=1}^{s-1}(n+1)\zeta(s-n+1)\zeta(n+2)$$
$$+\sum_{n=2}^{s-1}\zeta(s-n+1)\sum_{m=1}^{n-1}\zeta(n-m+1)\zeta(m+1)$$
$$-2\sum_{n=1}^{s}(s-n+1)\zeta(s-n+2)\zeta(n+1). \tag{3.21}$$

Equation (3.20) is a known result given in [11, p. 247, Equation (2.8)]. Equation (3.21), on the other hand, is equivalent to another known result [7, p. 128, Corollary 4.3]. Thus, for $r \in \mathbb{N} \setminus \{1, 2\}$, the assertion (3.9) of Theorem 1 is presumably new.

We next establish a recursion formula given by Theorem 2 below.

Theorem 2. *The following recursion formula holds true for* $\zeta_{MT,r}(0, \cdots, 0; s)$:

$$\zeta_{MT,r}(0, \cdots, 0; s) = \frac{1}{(r-1)!} \sum_{m=1}^{r} \mathfrak{s}(r, m)\zeta(s-m+1) \qquad \big(\Re(s) > r;\ r \in \mathbb{N}\big), \tag{3.22}$$

where $\mathfrak{s}(n, k)$ *denotes the Stirling numbers of the first kind given by* (1.8) *and* (1.9).

Proof. By integration by parts, it is easily seen that

$$\frac{s-1}{r} \int_0^\infty \frac{t^{s-2}}{(e^t-1)^r}\, dt = \int_0^\infty \frac{e^t t^{s-1}}{(e^t-1)^{r+1}}\, dt \qquad \big(\Re(s) > r+1;\ r \in \mathbb{N}\big),$$

so that

$$\begin{aligned} &\frac{1}{\Gamma(s)} \int_0^\infty \frac{t^{s-1}}{(e^t-1)^{r+1}}\, dt \\ &= \frac{1}{r\Gamma(s-1)} \int_0^\infty \frac{t^{s-2}}{(e^t-1)^r}\, dt - \frac{1}{\Gamma(s)} \int_0^\infty \frac{t^{s-1}}{(e^t-1)^r}\, dt \\ &\qquad \big(\Re(s) > r+1;\ r \in \mathbb{N}\big). \end{aligned} \tag{3.23}$$

Thus, by using the expression (2.5) in (3.23), we get the following recursion formula:

$$\zeta_{MT,r+1}(0, \cdots, 0; s) = \frac{1}{r}\, \zeta_{MT,r}(0, \cdots, 0; s-1) - \zeta_{MT,r}(0, \cdots, 0; s). \tag{3.24}$$

Since

$$\zeta_{MT,1}(0; s) = \zeta(s),$$

the proof of Theorem 2 can be completed by mathematical induction on r in light of (2.14) and (3.24). □

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