A SEPARABLE DEFORMATION OF THE QUATERNION GROUP ALGEBRA

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ABSTRACT. The Donald-Flanigan conjecture asserts that for any finite group $G$ and any field $k$, the group algebra $kG$ can be deformed to a separable algebra. The minimal unsolved instance, namely the quaternion group $Q_8$ over a field $k$ of characteristic 2 was considered as a counterexample. We present here a separable deformation of $kQ_8$. In a sense, the conjecture for any finite group is open again.

1. Introduction

In their paper [1], J. D. Donald and F. J. Flanigan conjectured that any group algebra $kG$ of a finite group $G$ over a field $k$ can be deformed to a semisimple algebra even in the modular case, namely where the order of $G$ is not invertible in $k$. A more customary formulation of the Donald-Flanigan (DF) conjecture is by demanding that the deformed algebra $[kG]_t$ should be separable; i.e., it remains semisimple when tensored with the algebraic closure of its base field. If, additionally, the dimensions of the simple components of $[kG]_t$ are in one-to-one correspondence with those of the complex group algebra $C^G$, then $[kG]_t$ is called a strong solution to the problem.

The DF conjecture was solved for different families of groups, using different methods:

- when the group $G$ has a cyclic $p$-Sylow subgroup and $k$ is an algebraically closed field of characteristic $p$ [12];
- when $G$ has a normal abelian $p$-Sylow subgroup in characteristic $p$ [6];
- dihedral groups [3] and semi-dihedral groups [2] in characteristic 2;
- reflection groups in any characteristic (with six exceptions) [7, 8, 11].

In [5], it is claimed that the group algebra $kQ_8$, where

$$Q_8 = \langle \sigma, \tau | \sigma^4 = 1, \tau \sigma = \sigma^3 \tau, \sigma^2 = \tau^2 \rangle$$

is the quaternion group of order 8 and $k$ is a field of characteristic 2, does not admit a separable deformation. This result allegedly gives a counterexample to the DF conjecture. However, as observed by M. Schaps, the proof apparently contains an error (see [7]).
The aim of this paper is to present a separable deformation of $kQ_8$, where $k$ is any field of characteristic 2, reopening the DF conjecture. The same approach works for the generalized quaternion group algebras and will be introduced separately.

2. Preliminaries

Let $k[[t]]$ be the ring of formal power series over $k$, and let $k(t)$ be its field of fractions. Recall that the deformed algebra $[kG]_t$ has the same underlying $k(t)$-vector space as $k(t)$, with multiplication defined on basis elements

\[(2.1) \quad g_1 * g_2 := g_1 g_2 + \sum_{i \geq 1} \Psi_i(g_1, g_2) t^i, \quad g_1, g_2 \in G \]

and extended $k((t))$-linearly (such that $t$ is central). Here $g_1 g_2$ is the group multiplication. The functions $\Psi_i : G \times G \to kG$ satisfy certain cohomological conditions induced by the associativity of $[kG]_t$. \cite[§§1, 2]{3}.

Note that the set of equations (2.1) determines a multiplication on the free $k[[t]]$-module $\Lambda_t$ spanned by the elements $\{g\}_{g \in G}$ such that $kG \simeq \Lambda_t/\langle t \Lambda_t \rangle$ and $[kG]_t \simeq \Lambda_t \otimes_{k[[t]]} k((t))$. In a more general context, namely over a domain $R$ which is not necessarily local, the $R$-module $\Lambda_t$ which determines the deformation, is required only to be flat rather than free \cite[§1]{3}.

In what follows, we shall define the deformed algebra $[kG]_t$ by using generators and relations. These will implicitly determine the set of equations (2.1).

3. Sketch of the construction

Consider the extension

\[(3.1) \quad [\beta] : 1 \to C_4 \to Q_8 \to C_2 \to 1, \]

where $C_2 = \langle \bar{\tau} \rangle$ acts on $C_4 = \langle \sigma \rangle$ by

\[
\eta : C_2 \to \text{Aut}(C_4), \\
\eta(\bar{\tau}) : \sigma \mapsto \sigma^2 = \sigma^{-1},
\]

and the associated 2-cocycle $\beta : C_2 \times C_2 \to C_4$ is given by

\[
\beta(1, 1) = \beta(1, \bar{\tau}) = \beta(\bar{\tau}, 1) = 1, \quad \beta(\bar{\tau}, \bar{\tau}) = \sigma^2.
\]

The group algebra $kQ_8$ ($k$ any field) is isomorphic to the quotient $kC_4[y; \eta]/\langle q(y) \rangle$, where $kC_4[y; \eta]$ is a skew polynomial ring \cite[§1.2]{10}, whose indeterminate $y$ acts on the ring of coefficients $kC_4$ via the automorphism $\eta(\bar{\tau})$ (extended linearly) and where

\[(3.2) \quad q(y) := y^2 - \sigma^2 \in kC_4[y; \eta]
\]

is central. The above isomorphism is established by identifying $\tau$ with the indeterminate $y$.

Suppose now that $\text{Char}(k) = 2$. The deformed algebra $[kQ_8]_t$ is constructed as follows.

In \cite[4.1]{1} the subgroup algebra $kC_4$ is deformed to a separable algebra $[kC_4]_t$ which is isomorphic to $K \oplus k((t)) \oplus k((t))$, where $K$ is a separable field extension of $k((t))$ of degree 2.

The next step (4.2) is to construct an automorphism $\eta_t$ of $[kC_4]_t$ which agrees with the action of $C_2$ on $kC_4$ when specializing $t = 0$. This action fixes all three
Then the quotient

\[ [kQ_S]_t := [kC_4][y; \eta_t]/\langle q_t(y) \rangle. \]

In \[4\] we show that \([kQ_S]_t\) as above is separable. Moreover, passing to the algebraic closure \(k((t))\) we have

\[ [kQ_S]_t \otimes_{k((t))} k((t)) \cong \bigoplus_{i=1}^{4} k((t)) \oplus M_2(k((t))). \]

This is a strong solution to the DF conjecture since its decomposition to simple components is the same as

\[ CQ_S \cong \bigoplus_{i=1}^{4} C \oplus M_2(C). \]

4. A DEFORMATION OF \(kC_4[y; \eta]\)

4.1. We begin by constructing \([kC_4], C_4 = \langle \sigma \rangle\). Recall that

\[ kC_4 \cong k[x]/\langle x^4 + 1 \rangle \]

by identifying \(\sigma\) with \(x + \langle x^4 + 1 \rangle\). We deform the polynomial \(x^4 + 1\) to a separable polynomial \(p_t(x)\) as follows.

Let \(k[[t]]^*\) be the group of invertible elements of \(k[[t]]\) and denote by

\[ U := \{1 + zt | z \in k[[t]]^* \} \]

its subgroup of 1-units (when \(k = \mathbb{F}_2\), \(U\) is equal to \(k[[t]]^*\)).

Let

\[ a \in k[[t]] \setminus k[[t]]^* \]

be a non-zero element, and let

\[ b, c, d \in U(c \neq d), \]

such that

\[ \pi(x) := x^2 + ax + b \]

is an irreducible (separable) polynomial in \(k((t))[x]\). Let

\[ p_t(x) := \pi(x)(x + c)(x + d) \in k((t))[x]. \]

Then the quotient \(k((t))[x]/\langle p_t(x) \rangle\) is isomorphic to the direct sum \(K \cong k((t)) \oplus k((t))\), where \(K := k((t))[x]/\langle \pi(x) \rangle\). The field extension \(K/k((t))\) is separable and of dimension 2.

Note that \(p_{t=0}(x) = x^4 + 1\) and that only lower order terms of the polynomial were deformed. Hence, the quotient \(k[[t]][x]/\langle p_t(x) \rangle\) is \(k[[t]]\)-free and \(k((t))[x]/\langle p_t(x) \rangle\) indeed defines a deformation \([kC_4], kC_4 \cong k[x]/\langle x^4 + 1 \rangle\). The new multiplication \(\sigma^1 \circ \sigma^1\) of basis elements \([2, 1]\) is determined by identifying \(\sigma^1\) with \(\bar{x} := x^4 + \langle p_t(x) \rangle\). We shall continue to use the term \(\bar{x}\) in \([kC_4]\), rather than \(\sigma\).

Assume further that there exists \(w \in k[[t]]\) such that

\[ (x + w)(x + c)(x + d) = x\pi(x) + a \]

primitive idempotents of \([kC_4]\). In that way, we obtain the skew polynomial ring \([kC_4], [y; \eta_t]\).

In \[3\] we deform \(q(y) = y^2 + \sigma^2\) to \(q_t(y)\), a separable polynomial of degree 2 in the center of \([kC_4], [y; \eta_t]\).

By factoring out the two-sided ideal generated by \(q_t(y)\), we establish the deformation

\[ [kQ_S]_t := [kC_4][y; \eta_t]/\langle q_t(y) \rangle. \]
(see Example 4.3). Then $K \simeq ([kC_4][t])e_1$, where

$$e_1 = \frac{(\bar{x} + w)(\bar{x} + c)(\bar{x} + d)}{a}.$$  

(4.2)

The other two primitive idempotents of $[kC_4][t]$ are

$$e_2 = \frac{c(\bar{x} + d)\pi(\bar{x})}{a(c + d)}, \quad e_3 = \frac{d(\bar{x} + c)\pi(\bar{x})}{a(c + d)}.$$  

(4.3)

4.2. Let

$$\eta : k((t))[x] \to k((t))[x]$$

be an algebra endomorphism determined by its value on the generator $x$ as follows:

$$\eta(x) := x\pi(x) + x + a.$$  

(4.4)

We compute $\eta(\pi(x))$, $\eta(x + c)$ and $\eta(x + d)$:

$$\eta(\pi(x)) = \eta(x)^2 + a\eta(x) + b = x^2\pi(x)^2 + x^2 + a^2 + ax\pi(x) + ax + a^2 + b$$

$$= \pi(x)(x^2\pi(x) + ax + 1).$$

By (4.1),

$$\eta(x) = x\pi(x) + x + a.$$  

(4.5)

Next,

$$\eta(x + c) = x\pi(x) + x + a + c.$$  

(4.6)

Similarly,

$$\eta(x + d) = (x + d)[(x + w)(x + c) + 1] \in \langle x + d \rangle.$$  

(4.7)

By (4.5), (4.6) and (4.7), we obtain that $\eta(p_t(x)) \in \langle p_t(x) \rangle$, and hence $\eta$ induces an endomorphism of $k((t))[x]/\langle p_t(x) \rangle$, which we continue to denote by $\eta$. As can easily be verified, the primitive idempotents given in (4.2) and (4.3) are fixed under $\eta$:

$$\eta(e_i) = e_i, \quad i = 1, 2, 3,$$

(4.8)

whereas

$$\eta(\bar{x}e_1) = \eta(\bar{x})e_1 = (\bar{x}\pi(\bar{x}) + \bar{x} + a)e_1 = (\bar{x} + a)e_1.$$  

(4.9)

Hence, $\eta$ induces an automorphism of $K$ of order 2 while fixing the two copies of $k((t))$ pointwise. Furthermore, one can easily verify that

$$\eta_{t=0}(\bar{x}) = \bar{x}^3.$$  

Consequently, the automorphism $\eta_t$ of $[kC_4][t]$ agrees with the automorphism $\eta(\bar{\tau})$ of $kC_4$ when $t = 0$. The skew polynomial ring

$$[kC_4][y; \eta] = (k((t))[x]/\langle p_t(x) \rangle)[y; \eta]$$

is therefore a deformation of $kC_4[y; \eta]$.

Note that by (4.8), the idempotents $e_i, i = 1, 2, 3$ are central in $[kC_4][y; \eta]$ and hence

$$[kC_4][y; \eta] = \bigoplus_{i=1}^{3} [kC_4][y; \eta] e_i.$$  

(4.10)
4.3. Example. The following is an example for the above construction.

Put
\[ a := \frac{t + t^2 + t^3}{1 + t}, \quad b := 1 + t^2 + t^3, \quad c := \frac{1}{1 + t}, \quad d := 1 + t + t^2, \quad w := t. \]

These elements satisfy equation (4.1):
\[
(x + w)(x + c)(x + d) = (x + t)(x + \frac{1}{1 + t})(x + 1 + t + t^2)
\]
\[ = x^3 + \frac{t + t^2 + t^3}{1 + t} x^2 + \frac{(1 + t^2 + t^3)x + t + t^2 + t^3}{1 + t} = x\pi(x) + a. \]

The polynomial
\[ \pi(x) = x^2 + \frac{t + t^2 + t^3}{1 + t} x + 1 + t^2 + t^3 \]

does not admit roots in \( k[[t]]/(t^2) \); thus it is irreducible over \( k((t)) \).

5. A deformation of \( q(y) \)

The construction of \( [kQ_8]_t \) will be completed once the product \( \bar{\tau} \ast \bar{\eta} \) is defined. For this purpose the polynomial \( q(y) \) (3.2), which determined the ordinary multiplication \( \tau^2 \), will now be developed in powers of \( t \).

For any non-zero element \( z \in k[[t]] \setminus k[[t]]^* \), let
\[ q_t(y) := y^2 + z\bar{\pi}(\bar{x})y + \bar{x}^2 + a\bar{x} \in [kC_4]_t[y; \eta_t]. \]

The decomposition of (5.1) with respect to the idempotents \( e_1, e_2, e_3 \) yields
\[ q_t(y) = (y^2 + b)e_1 + [y^2 + zay + c(c + a)]e_2 + [y^2 + zay + d(d + a)]e_3. \]

We now show that \( q_t(y) \) is in the center of \( [kC_4]_t[y; \eta_t] \).

First, the leading term \( y^2 \) is central since the automorphism \( \eta_t \) is of order 2. Next, by (1.8), the free term \( b e_1 + c(c + a) e_2 + d(d + a) e_3 \) is invariant under the action of \( \eta_t \) and hence central. It is left to check that the term \( za(e_2 + e_3)y \) is central. Indeed, since \( e_2 \) and \( e_3 \) are \( \eta_t \)-invariant, then \( za(e_2 + e_3)y \) commutes both with \( [kC_4]_t[y; \eta_t]e_2 \) and \( [kC_4]_t[y; \eta_t]e_3 \). Furthermore, by orthogonality,
\[ za(e_2 + e_3)y \cdot [kC_4]_t[y; \eta_t]e_1 = [kC_4]_t[y; \eta_t]e_1 \cdot za(e_2 + e_3)y = 0, \]
and hence \( za(e_2 + e_3)y \) commutes with \( [kC_4]_t[y; \eta_t] \).

Consequently, \( \langle q_t(y) \rangle = q_t(y)[kC_4]_t[y; \eta_t] \) is a two-sided ideal.

Now, as can easily be deduced from (5.1),
\[ q_{t=0}(y) = y^2 + \bar{x}^2 = q(y), \]
where the leading term \( y^2 \) remains unchanged. Then
\[ [kQ_8]_t := [kC_4]_t[y; \eta_t]/\langle q_t(y) \rangle \]
is a deformation of \( kQ_8 \), identifying \( \bar{\tau} \) with \( \bar{y} := y + \langle q_t(y) \rangle \).
6. Separability of $[kQ_8]_t$

Finally, we need to prove that the deformed algebra $[kQ_8]_t$ is separable. Moreover, we prove that its decomposition to simple components over the algebraic closure of $k((t))$ resembles that of $CQ_8$. By (4.10), we obtain

\[
[kQ_8]_t = \bigoplus_{i=1}^3 [kC_4]_t[y; \eta_i]e_i/(q_t(y)e_i).
\]

We handle the three summands in (6.1) separately.

By (6.2),

\[
[kC_4]_t[y; \eta_i]e_i/(q_t(y)e_i) \cong K[y; \eta_i]/(y^2 + b) \cong K \ast f C_2.
\]

The rightmost term is the crossed product of the group $C_2 := \langle \tau \rangle$ acting faithfully on the field $K = [kC_4]_t e_1$ via $\eta_i$ (4.9), with a twisting determined by the 2-cocycle $f : C_2 \times C_2 \rightarrow K^*$:

\[
f(1, 1) = f(1, \bar{\tau}) = f(\bar{\tau}, 1) = 1, \quad f(\bar{\tau}, \bar{\tau}) = b.
\]

This is a central simple algebra over the subfield of invariants $k((t))$. Theorem 4.4.1. Evidently, this simple algebra is split by $k((t))$, i.e.

\[
[kC_4]_t[y; \eta_i]e_i/(q_t(y)e_i) \otimes_{k((t))} k((t)) \cong M_2(k((t))).
\]

(The fact, $K \ast f C_2$ splits already over $k((t))$, since $b$ is a $C_2$-norm of a root of the irreducible polynomial $\pi(x)$ and therefore $f$ is cohomologically trivial.)

Next, since $\eta_i$ is trivial on $[kC_4]_t e_2$, the skew polynomial ring $[kC_4]_t e_2[y; \eta_i]$ is actually an ordinary polynomial ring $k((t))[y]$. Again by (5.2),

\[
[kC_4]_t[y; \eta_i]e_2/(q_t(y)e_2) \cong k((t))[y]/(y^2 + zy + c(c + a)).
\]

Similarly,

\[
[kC_4]_t[y; \eta_i]e_3/(q_t(y)e_3) \cong k((t))[y]/(y^2 + zy + c(c + a)).
\]

The polynomials $y^2 + zy + c(c + a)$ and $y^2 + zy + d(d + a)$ are separable (since $za$ is non-zero). Thus, both $[kC_4]_t[y; \eta_i]e_2/(q_t(y)e_2)$ and $[kC_4]_t[y; \eta_i]e_3/(q_t(y)e_3)$ are separable $k((t))$-algebras, and for $i = 2, 3,$

\[
[kC_4]_t[y; \eta_i]e_i/(q_t(y)e_i) \otimes_{k((t))} k((t)) \cong k((t)) \oplus k((t)).
\]

Equations (6.1), (6.2) and (6.3) yield

\[
[kQ_8]_t \otimes_{k((t))} k((t)) \cong \bigoplus_{i=1}^4 k((t)) \oplus M_2(k((t))),
\]

as required.

7. Acknowledgement

We wish to thank M. Schaps for pointing out to us that there is an error in the attempted proof in [5] that the quaternion group is a counterexample to the DF conjecture. Here is her explanation: The given relations for the group algebra are incorrect. Using the notation in pages 166-7 of [5], if $a = 1 + i$, $b = 1 + j$ and $z = i^2 = j^2$, then $ab + ba = ij(1 + z)$, while $a^2 = b^2 = 1 + z$. There is a further error later on when the matrix algebra is deformed to four copies of the field, since a non-commutative algebra can never have a flat deformation to a commutative algebra.
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