THE LAPLACE TRANSFORM OF THE DIGAMMA FUNCTION: AN INTEGRAL DUE TO GLASSER, MANNA AND OLOA

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ABSTRACT. The definite integral
\[ M(a) := \frac{4}{\pi} \int_0^{\pi/2} \frac{x^2 \, dx}{x^2 + \ln^2(2e^{-a} \cos x)} \]
is related to the Laplace transform of the digamma function
\[ L(a) := \int_0^\infty e^{-as} \psi(s + 1) \, ds, \]
by \( M(a) = L(a) + \gamma/a \) when \( a > \ln 2 \). Certain analytic expressions for \( M(a) \)
in the complementary range, \( 0 < a \leq \ln 2 \), are also provided.

1. Introduction

The classical table of integrals by I. S. Gradshteyn and I. M. Ryzhik \[7\] contains
a large collection organized in sections according to the form of the integrand. In
each section one finds a significant variation on the complexity of the integrals. For
example, section 4.33–4.34, with the title \textit{Combinations of logarithms and exponen-
tials}, presents the elementary formula 4.33.1:

\[ \int_0^\infty e^{-ax} \ln x \, dx = -\frac{\gamma + \ln a}{a}, \] where \( \gamma \) is the \textit{Euler constant}

\[ \gamma = \lim_{n \to \infty} \sum_{k=1}^{n} \frac{1}{k} - \ln n. \]

Contained in the same section are the more elaborate 4.332.1 and 4.325.6:

\[ \int_0^\infty \frac{\ln x \, dx}{e^x + e^{-x} - 1} = \int_0^1 \ln \left( \frac{1}{x} \right) \frac{dx}{x^2 - x + 1} = \frac{2\pi}{\sqrt{3}} \ln 2\pi - \ln \Gamma \left( \frac{1}{6} \right). \]

The difficulty involved in the evaluation of a definite integral is hard to measure from
the complexity of the integrand. For instance, the evaluation of \textit{Vardi's integral},

\[ \int_{\pi/4}^{\pi/2} \ln \tan x \, dx = \int_0^1 \ln \left( \frac{1}{x} \right) \frac{dx}{1 + x^2} = \frac{\pi}{2} \ln \left( \frac{\Gamma(\frac{1}{4}) \sqrt{2\pi}}{\Gamma(\frac{1}{4})} \right), \]
that appears as 4.229.7 in [7], requires a reasonable amount of number theory. The second integral form is 4.325.4, found in the section entitled *Combinations of logarithmic functions of more complicated arguments and powers*. The reader will find in [15] a discussion of this formula.

It is a remarkable fact that combinations of elementary functions in the integrand often exhibit definite integrals whose evaluation is far from elementary. A systematic study of the formulas in [7] has been initiated in the series [1, 2, 9, 10, 11, 12]. These papers are organized according to the combinations appearing in the integrand. Even the elementary cases, such as the combination of logarithms and rational functions discussed in [2], entail interesting results. The evaluation

\[ \int_0^b \frac{\ln t}{(1 + t)^{n+1}} \, dt = \frac{1}{n} \left[ 1 - (1 + b)^{-n} \right] \ln b - \frac{1}{n} \ln(1 + b) - \frac{1}{n(1 + b)^{n-1}} \sum_{j=1}^{n-1} \frac{1}{j!} \binom{n-1}{j} |s(j + 1, 2)| b^j, \]

for \( b > 0 \) and \( n \in \mathbb{N} \), produces an explicit formula for the case where the rational function has a single pole. Here, \(|s(n, k)|\) are the unsigned Stirling numbers of the first kind, which count the number of permutations of \( n \) letters having exactly \( k \) cycles. The case of a purely imaginary pole,

\[ \int_0^x \frac{\ln t}{(1 + t^2)^{n+1}} \, dt = \left( \frac{2n}{2^n} \right) g_0(x) + p_n(x) \ln x - \sum_{k=0}^{n-1} \frac{\tan^{-1} x + p_k(x)}{2k + 1}, \]

is expressed in terms of the rational function

\[ p_n(x) = \sum_{j=1}^n \frac{2^{2j}}{2j(\,)^2 (1 + x^2)^j}, \]

and with

\[ g_0(x) = \ln x \tan^{-1} x - \int_0^x \frac{\tan^{-1} t}{t} \, dt. \]

The special case \( x = 1 \) becomes

\[ \int_0^1 \frac{\ln t}{(1 + t^2)^{n+1}} \, dt = -2^{-2n} \frac{2n}{n} \left( G + \sum_{k=0}^{n-1} \frac{\frac{\pi}{2} + p_k(1)}{2k + 1} \right), \]

where

\[ G = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k + 1)^2} \]

is Catalan’s constant. The values

\[ p_k(1) = \sum_{j=1}^k \frac{2^j}{2j(\,)^2} \]

do not admit a closed form (in the sense of [14]), but they do satisfy the three-term recurrence

\[ (2k + 1)p_{k+1}(1) - (3k + 1)p_k(1) + kp_{k-1}(1) = 0. \]
The study of definite integrals, where the integrand is a combination of powers, logarithms and trigonometric functions, was initiated by Euler \[5\], with the evaluation of
\[
\int_0^{\pi/2} x \ln(2 \cos x) \, dx = -\frac{7}{16} \zeta(3)
\]
and
\[
\int_0^{\pi/2} x^2 \ln(2 \cos x) \, dx = -\frac{\pi}{4} \zeta(3),
\]
which appear in his study of the Riemann zeta function at the odd integers. These type of integrals have been investigated in \[8\], \[16\]. The intriguing integral of D. and J. Borwein \[3\],
\[
\int_0^{\pi/2} x^2 \ln^2(2 \cos x) \, dx = \frac{11 \pi^4}{16} \zeta(4) = \frac{11 \pi^8}{1440},
\]
was first conjectured on the basis of a numerical computation by Enrico Au-Yueng while he was an undergraduate student at the University of Waterloo. A nice example of experimental mathematics in action.

Recently O. Oloa considered the integral
\[
M(a) := \frac{4}{\pi} \int_0^{\pi/2} \frac{x^2 \, dx}{x^2 + \ln^2(2e^{-a} \cos x)},
\]
and the special value
\[
M(0) = \frac{1}{2} \left( 1 + \ln(2\pi) - \gamma \right)
\]
is established in \[13\].

Oloa’s method of proof relies on the expansion
\[
\frac{x^2}{x^2 + \ln^2(2 \cos x)} = x \sin 2x + \sum_{n=1}^{\infty} (-1)^{n-1} \left( \frac{a_n}{n!} - \frac{a_{n+1}}{(n+1)!} \right) x \sin(2nx).
\]
Here
\[
a_n := \int_0^1 (t)_n \, dt,
\]
where \((t)_n = t(t+1) \cdots (t+n-1)\) is the Pochhammer symbol. The standard relation
\[
(t)_n = \sum_{k=1}^{n} |s(n,k)| t^k
\]
gives
\[
a_n = \sum_{k=1}^{n} \frac{|s(n,k)|}{k+1}.
\]

M. L. Glasser and D. Manna \[6\] introduced the function
\[
L(a) := \int_0^{\infty} e^{-as} \psi(s + 1) \, ds,
\]
where \( \psi(x) = \frac{d}{dx} \ln \Gamma(x) \) is the digamma function. After integrating by parts and making use of (1.1), one finds

\[
L(a) = -\gamma - \ln a + a \int_0^\infty e^{-at} \ln \Gamma(t) \, dt.
\]

The main result in [6] gives a relation between \( M(a) \) and \( L(a) \).

**Theorem 1.1.** If \( a > \ln 2 \), then

\[
M(a) = L(a) + \frac{\gamma}{a}.
\]

That is, for \( a > \ln 2 \),

\[
(1.22) \quad M(a) = \frac{\gamma}{a} - \gamma - \ln a + a \int_0^\infty e^{-at} \ln \Gamma(t) \, dt.
\]

The proof in [6] begins with the representation

\[
(1.23) \quad \int_0^\pi/2 \cos^\nu x \cos ax \, dx = \frac{\pi \Gamma(\nu + 2)}{2^{\nu+1}(\nu + 1) \Gamma(1 + \frac{\nu}{2} + \frac{a}{2}) \Gamma(1 + \frac{\nu}{2} - \frac{a}{2})} \tag{3.631.9 in [7]}
\]

Differentiating with respect to \( a \), evaluating at \( a = s \), and using \( \psi(1) = -\gamma \) yields

\[
(1.24) \quad \psi(s + 1) = \frac{2s+2}{\pi} \int_0^{\pi/2} x \cos^s x \sin(sx) \, dx - \gamma.
\]

Replacing (1.24) into (1.20) produces

\[
(1.25) \quad L(a) + \frac{\gamma}{a} = -\frac{4}{\pi} \Im \int_0^\infty \int_0^{\pi/2} xe^{s(\ln[2e^{-a} \cos x] - ix)} \, dx \, ds.
\]

The identity (1.22) follows from evaluating the \( s \)-integral as

\[
(1.26) \quad \int_0^\infty e^{s(\ln[2e^{-a} \cos x] - ix)} \, ds = \frac{1}{ix - \ln[2e^{-a} \cos x]}.\tag{1.26}
\]

The authors of [6] produced a series expansion for \( M(a) \), which they recognize as a hypergeometric function in two variables, and state that this strongly suggests that for a general value of \( a \), no further progress is possible. The hypergeometric expression gives

\[
(1.27) \quad M(0) = 1 + \frac{1}{2} \int_0^1 t(1-t) \, _3F_2(1, 1, 2-t; 2, 3; 1) \, dt,
\]

on which they invoke

\[
(1.28) \quad _3F_2(1, 1, 2-t; 2, 3; 1) = \frac{2(1 - \gamma - \psi(t+1))}{1-t}
\]

to give a new proof of (1.15).

The graph of \( M(a) \) shown in Figure 1, obtained by the numerical integration of (1.14), has a well-defined cusp at \( a = \ln 2 \). In this paper, analytic expressions for both branches of \( M(a) \) are provided. The region \( a > \ln 2 \), determined in [6], has been reviewed in the present section. The corresponding expressions for \( 0 < a < \ln 2 \) is the content of the next section.
2. The case $0 < a < \ln 2$

The representation

\[ M(a) = -\frac{e^a}{2\pi} \operatorname{Im} \int_0^1 e^{-at} \int_{-\pi}^{\pi} \frac{x(1 + e^{ix})}{1 - e^a + e^{ix}} \, dx \, dt \]

was established in \[6\]. Their proof is replicated here for the sake of the reader’s convenience. The identity

\[ \operatorname{Im} \frac{x}{ix + \ln [2e^{-a} \cos x]} = \frac{x^2}{x^2 + \ln^2 [2e^{-a} \cos x]} \]

yields

\[ M(a) = \frac{4}{\pi} \operatorname{Im} \int_{0}^{\pi/2} \frac{x \, dx}{ix + \ln [2e^{-a} \cos x]} . \]

If $a > \ln 2$, then

\[ \int_{0}^{\infty} e^{s \ln [2e^{-a} \cos x] + ix} \, ds = \frac{1}{ix + \ln [2e^{-a} \cos x]} . \]

This implies

\[ M(a) = \frac{2}{\pi} \operatorname{Im} \int_{-\pi/2}^{\pi/2} \int_{0}^{\infty} xe^{s \ln [2e^{-a} \cos x] + ix} \, dx \, ds , \]

where one uses the fact that the imaginary part of the integrand is an even function of $x$. One more identity,

\[ e^{ix} \cdot e^{s \ln [2e^{-a} \cos x]} = e^{s \ln [e^{-a}(1 + e^{ix})]} , \]

and the change of variables $x \mapsto x/2$, give the equality

\[ M(a) = \frac{1}{2\pi} \operatorname{Im} \int_{-\pi}^{\pi} \int_{0}^{\infty} xe^{s \ln [e^{-a}(1 + e^{ix})]} \, ds \, dx . \]

Evaluating the $s$-integral yields

\[ M(a) = -\frac{1}{2\pi} \operatorname{Im} \int_{-\pi}^{\pi} \frac{x \, dx}{\ln [e^{-a}(1 + e^{ix})]} . \]
The formula
\[
\frac{1}{\ln u} = \int_0^1 \frac{u^t}{u - 1} \, dt
\]
now gives (2.1) from (2.8).

Note 2.1. The proof outlined above is valid for \( a > \ln 2 \), but (2.1) holds for \( a > 0 \).

Notation. Define \( b := e^a - 1 \) and let \( 0 < a < \ln 2 \) so that \( 0 < b < 1 \).

The terms \((1 + e^{ix})^t\) and \(1/(1 - be^{-ix})\) from (2.1) are now expanded in a power series to produce
\[
M(a) = -\frac{e^a}{2\pi} \int_0^1 \int_{-\pi}^\pi x e^{-at} \sum_{j=0}^\infty \sum_{k=0}^\infty b^j \binom{t}{k} \sin[x(k-j-1)] \, dx \, dt.
\]
The term corresponding to \( k = j+1 \) disappears, and a computation of the \( x \)-integral gives
\[
M(a) = -\frac{e^a}{2\pi} \int_0^1 \int_{-\pi}^\pi x e^{-at} \sum_{j=0}^\infty \sum_{k=0}^\infty b^j \binom{t}{k} \sin[x(k-j-1)] \, dx \, dt.
\]

Lemma 2.1. Let \( t \in \mathbb{R} \) and \( j \in \mathbb{N} \cup \{0\} \). Then
\[
\sum_{\nu=1}^\infty \frac{(-1)^\nu}{\nu} \binom{t}{\nu+j} = \binom{t}{j} \left[ \psi(j+1) - \psi(t+1) \right].
\]

Proof. The integral representation (3.268.2 in [7])
\[
\psi(p+1) - \psi(q+1) = -\int_0^1 \frac{x^p - x^q}{1-x} \, dx
\]
yields
\[
\psi(p+1) - \psi(q+1) = \sum_{j=1}^\infty (-1)^{j-1} \binom{p}{j} - \binom{q}{j}.
\]
The result now follows from the identity
\[
\binom{t}{k} = \sum_{m=1}^k \frac{(-1)^m}{m} \binom{t}{m+k} - \sum_{m=1}^k \frac{(-1)^m}{m} \binom{t}{m} = \sum_{m=1}^k \frac{(-1)^{m-1}}{m} \binom{k}{m}.
\]
Apply the difference operator \( \Delta a(k) := a(k+1) - a(k) \) and use
\[
\binom{t}{k+1} = \binom{t}{m+k+1} - \binom{t}{m+k} = -\frac{m}{k+1} \binom{t}{k+1} - \binom{t+1}{m+k+1}
\]
to write the derived equation as
\[
\sum_{m=1}^k \frac{(-1)^m}{m} \binom{k}{m} = \Delta \sum_{m=1}^k \frac{(-1)^m}{m} \binom{k}{m}.
\]
The left hand side of (2.14) reduces to $-1/(k+1)$ in view of the classical identity
\begin{equation}
\sum_{m=1}^{\infty} (-1)^{m-1} \binom{t+1}{m+k+1} = \binom{t}{k+1}.
\end{equation}
A simple evaluation of the right hand side in (2.14) also produces $-1/(k+1)$. Therefore both sides of (2.14) are, up to a constant, the harmonic number $H_k$. The special case $k = 0$ shows that this constant vanishes.

Continuing from (2.10), it follows that
\begin{equation}
M(a) = e^a \int_{0}^{1} \sum_{j=0}^{\infty} b^j \sum_{k=0}^{j} \frac{(-1)^{j-k} \binom{j}{k}}{j+1-k} \frac{e^{-at}}{b} \int_{0}^{1} e^{-at} \sum_{j=1}^{\infty} b^j \binom{t}{j} \psi(j+1) \ dt - \frac{e^a}{b} \int_{0}^{1} e^{-at} \sum_{j=1}^{\infty} b^j \binom{t}{j} \psi(t+1) \ dt = M_1 + M_2 + M_3.
\end{equation}

To simplify $M_1$, observe
\begin{equation}
\sum_{j=0}^{\infty} b^j \sum_{k=0}^{j} \frac{(-1)^{j-k} \binom{j}{k}}{j+1-k} = \sum_{k=0}^{\infty} b^k \binom{t}{k} \sum_{\nu=0}^{\infty} \frac{(-1)^{\nu} b^{\nu}}{\nu+1} \sum_{\nu=0}^{\infty} \frac{(-1)^{\nu} b^{\nu}}{\nu+1} = \frac{\ln(1+b)}{b} \sum_{k=0}^{\infty} \binom{t}{k} b^k = \frac{ae^a}{b}.
\end{equation}
Thus, $M_1 = a/(1 - e^{-a})$.

The reduction of $M_2$ employs the following result.

Lemma 2.2. If $0 < a < \ln 2$, then
\begin{equation}
\int_{0}^{1} e^{-at} \sum_{j=0}^{\infty} b^j \binom{t}{j} \psi(j+1) \ dt = \ln(1 - e^{-a}) + \int_{1}^{\infty} \frac{e^{-at}}{t} \ dt.
\end{equation}

Proof. The Stirling numbers $s(j,k)$ satisfy
\begin{equation}
\binom{t}{j} = \sum_{k=0}^{j} s(j,k) t^k,
\end{equation}
so that
\begin{equation}
\int_{0}^{1} e^{-at} \sum_{j=0}^{\infty} b^j \binom{t}{j} \psi(j+1) \ dt = \frac{e^{-a} b^{\gamma}}{a} + e^{-a} \sum_{j=1}^{\infty} (b^{j+1} \alpha_j - b^j \alpha_{j-1}) \psi(j+1),
\end{equation}
with
\begin{equation}
\alpha_j(a) = \frac{1}{j!} \sum_{k=0}^{j} \frac{s(j,k) k!}{a^{k+1}}.
\end{equation}
The result now follows from integration by parts and the identity
\begin{equation}
\sum_{j=k}^{\infty} s(j,k) b^j = \frac{\ln^k(1+b)}{k!}.
\end{equation}
Therefore,
\begin{equation}
M_2 = \frac{\ln(1 - e^{-a})}{1 - e^{-a}} + \frac{1}{1 - e^{-a}} \int_1^\infty \frac{e^{-at}}{t} \, dt.
\end{equation}

Finally,
\begin{equation}
M_3 = -\frac{e^a}{b} \int_0^1 e^{-at} \left( \sum_{j=1}^{\infty} \binom{t}{j} b^j \right) \psi(t + 1) \, dt = -\frac{e^a}{b} \int_0^1 (1 - e^{-at}) \psi(t + 1) \, dt.
\end{equation}

A direct computation shows that \( \int_0^1 \psi(t + 1) \, dt = 0 \), and integration by parts gives
\begin{equation}
M_3 = \frac{a}{1 - e^{-a}} \int_0^1 e^{-at} \ln(t + 1) \, dt.
\end{equation}

The identity \( \ln \Gamma(t + 1) = \ln \Gamma(t) + \ln t \) now yields
\begin{equation}
M_3 = \frac{a}{1 - e^{-a}} \left( \int_0^1 e^{-at} \ln t \, dt + \int_0^1 e^{-at} \ln \Gamma(t) \, dt \right).
\end{equation}

Replacing (2.17), (2.19) and (2.21) into (2.17) provides an expression for \( M(a) \):
\begin{equation}
M(a) = \frac{a}{1 - e^{-a}} + \frac{\ln(1 - e^{-a})}{1 - e^{-a}} + \frac{a}{1 - e^{-a}} \int_0^1 e^{-at} \ln \Gamma(t) \, dt.
\end{equation}

The term \( \gamma/a \) comes from the index \( j = 0 \) in the sum (2.18). The main result presented here now follows from (2.1). This settles a conjecture of O. Oloa presented in [13].

**Theorem 2.1.** If \( 0 < a < \ln 2 \), then
\begin{equation}
M(a) = \frac{\gamma}{a} + \frac{a + \ln(1 - e^{-a}) - \gamma - \ln a}{1 - e^{-a}} + \frac{a}{1 - e^{-a}} \int_0^1 e^{-at} \ln \Gamma(t) \, dt.
\end{equation}

The above result is complementary to Theorem 1.1.

**Corollary 2.1.** If \( 0 < a < \ln 2 \), then
\begin{equation}
M(a) = \frac{\gamma}{a} + \frac{a + \ln(1 - e^{-a}) + \Gamma(0,a)}{1 - e^{-a}} + \frac{1}{1 - e^{-a}} \int_0^1 e^{-at} \psi(t + 1) \, dt,
\end{equation}

where \( \Gamma(0,a) \) is the incomplete gamma function.

**Proof.** Split up the first integral in (2.26) and integrate by parts. \( \square \)

The derivative of (2.1) at \( a = 0 \), the classical values
\begin{equation}
\int_0^1 \ln \Gamma(t) \, dt = \frac{1}{2} \ln 2 \pi
\end{equation}

and
\begin{equation}
\int_0^1 t \ln \Gamma(t) \, dt = \frac{\zeta'(2)}{2\pi^2} + \frac{1}{6} \ln 2 \pi - \frac{\gamma}{12},
\end{equation}

obtained in [4], give
\begin{equation}
\int_0^{\pi/2} \frac{x^2 \ln(2 \cos x) \, dx}{(x^2 + \ln^2(2 \cos x))^2} = \frac{7\pi}{192} + \frac{\pi \ln 2 \pi}{96} - \frac{\zeta'(2)}{16\pi}.
\end{equation}
Further differentiation of (2.21) produces the evaluation of a family of integrals similar to (2.28).

The integral in (2.21) can be expressed in an alternative form. Define

\[ \Lambda(z) := \lim_{n \to \infty} \left( \sum_{j=1}^{n} \frac{j}{j^2 + z^2} - \ln n \right). \]

Observe that \( \Lambda(0) = \gamma \), so \( \Lambda(z) \) is a generalization of Euler’s constant.

**Lemma 2.3.** Let \( a > 0 \), \( c = 1 - e^{-a} \) and define \( A := \ln 2 + \gamma \). Then

\[ \int_{0}^{1} e^{-at} \ln \Gamma(t) \, dt = \frac{A(a - c)}{a^2} - \frac{c}{2a} \Lambda \left( \frac{a}{2\pi} \right) + 2c \sum_{j=1}^{\infty} \frac{\ln j}{a^2 + 4\pi^2 j^2}. \]

**Proof.** Expand the exponential into a Maclaurin series and use the value

\[ \int_{0}^{1} t^n \ln \Gamma(t) \, dt = \frac{1}{n+1} \sum_{k=1}^{n+1} (-1)^k \binom{n+1}{k-1} \frac{(2k)!}{k!(2\pi)^{2k}} [A\zeta(2k) - \zeta'(2k)] \]

\[ - \sum_{k=1}^{\infty} \frac{(2k)!}{(2\pi)^{2k} \zeta(2k+1)} + \frac{\ln \sqrt{2\pi}}{n+1} \]

given as (6.14) in [3]. Then interchange the resulting double sums. \qed

The next corollary follows from the identity \( M(a) = L(a) + \gamma/a \).

**Corollary 2.2.** If \( 0 < a < \ln 2 \) and \( c = 1 - e^{-a} \), then

\[ \int_{0}^{\infty} e^{-at} \ln \Gamma(t) \, dt = -1 + \gamma + \ln a \frac{a}{ae^a} + \frac{A(a - c)}{a^2} - \frac{c}{2a} \Lambda \left( \frac{a}{2\pi} \right) + 2 \sum_{j=1}^{\infty} \frac{\ln j}{a^2 + 4\pi^2 j^2}. \]

**Lemma 2.4.** Let \( f(t) = 2^{-t} \ln \Gamma(t) \). Then

\[ \int_{0}^{\infty} f(t) \, dt = 2 \int_{0}^{1} f(t) \, dt - \gamma + \ln \ln 2 \frac{\ln 2}{\ln 2}, \]

\[ \int_{0}^{\infty} t f(t) \, dt = 2 \int_{0}^{1} (t + 1) f(t) \, dt - \frac{(\gamma + \ln \ln 2)(1 + \ln 2) - 1}{\ln^2 2}. \]

**Proof.** The function \( f(t) \) satisfies \( f(t+1) = \frac{1}{2} f(t) + \frac{1}{2} 2^{-t} \ln t \). Splitting the integral

\[ \int_{0}^{\infty} f(t) \, dt = \int_{0}^{1} f(t) \, dt + \int_{1}^{\infty} f(t+1) \, dt \]

and using (1.1) gives the first result. The proof of (2.33) is similar; it only requires differentiating (1.1) with respect to the parameter \( a \). \qed

The reader will check that (2.32) is equivalent to the continuity of \( M(a) \) at \( a = \ln 2 \). The identity (2.28) provides a proof of the next theorem, which in itself is worthy of singular (pun intended) interest.

**Theorem 2.2.** The jump of \( M'(a) \) at \( a = \ln 2 \) is 4.
3. Conclusions

The integral
\[ M(a) := \frac{4}{\pi} \int_0^{\pi/2} \frac{x^2 \, dx}{x^2 + \ln^2(2e^{-a} \cos x)} \]
satisfies
\[ M(a) = \frac{\gamma}{a} + \int_0^\infty e^{-at} \psi(t+1) \, dt \]
for \( a > \ln 2 \) and
\[ M(a) = \frac{\gamma}{a} + \frac{a + \ln(1 - e^{-a}) + \Gamma(0, a)}{1 - e^{-a}} + \int_0^1 e^{-at} \psi(t+1) \, dt \]
for \( 0 < a \leq \ln 2 \).

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