TOPOLOGY OF THREE-MANIFOLDS
WITH POSITIVE P-SCALAR CURVATURE

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ABSTRACT. Consider an n-dimensional smooth Riemannian manifold (M^n, g) with a given smooth measure m on it. We call such a triple (M^n, g, m) a Riemannian measure space. Perelman introduced a variant of scalar curvature in his recent work on solving Poincaré's conjecture $P(g) = R_\infty^m(g) = R(g) - 2\Delta_g log\phi - |\nabla log\phi|_g^2$, where $dm = \phi\ dvol(g)$ and R is the scalar curvature of (M^n, g). In this note, we study the topological obstruction for the ϕ-stable minimal submanifold with positive P-scalar curvature in dimension three under the setting of manifolds with density.

1. MANIFOLD WITH DENSITY

By a Riemannian measure space we mean a triple (M^n, g, m), where (M^n, g) is an n-dimensional smooth oriented Riemannian manifold and m is a smooth measure defined on M^n. Given a Riemannian manifold, there is a natural measure associated with it, i.e. the Riemannian volume measure $dvol(g)$. By the Radon-Nikodým theorem, there exists a smooth function $\phi > 0$ on M^n such that

$$dm = \phi\ dvol(g).$$

Here $\phi > 0$ is called the density of the manifold. The triple (M^n, g, ϕ) is called the Riemannian manifold with density ϕ. Clearly, the study of Riemannian measure space is equivalent to the study of manifolds with density.

The study of manifolds with density traces back to the work of Bakry-Émery [1] in the early 1980s, in which they introduced the Bakry-Émery-Ricci tensor in the study of the diffusion process:

$$Rc_\infty^m(g) = Rc(g) - \nabla_g^2 log\phi,$$

where $dm = \phi\ dvol(g)$. Gromov [5] studied the manifold with density (M^n, g, ϕ) and introduced the generalized mean curvature

$$H_\infty^m(g) = H(g) + \langle N, \nabla \log \phi \rangle,$$

where H denotes the mean curvature and N is the unit normal vector field. Lott [8] studied the topological consequences of positive or nonnegative $Rc_\infty^m(g)$. Morgan [9]

studied the isoperimetric inequalities of manifolds with density. In the recent work of Perelman [10] on solving Poincaré's conjecture, he introduced the new functional

$$\mathcal{F}(g) = \int_M R_\infty^m \ dm$$

and showed that the variation of his $\mathcal{F}$-functional is

$$\delta\mathcal{F}(v_{ij}, h) = \int_M -\langle v, Rc_\infty^m \rangle + (\frac{v}{2} - h)(R_\infty^m) \ dm,$$

where $dm = \phi \ dvol(g)$ and $R_\infty^m(g) = R(g) - 2\Delta_g \log \phi - |\nabla \log \phi|_g^2$. We call $R_\infty^m(g)$ the P-scalar curvature. Perelman showed that P-scalar curvature is not the trace of the Barkry-Émery-Ricci tensor, but it relates to the Bakry-Émery-Ricci tensor under the Bianchi identity:

$$\nabla^{*m} Rc_\infty^m = \frac{1}{2} R_\infty^m,$$

where ∇^{*m} is the L^2 adjoint of ∇ with respect to the measure dm. Both the Bakry-Émery-Ricci tensor and P-scalar curvature are, in some sense, the natural "infinite-dimensional" space analog of Ricci and scalar curvatures, respectively, in finite-dimensional space. More recently, Chang-Gursky-Yang [3], [4] introduced new notions for Ricci and scalar curvatures for manifolds with density $Rc_n^m(g)$ and $R_n^m(g)$, respectively:

$$Rc_n^m(g) = Rc(g) - \frac{2(n-1)}{n}\Delta_g log\phi - \frac{(n-1)(n-2)}{n^2}|\nabla log\phi|_g^2$$

and

$$R_n^m(g) = R(g) - \frac{2(n-1)}{n}\Delta_g log\phi - \frac{(n-1)(n-2)}{n^2}|\nabla log\phi|_g^2,$$

where n is the dimension of (M^n, g) and $dm = \phi \ dvol(g)$. They showed that $Rc_n^m \to Rc_\infty^m$ and $R_n^m \to R_\infty^m$ as the dimension $n \to \infty$. (For the rigorous statement, cf. [3], [4].) These new tensors have the properties of being conformally invariant under conformal change of metrics with fixed measure dm.

Given a Riemannian manifold with density (M^n, g, ϕ), we consider a smooth immersed hypersurface $\Sigma^{n-1} \subset M^n, \partial\Sigma^{n-1} = \emptyset$. Let $F : \Sigma \times (-\epsilon, \epsilon) \to M$ be a map with compact support such that $F(x, 0) = x$ for all $x \in \Sigma$. We call such a map a variation of Σ.

Definition 1.1. An immersed submanifold $\Sigma^{n-1} \subset M^n$ is called ϕ-minimal if

$$\frac{d}{dt}|_{t=0} Vol_\phi(F(\Sigma, t)) = \frac{d}{dt}|_{t=0} \int_\Sigma F^*\phi \ dvol(F^*g) = 0 \ \text{ for all variations } F.$$

Definition 1.2. An immersed submanifold $\Sigma^{n-1} \subset M^n$ is called ϕ-stable minimal if it is ϕ-minimal and

$$\frac{d^2}{dt^2}|_{t=0} Vol_\phi(F(\Sigma, t)) \geq 0$$

for all variations F.

Bayle [2] derived the first and second variation formulae for the weighted volume functional (cf. [13] for a detailed presentation of such derivation). From the first variation formula, it's easy to see that an immersed submanifold $\Sigma^{n-1} \subset M^n$ is ϕ-minimal if and only if the generalized mean curvature $H_\phi = H(g) + \langle N, \nabla \log(\phi) \rangle = 0$, where H is the mean curvature with respect to a unit normal N to Σ. From the second variation formula, it is also straightforward to obtain the following lemma.

Lemma 1.3. *For a ϕ-stable minimal submanifold $s : \Sigma^{n-1} \to (M^n, g, m), dm = \phi \ dvol(g)$, the following inequality holds for any smooth function η with compact support in Σ^{n-1}:*

$$\int_\Sigma Rc_\infty^m(N, N)\eta^2 s^*(dm) + \int_\Sigma |S|^2 \eta^2 s^*(dm) \leq \int_\Sigma |\nabla_\Sigma \eta|^2 s^*(dm),$$

where N is the unit normal vector field on Σ^{n-1}, S is the shape operator on Σ and $Rc_\infty^m(g) = Rc(g) - \nabla^2 log\phi$ is the Bakry-Émery-Ricci curvature.

Corollary 1.4. *Suppose (M, g, ϕ) has Bakry-Émery-Ricci curvature $Rc_\infty^m > 0$. Then there exist no compact ϕ-stable minimal immersed submanifolds of codimension one.*

Proof. Take $\eta = 1$ in Lemma 1.3. Then the conclusion follows. $\qquad\square$

Remark 1.5. In the case $\phi = constant$, this reduces to the well-known result that (M^n, g) with positive Ricci curvature has no compact immersed stable minimal submanifolds of co-dimension 1.

2. Main results

In this section, we generalize two results of Schoen and Yau on stable minimal submanifolds and characterize totally geodesic submanifold of codimension one, all in the setting of manifolds with density.

Schoen and Yau [15] proved that if (M^3, g) is oriented and has positive scalar curvature, then it has no compact, immersed stable minimal surfaces of positive genus. We can generalize this result as follows.

Theorem 2.1. *Let (M^3, g, m) be a smooth oriented three-manifold with measure $dm = \phi \ dvol(g)$. If $R_\infty^m > 0$, then there is no compact immersed ϕ-stable minimal two-dimensional submanifold with positive genus.*

Proof. Let $\Sigma^2 \subset M^3$ be a compact immersed ϕ-stable minimal submanifold. Taking $\eta = \phi^{-\frac{1}{2}}$ in Lemma 1.3, we get

$$(2.1) \qquad \int_\Sigma Rc_\infty^m(N, N)dvol(g|_\Sigma) + \int_\Sigma |S|^2 dvol(g|_\Sigma) \leq \frac{1}{4}\int_\Sigma |\nabla_\Sigma log\phi|^2 dvol(g|_\Sigma).$$

Let K denote the Gauss curvature of Σ^2, and take an orthonormal frame e_i with respect to (M^3, g), where the index $i = 1, 2$ denotes the vector fields which are tangential to Σ^2 and $i = 3$ denotes the vector field in the normal direction. Let

$$h_{ij} = S(e_i, e_j) = \langle \nabla_{e_i} N, e_j \rangle \quad (i = 1, 2)$$

and let R_{ij} denote the sectional curvature of $e_i \wedge e_j$ and $R = tr(Rc)$ the scalar curvature. As in Schoen-Yau [15], we have

$$\begin{aligned} Rc(N, N) + |S|^2 &= R_{13} + R_{23} + \Sigma_{i,j=1}^2 h_{ij}^2 \\ &= \frac{1}{2}R - K + h_{11}h_{22} + h_{12}^2 + h_{11}^2 + h_{22}^2 \\ &= \frac{1}{2}R - K + \frac{1}{2}(h_{11} + h_{22})^2 + \frac{1}{2}\Sigma_{i=1}^2 h_{ij}^2 \end{aligned}$$

and therefore it follows from (2.1) that

$$\int_{\Sigma}(\frac{1}{2}R - \langle\nabla_N\nabla log\phi, N\rangle + \frac{1}{2}H^2 + \frac{1}{2}|S|^2)dvol(g|_{\Sigma})$$

$$\leq \int_{\Sigma} K \ dvol(g_{\Sigma}) + \frac{1}{4}\int_{\Sigma}|\nabla_{\Sigma}log\phi|^2 dvol(g|_{\Sigma}).$$

On the other hand, apply the divergence theorem:

$$\int_{\Sigma}\Delta_g log\phi \ dvol(g|_{\Sigma})$$

$$= \int_{\Sigma}\Sigma_{i=1}^2(\langle\nabla_{e_i}(\nabla log\phi)^T, e_i\rangle + \langle\nabla_{e_i}(\langle\nabla log\phi, N\rangle N), e_i\rangle)dvol(g|_{\Sigma})$$

$$+ \int_{\Sigma}\langle\nabla_N\nabla log\phi, N\rangle \ dvol(g|_{\Sigma})$$

$$= \int_{\Sigma}div_{\Sigma}(\nabla log\phi)^T dvol(g|_{\Sigma}) + \int_{\Sigma}(\langle\nabla log\phi, N\rangle H + \langle\nabla_N\nabla log\phi, N\rangle) \ dvol(g|_{\Sigma})$$

$$= \int_{\Sigma}(\langle\nabla log\phi, N\rangle H + \langle\nabla_N\nabla log\phi, N\rangle) \ dvol(g|_{\Sigma}).$$

Combining the above calculations, we obtain

$$\frac{1}{2}\int_{\Sigma}R_{\infty}^m \ dvol(g|_{\Sigma})$$

$$= \frac{1}{2}\int_{\Sigma}R \ dvol(g|_{\Sigma}) - \int_{\Sigma}\Delta_g log\phi \ dvol(g|_{\Sigma}) - \frac{1}{2}\int_{\Sigma}|\nabla log\phi|^2 dvol(g|_{\Sigma})$$

$$= \int_{\Sigma}(\frac{1}{2}R - \langle\nabla_N\nabla log\phi, N\rangle + \frac{1}{2}H^2 + \frac{1}{2}|S|^2 - \frac{1}{4}|\nabla_{\Sigma}log\phi|^2)dvol(g|_{\Sigma})$$

$$+ \int_{\Sigma}-\frac{1}{2}H^2 - \frac{1}{2}|S|^2 - \langle\nabla log\phi, N\rangle H - \frac{1}{2}|\nabla log\phi|^2 + \frac{1}{4}|\nabla_{\Sigma}log\phi|^2)dvol(g|_{\Sigma})$$

$$\leq \int_{\Sigma}K \ dvol(g|_{\Sigma}) - \frac{1}{2}\int_{\Sigma}(H_{\phi}^2 + |S|^2 + \frac{1}{2}|\nabla_{\Sigma}log\phi|^2)dvol(g|_{\Sigma})$$

$$= \int_{\Sigma}Kdvol(g|_{\Sigma}) - \frac{1}{2}\int_{\Sigma}(|S|^2 + \frac{1}{2}|\nabla_{\Sigma}log\phi|^2)dvol(g|_{\Sigma})$$

$$\leq \int_{\Sigma}Kdvol(g|_{\Sigma}).$$

By the Gauss-Bonnet theorem,

$$2\pi(2 - 2g) = \int_{\Sigma}Kdvol(g|_{\Sigma}), \quad \text{where } g \text{ is the genus of the surface } \Sigma.$$

Hence

$$\frac{1}{2}\int_{\Sigma}R_{\infty}^m \ dvol(g|_{\Sigma}) \leq 2\pi(2 - 2g).$$

The conclusion follows. $\square$

Remark 2.2. In dimension three, we can replace Perelman's P-scalar curvature R_{∞}^m with the CGY-scalar curvature R_n^m in the above theorem and the conclusion still holds (cf. [3]).

We recall the definition of harmonic maps on surfaces. Let e_i be an orthonormal frame on (Σ^2, h), and let $e(s) = \sum_{i=1}^2\langle ds(e_i), ds(e_i)\rangle_g$.

Definition 2.3. A smooth map $s : (\Sigma^2, h) \rightarrow (M^3, g)$ is harmonic if for any 1-parameter family of maps $u : (\Sigma^2, h) \times (-\epsilon, \epsilon) \rightarrow (M^3, g)$ with $u(x, 0) = s(x)$, we have

$$\frac{d}{dt} E(u(t))|_{t=0} = \frac{d}{dt}|_{t=0} \int_\Sigma e(u(t)) dvol(h) = 0.$$

A harmonic map on a surface is a nice object to study because of its complex structure. We can complexify the tangent bundle of M^3, $T^\mathbb{C}(M) = T(M) \otimes \mathbb{C}$. We have

$$\partial_z s = \frac{1}{2}(\partial_x s - i\partial_y s),$$

$$\partial_{\bar{z}} s = \frac{1}{2}(\partial_x s + i\partial_y s).$$

The map s is called weakly conformal if its quadratic differential

$$(s^* g)^{(2,0)} = \langle \partial_z s, \partial_z s \rangle^\mathbb{C}_g = 0.$$

Gulliver-Osserman-Royden [7] proved that if s is harmonic and weakly conformal, then it is a branched immersion. It is well-known that in dimension three a branched minimal immersion is in fact immersion (cf. [6]). Based on this important observation which enables one to link harmonic maps with immersions, Sacks-Uhlenbeck [14] and Schoen-Yau [15] prove that under some non-degeneracy conditions on a map $u : \Sigma^2 \rightarrow M^3$, there exists a minimal harmonic map s homotopic to u. Schoen-Yau further used this result to show two important existence theorems of minimal immersions for three-manifolds. We state their results below.

Theorem 2.4 (Theorem 3.1 of [15])**.** *Let Σ^2 denote a compact surface of genus $g > 1$ and let $(M^3, \tilde{g})$ denote a compact Riemannian manifold. Suppose $u : \Sigma^2 \rightarrow (M^3, \tilde{g})$ is a map which is injective on fundamental groups. Then there exist a conformal structure on Σ^2 and an immersion $s : \Sigma^2 \rightarrow (M^3, \tilde{g})$ whose action on π_1 is conjugate to that of u and which minimizes the area over all such maps.*

Theorem 2.5 (Theorem 4.1 of [15])**.** *Let T^2 denote the two-dimensional torus and let $(M^3, \tilde{g})$ denote a compact Riemannian manifold. Suppose $\pi_1(M)$ contains a finitely generated non-cyclic abelian group. Then there exists a minimal immersion $s : T^2 \rightarrow (M^3, \tilde{g})$ whose induced map on π_1 has rank 2 image in $\pi_1(M)$. Moreover, s minimizes the area over all maps $u : T^2 \rightarrow (M^3, \tilde{g})$ which satisfy $u_\sharp = \tau^{-1} s_\sharp \tau : \pi_1(T^2, p) \rightarrow \pi_1(M, u(p))$ for some curve τ from $u(p)$ to $s(p)$.*

We can generalize Schoen-Yau's [15] work on the topological constraints of the existence of positive scalar curvature on three-manifolds as follows.

Theorem 2.6. *Let M^3 be a compact, oriented, smooth manifold. Suppose either of the following two conditions holds for M^3:*

   (i) $\pi_1(M)$ contains a finitely generated non-cyclic abelian subgroup or

   (ii) $\pi_1(M)$ contains a subgroup abstractly isomorphic to the fundamental group of a surface of genus $g > 1$.

Then M^3 admits no metrics of positive P-scalar curvature w.r.t. any density function; i.e. M^3 can have no metrics g with $R_\infty(g, \phi) = R(g) - 2\Delta_g log\phi - |\nabla log\phi|^2_g > 0$ for any smooth positive function ϕ.

Proof. We closely follow the proof of Theorem 5.2 in Schoen-Yau ([15], p. 139).

In our setting, (M^3, g, ϕ), we let $\tilde{g} = \phi g$. Under the hypotheses of Theorem 2.6, we can easily construct the map $s : \Sigma^2 \to (M^3, \tilde{g})$ with genus of Σ^2 being positive. We apply Theorem 2.4 or Theorem 2.5. In either case, the map s satisfies

$$\frac{d}{dt} vol(F(\Sigma, t))|_{t=0} = \frac{d}{dt}|_{t=0} \int_\Sigma dvol(F^* \tilde{g}) = 0,$$

for all variations $F : \Sigma \times (-\epsilon, \epsilon) \to (M^3, \tilde{g}), F(x, 0) = s$. Since Σ^2 is of dimension two,

$$\int_\Sigma dvol(F^* \tilde{g}) = \int_\Sigma dvol(F^*(\phi g)) = \int_\Sigma \phi(F) \, dvol(F^* g) = vol_\phi(F(\Sigma, t)).$$

This implies that s is a ϕ-stable minimal immersion from Σ^2 to (M^3, g). If M^3 has a metric g such that $R_\infty^m(g) > 0$, applying Theorem 2.1, we know that the map s we constructed above cannot exist. This is a contradiction and completes the proof. $\qquad\square$

Remark 2.7. The P-scalar curvature is the limit of the scalar curvature of Chang-Gursky-Yang [3], $R_n^m(g) \to R_\infty^m$, as the dimension $n \to \infty$. (See Chang-Gursky-Yang for the precise statement.) Even though there exists a "canonical metric" $g_n \in [g]$ such that $R_n^m(g) = R(g_n)$, this is not true for the P-scalar curvature in general. So the topological constraints on fundamental groups indeed give a more restrictive class of metrics which can be put on M^3, thus generalizing the result of Schoen and Yau.

The Bonnet-Myers Theorem states that a complete Riemannian manifold (M^n, g) with Ricci curvature $Rc(g) \geq \lambda > 0$ is compact (cf. [11]). On the other hand, Morgan ([9], Corollary 4) showed that any complete smooth Riemannian manifold with density (M^n, g, ϕ) with $Rc_\infty^m(g) \geq \lambda > 0$ must have finite weighted volume even though it is not necessarily compact. A simple example of such a case is the Gauss space. If $\Sigma^{n-1} \subset M^n$ is non-compact, utilizing Lemma l.3 and Morgan ([9], Corollary 4), we have the following characterization of totally geodesic submanifolds.

Theorem 2.8. *Suppose $\Sigma^{n-1} \subset (M^n, g, \phi)$ is a complete non-compact ϕ-stable minimal hypersurface, and let $Rc_\infty^m(M^n, g, \phi) \geq \lambda > 0$. Then $\Sigma^{n-1} \subset M^n$ is totally geodesic if and only if $vol_\phi(\Sigma, g|_\Sigma) = \int_\Sigma \phi \, dvol(g|_\Sigma) < \infty$.*

Proof. Take a ball $B_r(x) \subset \Sigma$. Let η be a cutoff function such that it is equal to 1 in $B_r(x)$ and vanishes outside $B_{3r}(x)$ and $|\nabla \eta| < \frac{1}{r}$. Lemma 1.3 implies

$$\int_\Sigma Rc_\infty^m(N, N) \eta^2 v_0 + \int_\Sigma |S|^2 \eta^2 v_0 \leq \int_\Sigma |\nabla_\Sigma \eta|^2 v_0,$$

where $v_0 = \phi \, dvol(g|_\Sigma)$. By assumption, $Rc_\infty^m(M^n, g, \phi) > 0$; therefore

$$\int_{B_r(x)} |S|^2 v_0 \leq \int_\Sigma \frac{1}{r^2} v_0 = \frac{vol_\phi(\Sigma)}{r^2} \to 0, \quad \text{as } r \to \infty,$$

provided $vol_\phi(\Sigma) < \infty$. This implies

$$|S| \equiv 0 \quad \text{on } \Sigma,$$

which means that $\Sigma^{n-1} \subset M^n$ is totally geodesic. Conversely, if $Rc_\infty^m(M, g, \phi) \geq \lambda > 0$ and the immersion is totally geodesic, then $Rc_\infty^m(\Sigma, g|_\Sigma, \phi|_\Sigma) \geq \lambda' > 0$. By Corollary 4 of Morgan [9], $vol_\phi \Sigma < \infty$. $\qquad\square$

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