THE "FUNDAMENTAL THEOREM"
FOR THE ALGEBRAIC K-THEORY OF SPACES.
III. THE NIL-TERM

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Abstract. In this paper we identify the “nil-terms” for Waldhausen’s algebraic $K$-theory of spaces functor as the reduced $K$-theory of a category of equivariant spaces equipped with a homotopically nilpotent endomorphism.

1. Introduction

This is the third in a series of papers which concerns the decomposition

$$A^dfd(X) \times S^1 \cong A^dfd(X) \times B A^dfd(X) \times N_+ A^dfd(X) \times N_+ A^dfd(X).$$

Here, $A^dfd(X)$ is Waldhausen’s algebraic $K$-theory of the space $X$ and $B A^dfd(X)$ is a certain nonconnective delooping of it. The remaining factors on the right, called “nil-terms”, are homotopy equivalent $[H^+_1]$, $[H^+_2]$. They have not been given a $K$-theoretic description thus far.

In this installment, we will identify the nil-terms as a shifted copy of the reduced $K$-theory of a category whose objects are equivariant spaces equipped with a homotopically nilpotent endomorphism.

Let $X$ be a connected based space. Let $G$ denote the Kan loop group of the total singular complex of $X$, and let $G$ denote the geometric realization of $G_\cdot$. Then the classifying space $BG$ has the weak homotopy type of $X$.

Define a category $\text{nil}(X)$ in which an object consists of a pair

$$(Y, f)$$

such that $Y$ is a based space with $G$-action and $f: Y \rightarrow Y$ is an equivariant map which is homotopically nilpotent under composition. Additionally, we assume that $Y$ admits the structure of a based $G$-cell complex in which the action of $G$ is free away from the basepoint. A morphism $(Y, f) \rightarrow (Z, g)$ is a based $G$-map $e: Y \rightarrow Z$ such that $g \circ e = e \circ f$.

There is a full subcategory $\text{nil}_dfd(X)$ of $\text{nil}(X)$ whose objects are those $Y$ which are finitely dominated in the sense that $Y$ is a retract up to homotopy of an object which is built up from a point by attaching a finite number of free $G$-cells. A morphism of $\text{nil}_dfd(X)$ is a weak equivalence if and only if its underlying map of topological spaces is a weak homotopy equivalence. It is a cofibration if its underlying map of spaces is obtained up to isomorphism by attaching free $G$-cells.

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With the above structure, it turns out that nil\(_{fd}(X)\) is a category with cofibrations and weak equivalences. It therefore has a \(K\)-theory, which is denoted \(K^{fd}(\text{nil}(X))\).

The forgetful functor \((Y,f) \rightarrow Y\) gives rise to a map on \(K\)-theories

\[ K^{fd}(\text{nil}(X)) \rightarrow A^{fd}(X). \]

Here we are using the model for \(A^{fd}(X)\) given by the algebraic \(K\)-theory of the category of finitely dominated based \(G\)-spaces ([W] §2.1, [H\_+ 1.5]). Let

\[ \overline{K}^{fd}(\text{nil}(X)) \]

denote the homotopy fiber of the map \(K^{fd}(\text{nil}(X)) \rightarrow A^{fd}(X)\).

Our main result establishes the other half of the “fundamental theorem” for \(A^{fd}(X)\):

**Main Theorem.** There is a homotopy equivalence of functors

\[ \overline{K}^{fd}(\text{nil}(X)) \simeq \Omega_{N^{+}}A^{fd}(X). \]

**Remark.** The above result is used in the paper [GKM], where it is shown that the homotopy groups of \(N^{+}A^{fd}(X)\) are either trivial or infinitely generated. Another result of that paper determines the \(p\)-complete homotopy type of \(N^{+}A^{fd}(*)\) in degrees \(\leq 4p - 7\), for \(p\) an odd prime.

2. Preliminaries

In what follows, we assume that the reader is familiar with the material of [H\_+].

The spaces in this paper are to be given the compactly generated topology. Products are taken in the compactly generated sense. Let \(M\) be a simplicial monoid, and let \(M = |M|\) denote its geometric realization. Let \(T(M)\) denote the category of based (left) \(M\)-spaces and based \(M\)-maps. We say that a based morphism \(Y \rightarrow Z\) of \(T(M)\) is a weak equivalence if (and only if) it is a weak homotopy equivalence of underlying topological spaces. Similarly, we say that \(Y \rightarrow Z\) is a fibration if it is a Serre fibration after forgetting actions. A morphism is a cofibration if and only if it satisfies the left lifting property with respect to the acyclic fibrations (i.e., those fibrations which are weak equivalences). Then \(T(M)\) is a Quillen model category (see, e.g., [VS]).

Then every object of \(T(M)\) is fibrant, and the cofibrant objects are precisely the retracts of those objects which are built up from a point by cell attachments, where the cell of dimension \(n\) is given by

\[ D^n \times M \]

with action defined by left translation.

Recall from [H\_+] that \(\mathcal{C}(M)\) denotes the full subcategory of \(T(M)\) consisting of the cofibrant objects. Then \(\mathcal{C}(M)\) is a category with cofibrations and weak equivalences in the sense of Waldhausen [W]. For objects \(Y\) and \(Z\) of \(\mathcal{C}(M)\), we let

\[ [Y,Z] \]

denote the homotopy classes of morphisms in \(\mathcal{C}_{fd}(G)\), i.e., the based equivariant homotopy classes.

We next recall the various finiteness notions. An object of \(\mathcal{C}(M)\) is finite if it is built up from a point by finitely many cell attachments (up to isomorphism). An object of \(\mathcal{C}(M)\) is said to be homotopy finite if there exists a weak equivalence to a
finite object. An object of \( \mathbb{C}(M) \) is said to be \textit{finitely dominated} if it is a retract of a homotopy finite object. Let \( \mathbb{C}_{fd}(M) \) denote the full subcategory of \( \mathbb{C}(M) \) whose objects are finitely dominated.

We let \( h\mathbb{C}_{fd}(M) \) denote the subcategory of \( \mathbb{C}_{fd}(M) \) defined by the weak equivalences. Then the associated \( K \)-theory space is given by

\[
A^{fd}(\ast;M) := \Omega h\mathbb{S}\mathbb{C}_{fd}(M),
\]

where the right side is the based loop space of the geometric realization of Waldhausen’s \( \mathcal{S} \)-construction of \( \mathbb{C}_{fd}(M) \) (\cite{W} p. 330). If \( M \) is the realization of a simplicial group, then \( A^{fd}(\ast;M) \) is one of the definitions of \( A^{fd}(BM) \) (cf. \cite{W} p. 379), \cite{H}, 1.6).

The category \( \text{nil}_{fd}(X) \) has objects specified by pairs \((Y,f)\) with \( Y \in \mathbb{C}_{fd}(G) \) and object \( f: Y \to Y \) a morphism which is homotopically nilpotent under composition, i.e., the associated homotopy class

\[
[f] \in [Y,Y]
\]
is nilpotent in the sense that some iterate \([f^{ok}] = [f]^o k\) is trivial.

A \textit{morphism} \( (Y,f) \to (Z,g) \) of \( \text{nil}_{fd}(X) \) is a map \( e: Y \to Z \) such that \( g \circ e = e \circ f \).

A \textit{cofibration} of \( \text{nil}_{fd}(X) \) is a morphism \( (Y,f) \to (Z,g) \) such that \( Y \to Z \) is a cofibration of \( \mathbb{C}_{fd}(G) \). A \textit{weak equivalence} is a morphism whose underlying map of spaces is a weak homotopy equivalence.

\textbf{Lemma 2.1.} With respect to the above conventions, \( \text{nil}_{fd}(X) \) is a category with cofibrations and weak equivalences.

\textit{Proof.} The nontrivial thing to be verified is that the cobase change axiom holds. Given a diagram

\[
(B, f_1) \leftarrow (A, f_0) \rightarrow (C, f_2)
\]

we define the pushout to be \((B \cup_A C, f)\), where \( f \) denotes \( f_1 \cup f_0 f_2 \). Choose a positive integer \( k \) such that \([f_i]^{ok}\) is trivial, for \( i = 0, 1, 2 \). It will be sufficient to check that \([f]\) is nilpotent. Let us rename \( g_i = f_i^k \) and \( g = f^{ok} \). Then, using the model structure, one has a Barratt-Puppe cofiber sequence

\[
B \vee C \xrightarrow{j} B \cup_A C \xrightarrow{\delta} \Sigma A
\]
in \( \mathbb{T}(M) \), where \( \vee \) means wedge and \( \Sigma \) is suspension. Consequently, there is an exact sequence of pointed sets

\[
[\Sigma A, B \cup_A C] \xrightarrow{\delta^*} [B \cup_A C, B \cup_A C] \xrightarrow{j^*} [B \vee C, B \cup_A C].
\]

Then

\[
j^*([g]) = [g \circ j] = [g_1 \vee g_2] = 0,
\]

so there is a homotopy class

\[
\gamma \in [\Sigma A, B \cup_A C]
\]
such that \([g] = \delta^*(\gamma) = \gamma \circ [\delta]\). Then

\[
[g]^{o2} = \gamma \circ [\delta] \circ \gamma \circ [\delta]
\]
is trivial because \([\delta] \circ \gamma \circ [\delta] = [\delta] \circ [g] \) coincides with \([\Sigma g_0] \circ [\delta]\), and \([\Sigma g_0]\) is trivial. \( \square \)
3. Another look at the projective line

Let \( \mathbb{N}_- \) denote the monoid of negative integers with generator \( t^{-1} \) and \( \mathbb{N}_+ \) denote the monoid of positive integers with generator \( t \). Let \( G \) be the realization of a simplicial group \( G \).

Recall that the mapping telescope of an object \( Y_+ \in \mathbb{C}_{fd}(G \times \mathbb{N}_+) \) is the object \( Y_+(t^{-1}) \in \mathbb{C}_{fd}(G \times \mathbb{Z}) \) defined by taking the categorical colimit of the sequence

\[
\cdots \xrightarrow{t} Y_+ \xrightarrow{t} Y_+ \xrightarrow{t} \cdots.
\]

Similarly, if \( Y_- \in \mathbb{C}_{fd}(G \times \mathbb{N}_-) \) is an object, we have a mapping telescope \( Y_-(t) \) given by the colimit of

\[
\cdots \xrightarrow{t^{-1}} Y_+ \xrightarrow{t^{-1}} Y_+ \xrightarrow{t^{-1}} \cdots.
\]

Define \( \mathbb{D}_{fd}(G \times \mathbb{Z}) \) to be the category whose objects are diagrams

\[ Y_- \to Y \leftarrow Y_+ \]

in which \( Y_- \in \mathbb{C}_{fd}(G \times \mathbb{N}_-) \), \( Y \in \mathbb{C}_{fd}(G \times \mathbb{Z}) \) and \( Y_+ \in \mathbb{C}_{fd}(G \times \mathbb{N}_+) \), and where the maps \( Y_- \to Y \) and \( Y_+ \to Y \) are required to be based and equivariant. Moreover, the induced morphisms

\[ Y_-(t) \to Y(t) \cong Y \quad \text{and} \quad Y_+(t^{-1}) \to Y(t^{-1}) \cong Y \]

are required to be cofibrations of \( \mathbb{C}_{fd}(G \times \mathbb{Z}) \). We take the liberty of specifying the object as a diagram or as a triple \( (Y_-, Y, Y_+) \).

A morphism \( (Y_-, Y, Y_+) \to (Z_-, Z, Z_+) \) of \( \mathbb{D}_{fd}(G \times \mathbb{Z}) \) is a morphism \( Y_- \to Z_- \), a morphism \( Y \to Z \) and a morphism \( Y_+ \to Z_+ \) so that the evident diagram commutes. A cofibration is a morphism \( (Y_-, Y, Y_+) \to (Z_-, Z, Z_+) \) in which

- each of the maps \( Y_- \to Z_- \), \( Y_+ \to Z_+ \) and \( Y \to Z \)
- the induced maps \( Y \cup_{Y_-(t)} Z_-(t) \to Z \) and \( Y \cup_{Y_+(t^{-1})} Z_+(t^{-1}) \to Z \)

are cofibrations of \( \mathbb{C}_{fd}(G \times \mathbb{Z}) \).

The projective line \( \mathbb{P}_{fd}(G) \) of \( [\mathbb{H} \mathbb{L}_-] \) is given by the full subcategory of \( \mathbb{D}_{fd}(G \times \mathbb{Z}) \) whose objects \( (Y_-, Y, Y_+) \) satisfy an auxiliary condition, viz., that the induced maps \( Y_- \to Y \) and \( Y_+ \) are weak homotopy equivalences. A cofibration is a morphism which is a cofibration of \( \mathbb{D}_{fd}(G \times \mathbb{Z}) \). A weak equivalence is a morphism in which \( Y_- \to Z_- \), \( Y \to Z \) and \( Y_+ \to Z_+ \) are weak homotopy equivalences of spaces.

Let \( \mathbb{D}_{fd}(G \times \mathbb{N}_-) \subset \mathbb{D}_{fd}(G \times \mathbb{Z}) \) denote the full subcategory whose objects \( (Y_-, Y, Y_+) \) satisfy the condition that \( Y_- \to Y \) is a weak equivalence. Similarly, define \( \mathbb{D}_{fd}(G \times \mathbb{N}_+) \) to be the full subcategory whose objects \( (Y_-, Y, Y_+) \) satisfy the condition that \( Y_+ \to Y \) is a weak equivalence.

A morphism \( (Y_-, Y, Y_+) \to (Z_-, Z, Z_+) \) of \( \mathbb{D}_{fd}(G \times \mathbb{N}_+) \) is a weak equivalence if the map \( Y_+ \to Z_+ \) is a weak homotopy equivalence. It is a cofibration if it is so when considered in \( \mathbb{D}_{fd}(G \times \mathbb{Z}) \).

Let \( \mathbb{P}_{fd}^{\mathbb{N}_+}(G) \subset \mathbb{P}_{fd}(G) \) denote the full subcategory with objects \( (Y_-, Y, Y_+) \) such that \( Y_+ \) is acyclic.
Proposition 3.1. There is a homotopy fiber sequence
\[ \Omega|hS\mathbb{F}^h_{\mathcal{F}d}(G)| \rightarrow \Omega|hS\mathbb{F}_{\mathcal{F}d}(G)| \rightarrow \Omega|hS\mathbb{D}_{\mathcal{F}d}(G \times \mathbb{N}_+)|. \]

Proof. Define a coarser notion of weak equivalence on the projective line by specifying a morphism \((Y_-, Y, Y_+) \rightarrow (Z_-, Z, Z_+)) to be an \(h_{\mathbb{N}_+}\)-equivalence if (and only if) the map \(Y_+ \rightarrow Z_+\) is a weak equivalence. Application of the fibration theorem [W 1.6.5] shows that the sequence
\[ \Omega|hS\mathbb{F}^h_{\mathcal{F}d}(G)| \rightarrow \Omega|hS\mathbb{F}_{\mathcal{F}d}(G)| \rightarrow \Omega|hS\mathbb{D}_{\mathcal{F}d}(G)| \]

is a fibration up to homotopy. Let \(\mathbb{P}_{\mathcal{F}d}(G) \rightarrow \mathbb{D}_{\mathcal{F}d}(G \times \mathbb{N}_+)\) denote the inclusion functor. By [H, §4] we have that the induced map
\[ |hS\mathbb{F}_{\mathcal{F}d}(G)| \rightarrow |hS\mathbb{D}_{\mathcal{F}d}(G \times \mathbb{N}_+)| \]

induces an isomorphism on homotopy groups in degrees \(> 1\). Hence, the homotopy fiber of the induced map of loop spaces
\[ \Omega|hS\mathbb{F}_{\mathcal{F}d}(G)| \rightarrow \Omega|hS\mathbb{D}_{\mathcal{F}d}(G \times \mathbb{N}_+)| \]

is homotopically trivial.

It follows that the homotopy fiber of the map
\[ \Omega|hS\mathbb{F}_{\mathcal{F}d}(G)| \rightarrow \Omega|hS\mathbb{D}_{\mathcal{F}d}(G \times \mathbb{N}_+)| \]

is identified with the homotopy fiber of the map
\[ \Omega|hS\mathbb{F}_{\mathcal{F}d}(G)| \rightarrow \Omega|hS\mathbb{S}_{\mathcal{F}d}(G)|. \]

The result follows. \(\square\)

4. The “characteristic sequence”

Let \((Y, f) \in \text{nil}_{\mathcal{F}d}(X)\) be an object, and let \(Y \otimes \mathbb{N}_- \in \mathcal{C}_{\mathcal{F}d}(G)\) be the object given by
\[ (Y \times \mathbb{N}_-)/(\ast \times \mathbb{N}_-). \]

Then \(f\) induces a self-map of \(Y \otimes \mathbb{N}_-\) which is given by \((y, r) \mapsto (f(y), r)\). We will denote this self-map also by \(f\).

Let \(Y_f\) be the homotopy coequalizer of the pair of maps
\[ Y \otimes \mathbb{N}_- \xrightarrow{f} Y \otimes \mathbb{N}_-, \]

where \(t^{-1}\) denotes the map \((y, r) \mapsto (y, r-1)\). (Recall that the homotopy coequalizer of a pair of morphisms \(\alpha, \beta: U \rightarrow V\) is defined to be the quotient of the disjoint union \(V \amalg (U \times [0, 1])\) which is given by identifying \((u, 0)\) with \(\alpha(u)\), \((u, 1)\) with \(\beta(u)\) and \(\ast \times [0, 1]\) with the basepoint of \(V\).)

If we give \(Y\) the structure of a based \((G \times \mathbb{N}_-)\)-space by letting \(\mathbb{N}_-\) act by means of \(f\), then we also have a \((G \times \mathbb{N}_-)\)-equivariant map
\[ \pi_f: Y \otimes \mathbb{N}_- \rightarrow Y \]

which is given by \((y, r) \mapsto f^{-r}(y)\). Then \(\pi_f\) coequalizes \(f\) and \(t^{-1}\), so by the universal property of the homotopy coequalizer, there is an induced map
\[ Y_f \rightarrow Y, \]

which is \((G \times \mathbb{N}_-)\)-equivariant.
Lemma 4.1. The map \( Y_f \to Y \) induces an isomorphism in reduced singular homology.

Proof. Let \( p: S^1 \to S^1 \lor S^1 \) be the pinch map, and let \( \rho: S^1 \to S^1 \) be the reflection map. Then the composite

\[
S^1 \xrightarrow{p} S^1 \lor S^1 \xrightarrow{id \lor \rho} S^1 \lor S^1
\]

will be denoted by \((1, -1)\).

The homotopy coequalizer induces a homotopy cofiber sequence

\[
\Sigma(Y \otimes N_-) \xrightarrow{t^{-1} \cdot f} \Sigma(Y \otimes N_-) \to \Sigma Y_f
\]

where the first map is defined to be the composite

\[
\Sigma(Y \otimes N_-) \xrightarrow{(1, -1) \lor \text{id}} \Sigma(Y \otimes N_-) \lor \Sigma(Y \otimes N_-) \xrightarrow{t^{-1} \lor f} \Sigma(Y \otimes N_-).
\]

Taking reduced singular chains, we get an induced homotopy cofiber sequence of chain complexes

\[
(1) \quad C_\ast(Y) \otimes \mathbb{Z}[t^{-1}] \xrightarrow{t^{-1} \cdot f} C_\ast(Y) \otimes \mathbb{Z}[t^{-1}] \quad \to \quad C_\ast(Y_f).
\]

(Recall that a sequence \( A \xrightarrow{i} B \xrightarrow{j} C \) of chain complexes is a homotopy cofiber sequence when the composite \( j \circ i: A \to C \) is equipped with a null homotopy such that the induced map from the mapping cone \( T_{i,j} \) to \( C \) is a quasi-isomorphism.)

Now, for any \( \mathbb{Z} \)-module \( M \) equipped with a self-map \( f: M \to M \), we have an exact sequence of \( \mathbb{Z}[t^{-1}] \)-modules

\[
(2) \quad 0 \quad \to \quad M \otimes \mathbb{Z}[t^{-1}] \xrightarrow{t^{-1} \cdot f} M \otimes \mathbb{Z}[t^{-1}] \quad \to \quad M_f \quad \to \quad 0
\]

in which \( M_f \) denotes \( M \) considered as a \( \mathbb{Z}[t^{-1}] \)-module where \( t^{-1} \) acts via \( f \) (see [B, p. 630]). This implies that the sequence \((1)\) becomes exact when \( C_\ast(Y_f) \) is replaced by \( C_\ast(Y) \) by means of the chain map \( C_\ast(Y_f) \to C_\ast(Y) \) which is induced by the map \( Y_f \to Y \). Consequently, the five lemma implies that the chain map \( C_\ast(Y_f) \to C_\ast(Y) \) is a quasi-isomorphism. \( \square \)

Remark 4.2. The sequence \((1)\) is a chain complex version of the so-called, “characteristic sequence” \((2)\) of the module \( M \). Consequently, it is not inappropriate to think of the homotopy coequalizer diagram

\[
Y \otimes N_- \xrightarrow{f} Y \otimes N_- \xrightarrow{t^{-1} \cdot f} Y_f
\]

as a kind of nonlinear version of the characteristic sequence (of the object \( Y \)).

Preliminary identification of \( K(\text{nil}_{fd}(X)) \). Define an exact functor

\[
\text{nil}_{fd}(X) \xrightarrow{\Phi} \mathbb{P}_{fd}^b(G)
\]

by

\[
(Y, f) \quad \mapsto \quad (Y_f, Y_f(t), \ast),
\]

where \( Y_f \) is defined above.
In the other direction, define an exact functor
\[
P^h_{fd^+}(G) \xrightarrow{\Psi} \text{nil}_{fd}(X)
\]
by
\[
(Y_-, Y, Y_+) \mapsto (Y_-, t^{-1}).
\]
To see that \(\Psi\) is well-defined, let \((Y_-, Y, Y_+)\) be an object of \(P^h_{fd^+}(G)\). Then \(Y_+\) and \(Y\) are acyclic. Hence \(Y_-\) has an acyclic mapping telescope. This implies that there exists a \(k \in \mathbb{N}_-\) such that \(t^k : Y_- \to Y_-\) is (equivariantly) null homotopic (this follows for finite objects by the “small object” argument, and hence for finitely dominated ones since a retract of a null homotopic morphism is again null homotopic; compare [H+, p. 40 bottom]).

Let \(Z\) denote the quotient \(Y_-/t^k(Y_-)\) considered as an object of \(C(G)\). Then \(Z\) is finitely dominated. This is a consequence of a cell-by-cell induction when \(Y_-\) is a finite object of \(C(G \times \mathbb{N}_-)\). It is true for homotopy finite objects because the functor \(Y_+ \mapsto Y_+/t^k(Y_-)\) preserves weak equivalences. It is therefore also true when \(Y_-\) is finitely dominated since this functor also preserves retracts (cf. [H+, p. 41 top]).

Since \(t^k\) is \(G\)-equivariantly null homotopic, the identity map \(Y_- \to Y_-\) factors through \(Z\) up to homotopy. It follows that \(Y_-\) is also finitely dominated when considered as an object of \(C(G)\). This shows that \((Y_-, t^{-1})\) is an object of \(\text{nil}_{fd}(X)\).

**Lemma 4.3.** The functors \(\Psi\) and \(\Phi\) induce mutually inverse homotopy equivalences on \(K\)-theory.

**Proof.** The composite \(\Psi \circ \Phi\) is given by
\[
(Y, f) \mapsto (Y_f, t^{-1})
\]
and Lemma 4.1 implies that there is a morphism \((Y_f, t^{-1}) \to (Y, f)\) which is a weak equivalence after taking a suitable number of suspensions. Since suspension induces a homotopy equivalence on the level of \(K\)-theory [W, 1.6.2], it follows that \(\Psi \circ \Phi\) induces a homotopy equivalence.

The composite \(\Phi \circ \Psi\) is given by
\[
(Y_-, Y, Y_+) \mapsto (Y_-, Y_-(t), *).
\]
This admits an evident equivalence to the identity functor. Consequently \(\Phi \circ \Psi\) induces a map which is homotopic to the identity on the level of \(K\)-theory. \(\square\)

**5. PROOF OF THE MAIN THEOREM**

By Lemma 4.3 we have a homotopy equivalence,
\[
\Omega!|\mathcal{H}_\mathcal{S}\text{nil}_{fd}(X)| \simeq \Omega!|\mathcal{H}_\mathcal{S}\mathbb{P}_{fd}^{h\mathcal{S}^+}(G)|.
\]
Plugging this into Proposition 3.1 we obtain a homotopy fiber sequence
\[
\Omega|\mathcal{H}_\mathcal{S}\text{nil}_{fd}(X)| \to \Omega|\mathcal{H}_\mathcal{S}\mathbb{P}_{fd}(G)| \to \Omega|\mathcal{H}_\mathcal{S}\mathbb{D}_{fd}(G \times \mathbb{N}_+)|.
\]
Let \(\epsilon : \Omega|\mathcal{H}_\mathcal{S}\mathbb{D}_{fd}(G \times \mathbb{N}_+)| \to \Omega|\mathcal{H}_\mathcal{S}\mathbb{C}_{fd}(G)|\) denote the augmentation map of [H, 7.1], which is induced by
\[
(Y_-, Y, Y_+) \mapsto Y/\mathbb{Z},
\]
where $Y/\mathbb{Z}$ denotes the orbit space under the $\mathbb{Z}$-action. Recall that the nil-term $N_+A^{fd}(X)$ was defined to be the homotopy fiber of $\epsilon$. Similarly, $\epsilon$ restricts to a map on $\Omega h\mathcal{S}\mathbb{P}_{fd}(G)$. Denote the homotopy fiber of this restriction by $\Omega h\mathcal{S}\mathbb{P}_{fd}(G)|^\epsilon$. Consequently, we have an induced homotopy fiber sequence

$$\Omega h\mathcal{S}\text{nil}_{fd}(X)| \rightarrow \Omega h\mathcal{S}\mathbb{P}_{fd}(G)|^\epsilon \rightarrow N_+A^{fd}(X).$$

In was shown in \([H_+] 7.6\) that the second of these maps, $\Omega h\mathcal{S}\mathbb{P}_{fd}(G)|^\epsilon \rightarrow N_+A^{fd}(X)$, is null homotopic. Moreover, it was shown in \([H_+] 7.5\) that there is a homotopy equivalence

$$\Omega h\mathcal{S}\mathbb{P}_{fd}(G)|^\epsilon \simeq \Omega h\mathcal{S}\mathbb{C}_{fd}(G)|$$

induced by the global sections functor $\Gamma: \mathbb{P}_{fd}(G) \rightarrow \mathbb{C}_{fd}(G)$ defined by

$$(Y_-, Y, Y_+) \mapsto CY_- \cup Y \cup CY_+,$$

where $CY_-$ denotes the cone on $Y_-$. Assembling this information, we have a homotopy fiber sequence

$$\Omega h\mathcal{S}\text{nil}_{fd}(X)| \rightarrow \Omega h\mathcal{S}\mathbb{C}_{fd}(G)| \rightarrow \Omega h\mathcal{S}\mathbb{C}_{fd}(G)|^\epsilon \rightarrow N_+A^{fd}(X).$$

where $\alpha$ is induced by the functor $(Z, f) \mapsto \Sigma Z$. Since the suspension functor $\Sigma: \mathbb{C}_{fd}(G) \rightarrow \mathbb{C}_{fd}(G)$ induces a homotopy equivalence (by \([W, 1.6.2]\)), we see that the homotopy fiber of $\alpha$ is homotopy equivalent to the homotopy fiber of the map $\alpha'$ which is induced by the forgetful map $(Z, f) \rightarrow Z$.

On the one hand, the homotopy fiber of $\alpha'$ is $\tilde{K}^{fd}(\text{nil}(X))$, by definition. On the other hand, the homotopy fiber sequence (3) implies that the homotopy fiber of $\alpha$ is homotopy equivalent to $\Omega N_+A^{fd}(X)$. We conclude that there is a homotopy equivalence

$$\tilde{K}^{fd}(\text{nil}(X)) \simeq \Omega N_+A^{fd}(X).$$

This completes the proof of the theorem.

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