A SEPARABLE DEFORMATION OF THE QUATERNION GROUP ALGEBRA

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(Communicated by Martin Lorenz)

Abstract. The Donald-Flanigan conjecture asserts that for any finite group G and any field k, the group algebra kG can be deformed to a separable algebra. The minimal unsolved instance, namely the quaternion group Q_8 over a field k of characteristic 2 was considered as a counterexample. We present here a separable deformation of kQ_8. In a sense, the conjecture for any finite group is open again.

1. Introduction

In their paper [1], J. D. Donald and F. J. Flanigan conjectured that any group algebra kG of a finite group G over a field k can be deformed to a semisimple algebra even in the modular case, namely where the order of G is not invertible in k. A more customary formulation of the Donald-Flanigan (DF) conjecture is by demanding that the deformed algebra [kG]_t should be separable; i.e., it remains semisimple when tensored with the algebraic closure of its base field. If, additionally, the dimensions of the simple components of [kG]_t are in one-to-one correspondence with those of the complex group algebra C[G], then [kG]_t is called a strong solution to the problem.

The DF conjecture was solved for different families of groups, using different methods:

- when the group G has a cyclic p-Sylow subgroup and k is an algebraically closed field of characteristic p [12];
- when G has a normal abelian p-Sylow subgroup in characteristic p [6];
- dihedral groups [3] and semi-dihedral groups [2] in characteristic 2;
- reflection groups in any characteristic (with six exceptions) [7, 8, 11].

In [5], it is claimed that the group algebra kQ_8, where

Q_8 = \langle \sigma, \tau | \sigma^4 = 1, \tau \sigma = \sigma^{-3} \tau, \sigma^2 = \tau^2 \rangle

is the quaternion group of order 8 and k is a field of characteristic 2, does not admit a separable deformation. This result allegedly gives a counterexample to the DF conjecture. However, as observed by M. Schaps, the proof apparently contains an error (see [7]).

Received by the editors April 23, 2007.
2000 Mathematics Subject Classification. Primary 16S80.

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The aim of this paper is to present a separable deformation of $kQ_8$, where $k$ is any field of characteristic 2, reopening the DF conjecture. The same approach works for the generalized quaternion group algebras and will be introduced separately.

2. Preliminaries

Let $k[[t]]$ be the ring of formal power series over $k$, and let $k((t))$ be its field of fractions. Recall that the deformed algebra $[kG]_t$ has the same underlying $k((t))$-vector space as $k((t)) \otimes_k kG$, with multiplication defined on basis elements

\[(2.1) \quad g_1 * g_2 := g_1 g_2 + \sum_{i \geq 1} \Psi_i(g_1,g_2)t^i, \quad g_1, g_2 \in G\]

and extended $k((t))$-linearly (such that $t$ is central). Here $g_1g_2$ is the group multiplication. The functions $\Psi_i : G \times G \to kG$ satisfy certain cohomological conditions induced by the associativity of $[kG]_t$, \[\text{(\S1, 2)}\].

Note that the set of equations (2.1) determines a multiplication on the free $k[[t]]$-module $\Lambda_t$ spanned by the elements $\{g\}_{g \in G}$ such that $kG \simeq \Lambda_t/\langle t\Lambda_t \rangle$ and $[kG]_t \simeq \Lambda_t \otimes_{k[[t]]} k((t))$. In a more general context, namely over a domain $R$ which is not necessarily local, the $R$-module $\Lambda_t$ which determines the deformation, is required only to be flat rather than free \[\text{\S1}\].

In what follows, we shall define the deformed algebra $[kG]_t$ by using generators and relations. These will implicitly determine the set of equations (2.1).

3. Sketch of the construction

Consider the extension

\[(3.1) \quad [\beta] : 1 \to C_4 \to Q_8 \to C_2 \to 1,\]

where $C_2 = \langle \tau \rangle$ acts on $C_4 = \langle \sigma \rangle$ by

\[
\eta : \quad C_2 \to \text{Aut}(C_4), \\
\eta(\tau) : \quad \sigma \mapsto \sigma^2 = (\sigma^{-1}),
\]

and the associated 2-cocycle $\beta : C_2 \times C_2 \to C_4$ is given by

\[
\beta(1,1) = \beta(1,\tau) = \beta(\tau,1) = 1, \quad \beta(\tau,\tau) = \sigma^2.
\]

The group algebra $kQ_8$ ($k$ any field) is isomorphic to the quotient $kC_4[y;\eta]/\langle q(y) \rangle$, where $kC_4[y;\eta]$ is a skew polynomial ring [10, \S1.2], whose indeterminate $y$ acts on the ring of coefficients $kC_4$ via the automorphism $\eta(\tau)$ (extended linearly) and where

\[(3.2) \quad q(y) := y^2 - \sigma^2 \in kC_4[y;\eta]\]

is central. The above isomorphism is established by identifying $\tau$ with the indeterminate $y$.

Suppose now that $\text{Char}(k) = 2$. The deformed algebra $[kQ_8]_t$ is constructed as follows.

In \[\text{\S1.1}\] the subgroup algebra $kC_4$ is deformed to a separable algebra $[kC_4]_t$ which is isomorphic to $K \oplus k((t)) \oplus k((t))$, where $K$ is a separable field extension of $k((t))$ of degree 2.

The next step (\[\text{\S1.2}\]) is to construct an automorphism $\eta_t$ of $[kC_4]_t$ which agrees with the action of $C_2$ on $kC_4$ when specializing $t = 0$. This action fixes all three
Then the quotient $k$ is an irreducible (separable) polynomial in $[kC_4]_t/y; \eta_t$. In $[k]_t$ we deform $q(x) = y^2 + \sigma^2$ to $q_t(y)$, a separable polynomial of degree 2 in the center of $[kC_4]_t/y; \eta_t$.

By factoring out the two-sided ideal generated by $q_t(y)$, we establish the deformation

$$[kQ_8]_t := [kC_4]_t/y; \eta_t)/(q_t(y)).$$

In $[kC_4]_t$ we show that $[kQ_8]_t$ as above is separable. Moreover, passing to the algebraic closure $K((t))$ we have

$$[kQ_8]_t \otimes_{K((t))} K((t)) \simeq \bigoplus_{i=1}^4 K((t)) \oplus M_2(K((t))).$$

This is a strong solution to the DF conjecture since its decomposition to simple components is the same as

$$CQ_8 \simeq \bigoplus_{i=1}^4 C \oplus M_2(C).$$

4. A deformation of $kC_4[y; \eta]$

4.1. We begin by constructing $[kC_4]_t$, $C_4 = \langle \sigma \rangle$. Recall that

$$kC_4 \simeq k[x]/(x^4 + 1)$$

by identifying $\sigma$ with $x + (x^4 + 1)$. We deform the polynomial $x^4 + 1$ to a separable polynomial $p_t(x)$ as follows.

Let $k[[t]]^*$ be the group of invertible elements of $k[[t]]$ and denote by

$$U := \{1 + zt | z \in k[[t]]^* \}$$

its subgroup of 1-units (when $k = \mathbb{F}_2$, $U$ is equal to $k[[t]]^*$).

Let $a \in k[[t]] \setminus k[[t]]^*$

be a non-zero element, and let

$$b, c, d \in U(c \neq d),$$

such that

$$\pi(x) := x^2 + ax + b$$

is an irreducible (separable) polynomial in $k((t))[x]$. Let

$$p_t(x) := \pi(x)(x + c)(x + d) \in k((t))[x].$$

Then the quotient $k((t))[x]/(p_t(x))$ is isomorphic to the direct sum $K \oplus k((t)) \oplus k((t))$, where $K := k((t))[x]/(\pi(x))$. The field extension $K/k((t))$ is separable and of dimension 2.

Note that $p_{t=0}(x) = x^4 + 1$ and that only lower order terms of the polynomial were deformed. Hence, the quotient $k[[t]][x]/(p_t(x))$ is $k[[t]]$-free and $k((t))[x]/(p_t(x))$ indeed defines a deformation $[kC_4]_t$ of $kC_4 \simeq k[x]/(x^4 + 1)$. The new multiplication $\sigma^t * \sigma^t$ of basis elements (2.3) is determined by identifying $\sigma^t$ with $t := x^4 + \langle p_t(x) \rangle$.

We shall continue to use the term $x$ in $[kC_4]_t$, rather than $\sigma$.

Assume further that there exists $w \in k[[t]]$ such that

$$\langle x + w \rangle (x + c)(x + d) = x\pi(x) + a$$

(4.1)
4.2. Let
\[ e_1 = \frac{(\bar{x} + w)(\bar{x} + c)(\bar{x} + d)}{a}. \]
The other two primitive idempotents of \([kC_4]_t\) are
\[ e_2 = \frac{c(\bar{x} + d)\pi(\bar{x})}{a(c + d)}, \quad e_3 = \frac{d(\bar{x} + c)\pi(\bar{x})}{a(c + d)}. \]

Similarly,
\[ e = \frac{c(\bar{x} + d)\pi(\bar{x})}{a(c + d)} \]
be an algebra endomorphism determined by its value on the generator \(x\) as follows:
\[ \eta_t : k((t))[x] \to k((t))[x] \]
We compute \(\eta_t(\pi(x)), \eta_t(x + c)\) and \(\eta_t(x + d)\):
\[ \eta_t(\pi(x)) = \eta_t(x)^2 + a\eta_t(x) + b = x^2\pi(x)^2 + x^2 + a^2 + ax\pi(x) + ax + a^2 + b = \pi(x)(x^2\pi(x) + ax + 1). \]
By \((4.1)\),
\[ \eta_t(x) := x\pi(x) + x + a. \]
Next,
\[ \eta_t(x + c) = x\pi(x) + x + a + c. \]
By \((4.1)\),
\[ \eta_t(x + c) = (x + c)[(x + w)(x + d) + 1] \in \langle x + c \rangle. \]
Similarly,
\[ \eta_t(x + d) = (x + d)[(x + w)(x + c) + 1] \in \langle x + d \rangle. \]
By \((4.5), (4.6)\) and \((4.7)\), we obtain that \(\eta_t(p_t(x)) \in \langle p_t(x) \rangle\), and hence \(\eta_t\) induces an endomorphism of \(k((t))[x]/\langle p_t(x) \rangle\), which we continue to denote by \(\eta_t\). As can easily be verified, the primitive idempotents given in \((4.2)\) and \((4.3)\) are fixed under \(\eta_t\):
\[ \eta_t(e_i) = e_i, \quad i = 1, 2, 3, \]
whereas
\[ \eta_t(\bar{x}e_1) = \eta_t(\bar{x})e_1 = (\bar{x}\pi(\bar{x}) + \bar{x} + a)e_1 = (\bar{x} + a)e_1. \]
Hence, \(\eta_t\) induces an automorphism of \(K\) of order 2 while fixing the two copies of \(k((t))\) pointwise. Furthermore, one can easily verify that
\[ \eta_{t=0}(\bar{x}) = \bar{x}^3. \]
Consequently, the automorphism \(\eta_t\) of \([kC_4]_t\) agrees with the automorphism \(\eta(\bar{t})\) of \(kC_4\) when \(t = 0\). The skew polynomial ring
\[ [kC_4]_t[y; \eta_t] = (k((t))[x]/\langle p_t(x) \rangle)[y; \eta_t] \]
is therefore a deformation of \(kC_4[y; \eta]\).

Note that by \((4.8)\), the idempotents \(e_i, i = 1, 2, 3\) are central in \([kC_4]_t[y; \eta_t]\) and hence
\[ [kC_4]_t[y; \eta_t] = \bigoplus_{i=1}^3[kC_4]_t[y; \eta_t]e_i. \]
4.3. Example. The following is an example for the above construction. Put
\[ a := \frac{t + t^2 + t^3}{1 + t}, \quad b := 1 + t^2 + t^3, \quad c := \frac{1}{1 + t}, \quad d := 1 + t + t^2, \quad w := t. \]
These elements satisfy equation (4.1):
\[ (x + w)(x + c)(x + d) = (x + t)(x + \frac{1}{1 + t})(x + 1 + t + t^2) \]
\[ = x^3 + \frac{t + t^2 + t^3}{1 + t} x^2 + (1 + t^2 + t^3) x + \frac{t + t^2 + t^3}{1 + t} = x\pi(x) + a. \]
The polynomial
\[ \pi(x) = x^2 + \frac{t + t^2 + t^3}{1 + t} x + 1 + t^2 + t^3 \]
does not admit roots in \( k[[t]]/(t^2) \); thus it is irreducible over \( k((t)) \).

5. A deformation of \( q(y) \)

The construction of \( [kQ_8]_t \) will be completed once the product \( \bar{\tau} \ast \bar{r} \) is defined. For this purpose the polynomial \( q(y) \) (3.2), which determined the ordinary multiplication \( \tau \ast \), will now be developed in powers of \( t \).

For any non-zero element \( z \in k[[t]] \setminus k[[t]]^* \), let
\[ q_t(y) := y^2 + z\tilde{x}\pi(\tilde{x})y + \tilde{x}^2 + a\tilde{x} \in [kC_4]_t[y;\eta_t]. \]
The decomposition of (5.1) with respect to the idempotents \( e_1, e_2, e_3 \) yields
\[ q_t(y) = (y^2 + b)e_1 + [y^2 + zay + c(c + a)]e_2 + [y^2 + zay + d(d + a)]e_3. \]
We now show that \( q_t(y) \) is in the center of \( [kC_4]_t[y;\eta_t] \).

First, the leading term \( y^2 \) is central since the automorphism \( \eta_t \) is of order 2. Next, by (1.8), the free term \( b e_1 + c(c + a)e_2 + d(d + a)e_3 \) is invariant under the action of \( \eta_t \) and hence central. It is left to check that the term \( za(e_2 + e_3)y \) is central. Indeed, since \( e_2 \) and \( e_3 \) are \( \eta_t \)-invariant, then \( za(e_2 + e_3)y \) commutes both with \([kC_4]_t[y;\eta_t]e_2\) and \([kC_4]_t[y;\eta_t]e_3\). Furthermore, by orthogonality,
\[ za(e_2 + e_3)y \cdot [kC_4]_t[y;\eta_t]e_1 = [kC_4]_t[y;\eta_t]e_1 \cdot za(e_2 + e_3)y = 0, \]
and hence \( za(e_2 + e_3)y \) commutes with \([kC_4]_t[y;\eta_t]\).

Consequently, \( \langle q_t(y) \rangle = q_t(y)[kC_4]_t[y;\eta_t] \) is a two-sided ideal.

Now, as can easily be deduced from (5.1),
\[ q_{t=0}(y) = y^2 + \tilde{x}^2 = q(y), \]
where the leading term \( y^2 \) remains unchanged. Then
\[ [kQ_8]_t := [kC_4]_t[y;\eta_t]/\langle q_t(y) \rangle \]
is a deformation of \( kQ_8 \), identifying \( \tilde{\tau} \) with \( \tilde{y} := y + \langle q_t(y) \rangle \).
6. Separability of \([kQ_8]_t\)

Finally, we need to prove that the deformed algebra \([kQ_8]_t\) is separable. Moreover, we prove that its decomposition to simple components over the algebraic closure of \(k((t))\) resembles that of \(CQ_8\). By (4.10), we obtain

\[
[kQ_8]_t = \bigoplus_{i=1}^{3} [kC_4]_t[y; \eta_i]/(q_i(y)e_i).
\]

We handle the three summands in (6.1) separately.

By (6.2),

\[
[kC_4]_t[y; \eta_i]/(q_i(y)e_i) \simeq K[y; \eta_i]/(y^2 + b) \simeq K \ast_f C_2.
\]

The rightmost term is the crossed product of the group \(C_2 := \langle \tilde{\tau} \rangle\) acting faithfully on the field \(K = [kC_4]_t e_1\) via \(\eta_i\) (4.9), with a twisting determined by the 2-cocycle \(f : C_2 \times C_2 \rightarrow K^\ast\):

\[
f(1, 1) = f(1, \tilde{\tau}) = f(\tilde{\tau}, 1) = 1, \quad f(\tilde{\tau}, \tilde{\tau}) = b.
\]

This is a central simple algebra over the subfield of invariants \(k((t))\) [7, Theorem 4.4.1]. Evidently, this simple algebra is split by \(k((t))\), i.e.

\[
[kC_4]_t[y; \eta_i]/(q_i(y)e_i) \otimes_{k((t))} k((t)) \simeq M_2(k((t))).
\]

(In fact, \(K \ast_f C_2\) splits already over \(k((t))\), since \(b\) is a \(C_2\)-norm of a root of the irreducible polynomial \(\pi(x)\) and therefore \(f\) is cohomologically trivial.)

Next, since \(\eta_i\) is trivial on \([kC_4]_t e_2\), the skew polynomial ring \([kC_4]_t e_2[y; \eta_i]\) is actually an ordinary polynomial ring \(k((t))[y]\). Again by (5.2),

\[
[kC_4]_t[y; \eta_i]/(q_i(y)e_2) \simeq k((t))[y]/(y^2 + za + c(c + a)).
\]

Similarly,

\[
[kC_4]_t[y; \eta_i]/(q_i(y)e_3) \simeq k((t))[y]/(y^2 + za + d(d + a)).
\]

The polynomials \(y^2 + za + c(c + a)\) and \(y^2 + za + d(d + a)\) are separable (since \(za\) is non-zero). Thus, both \([kC_4]_t[y; \eta_i]/(q_i(y)e_2)\) and \([kC_4]_t[y; \eta_i]/(q_i(y)e_3)\) are separable \(k((t))-\)algebras, and for \(i = 2, 3\),

\[
[kC_4]_t[y; \eta_i]/(q_i(y)e_i) \otimes_{k((t))} k((t)) \simeq k((t)) \otimes k((t)).
\]

Equations (6.1), (6.2) and (6.3) yield

\[
[kQ_8]_t \otimes_{k((t))} k((t)) \simeq \bigoplus_{i=1}^{4} k((t)) \otimes M_2(k((t))),
\]

as required.

7. Acknowledgement

We wish to thank M. Schaps for pointing out to us that there is an error in the attempted proof in [5] that the quaternion group is a counterexample to the DF conjecture. Here is her explanation: The given relations for the group algebra are incorrect. Using the notation in pages 166-7 of [4], if \(a = 1 + i, b = 1 + j\) and \(z = i^2 = j^2\), then \(ab + ba = ij(1 + z)\), while \(a^2 = b^2 = 1 + z\). There is a further error later on when the matrix algebra is deformed to four copies of the field, since a non-commutative algebra can never have a flat deformation to a commutative algebra.
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