NEAR-SYMMETRY IN $A_\infty$ AND REFINED JONES FACTORIZATION

WINSTON OU

(Communicated by Michael T. Lacey)

Abstract. We use variants of the Hardy-Littlewood maximal and the Cruz-Uribe–Neugebauer minimal operators to give direct characterizations of $A_1$ and $RH_\infty$ that clarify their near symmetry and yield elementary proofs of various known results, including Cruz-Uribe and Neugebauer’s refinement of the Jones factorization theorem.

1. Introduction

In 1972, B. Muckenhoupt [9] demonstrated that the weights $w$ for which the Hardy-Littlewood maximal operator $M$ was bounded on $L^p(wdx)$ $(1 < p < \infty)$ were those which belonged to the $A_p$ class, i.e., satisfying the condition

$$A_p(w) = \sup_{Q} \left(\frac{1}{|Q|} \int_Q w(x) \left(\frac{1}{|Q|} \int_Q w(x)^\frac{1}{p-1}\right)^{p-1}\right) < \infty.$$  

The classical theory of such $A_p$ weights (whose union $\bigcup_p A_p$ is denoted by $A_\infty$) reached a peak in 1980 with the factorization theorem of P. Jones [8], which stated that $w \in A_p$ if and only if $w = w_0 w_1^{-1/p}$ for some $w_0, w_1 \in A_1$, where the limiting class $A_1$ of weights for which $Mw(x) \leq cw(x)$ a.e. was in turn characterized by Coifman and Rochberg as $A_2 \cap eBLO$ [2] that same year. (For the standard account of the theory, definitions, and notation, see the text of Garcia-Cuerva and Rubio de Francia [4].)

In their elegant 1995 paper [3], Cruz-Uribe and Neugebauer inverted the common view of $A_\infty$ by focusing on the structure of the reverse Hölder classes $RH_s$ (where $w \in RH_s$ if $RH_s(w) := \inf \left\{C \left(\frac{1}{|Q|} \int_Q w^n\right)^{1/s} \leq C \frac{1}{|Q|} \int_Q w \right\} < \infty$) rather than on the $A_p$ structure. In particular, they showed convincingly that their minimal operator $mf(x) = \inf_{Q \ni x} \frac{1}{|Q|} \int_Q |f|$ filled a role with respect to the $RH_s$ structure mirroring exactly that played by the maximal operator with respect to the $A_p$ structure: for example, the class $RH_\infty$, defined analogously with $A_1$ as $\{w \mid \exists c > 0 \text{ s.t. } cw(x) \geq w(x) \text{ a.e.}\}$, emerged as the limiting class of the $RH_s$, $s > 1$; $m$ mapped $\infty$ into $RH_\infty$, just as $M$ mapped $\infty$ into $A_1$ [5]; and further, $RH_\infty$ could be characterized as $eBLO$ (and thus $A_1 = \frac{1}{RH_\infty \cap A_2}$; i.e., the limiting classes were nearly reciprocal). Their paper culminated with a symmetric
version of the Jones factorization that encompassed both $A_p$ and $RH_s$ data: that $w \in A_p \cap RH_s$ if and only if $w = w_0 w_1$, where $w_0 \in A_1 \cap RH_s$ and $w_0 \in A_p \cap RH_\infty$.

In this paper, we use variations of the Hardy-Littlewood maximal and Cruz-Uribe–Neugebauer minimal operators to simplify our understanding of the fundamental, near-symmetric structure of $A_\infty$ and its connection with $BMO$. We begin in section 2 by introducing the natural minimal operator $m^\natural$, which possesses properties identical to the natural maximal operator studied in [10]: in particular, it commutes with the logarithm theorem on $A_\infty$ and can be used to characterize $BUO$. In section 3, we show that these properties form the crux of both a refined version of Coifman-Rochberg’s characterization of $A_1$ as $A_2 \cap e^{BLO}$ and, simultaneously, a direct proof of Cruz-Uribe–Neugebauer’s characterization $RH_\infty = e^{BUO}$ that cleanly reveals the source of the asymmetry between the limiting weight classes. In section 4, we show how the above characterizations immediately yield various known properties of $A_1$ and $RH_\infty$, including, significantly, Cruz-Uribe–Neugebauer’s improvement of the Jones factorization theorem. We conclude in section 5 with some comments about the reciprocal nature of the $A_p$ and $RH_s$ structures themselves.

The author would like to express his deep gratitude to the anonymous PAMS referee for his candor and generous comments and suggestions (in particular, he pointed out the connection with the Jones factorization and the remark at the end of section 3), Professor Y. Maeda and Ms. Y. Igarashi of the COE program at Keio University for providing a wonderful environment in which to work during the summers, and the members of the Real Analysis Seminar at the University of Minnesota for their kind encouragement.

## 2. Natural maximal and minimal operators

We first review and extend some previous work [10]. Let us recall the definition of the natural maximal operator,

$$M^\natural f(x) = \sup_{Q \ni x} \frac{1}{|Q|} \int_Q f,$$

where $f \in L^1_{\text{loc}}(\mathbb{R}^n)$, and in this paper $Q$ will always range over cubes in $\mathbb{R}^n$ with sides parallel to the coordinate axes. This variant of the Hardy-Littlewood maximal operator was introduced by Bennett [11] in 1982 to study $BLO$, the functions of bounded lower oscillation, i.e., functions $\phi$ such that over all cubes $Q$, \[ \frac{1}{|Q|} \int_Q \phi - \inf_Q \phi \leq C \] (we will use $||\phi||_{BLO}$ to denote the infimum of such $C$; $BUO$, the space of functions of bounded upper oscillation, and $||\phi||_{BUO}$ are defined analogously). $M^\natural$ was later re-introduced in [10] as the heart of a simple proof of the boundedness of $M$ on $BMO$, in which the following two properties were central.

**Lemma 2.1** (Commutation [10]). For $w \in A_\infty$, $0 \leq ||\log M^3 - M^\natural \log w||_\infty \leq \log A_\infty(w)$.

**Lemma 2.2** (Characterization of $BLO$ [11][10]). $\phi \in BLO \iff (M^3 - I)\phi \in L^\infty$, in which case $||\phi||_{BLO} = ||(M^3 - I)\phi||_\infty$.

It will be useful later to observe that if we analogously define the natural minimal operator as $m^\natural f(x) = \inf_{Q \ni x} \frac{1}{|Q|} \int_Q f$, then clearly $M^\natural f(x) = -m^\natural(-f)(x)$. Consequently, we have the following properties, which will be critical in the next section.
Lemma 2.3 (Commutation). For all $w \in A_\infty$, $0 \leq |\log m^3 - m^3 \log w(x)| \leq \log A_\infty(w)$.

Lemma 2.4 (Characterization of BUO). $\phi \in BUO \iff (I - m^3)\phi \in L^\infty$, in which case $\|\phi\|_{BUO} = \|(I - m^3)\phi\|_\infty$.

We also note, digressing slightly, that the observation further immediately implies (see [10]) the following set of results about the behavior of $m^3$ and $m$; thus the symmetry, mentioned in the introduction, between the behavior of $m$ and $M$ on $BMO$ and $A_\infty$ is nearly tautological.

Theorem 2.5. $m^3$ maps $BMO$ boundedly into $BUO$.

Corollary 2.6. $m$ maps $BMO$ boundedly into $BUO$.

Theorem 2.7. Boundedness of $m^3 : BMO \to BUO$ implies $m(A_\infty) \subset RH_\infty$, where $RH_\infty(mw)$ depends only on $RH_\delta(w)$ and $s$.

3. Asymmetric Characterizations of $A_1$ and $RH_\infty$

We now use the above properties of the natural maximal and minimal operators to give characterizations of $A_1$ and $RH_\infty$. First we present a somewhat surprising and further, sharp, characterization of $A_1$, refining the result of Coifman-Rochberg [2] that $w \in A_2 \cap e^{\text{BLO}} \iff w \in A_1$.

Theorem 3.1 (Characterization of $A_1$). $w \in A_1 \iff w \in A_\infty \cap e^{\text{BLO}}$. Precisely, $e^{||\log w||_{\text{BLO}}} \leq A_1(w) \leq A_\infty(w)e^{||\log w||_{\text{BLO}}}$.

Proof. $\Rightarrow$ This direction is well-known; we follow [3, p. 157]. $w \in A_1$ implies that for every $Q$,

$$\left(\frac{1}{|Q|}\int_Q e^{\log w(x)} dx\right)(\sup_{x \in Q} e^{\log w(x)}) \leq A_1(w),$$

i.e.,

$$\left(\frac{1}{|Q|}\int_Q e^{\log w(x)} dx\right)(e^{-\inf_{x \in Q} \log w(x)}) \leq A_1(w).$$

Then, by Jensen’s inequality,

$$e^{\frac{1}{|Q|}\int_Q \log w(x) dx - \inf_{x \in Q} \log w(x)} \leq A_1(w),$$

for every $Q$, i.e.,

$$e^{||\log w||_{\text{BLO}}} \leq A_1(w).$$

$\Leftarrow$ Say that $w \in e^{\text{BLO}} \cap A_\infty$; $\log w \in BLO$. By the characterization of $BLO$ (Lemma 2.2) above,

$$M^3 \log w(x) \leq \log w(x) + ||\log w||_{\text{BLO}} \text{ a.e.}$$

Further, since $w \in A_\infty$, the commutation lemma, Lemma 2.1, for $M^3$ implies

$$\log M^3 w(x) - \log A_\infty(w) \leq M^3 \log w(x) \text{ a.e.}$$

Combining the two yields

$$\log M^4 w(x) \leq \log A_\infty(w) + \log w(x) + ||\log w||_{\text{BLO}} \text{ a.e.,}$$

i.e.,

$$Mw(x) \leq [A_\infty(w)e^{||\log w||_{\text{BLO}}}] w(x) \text{ a.e.}$$

Considering the case of constant weights shows the bound to be sharp. \qed
Applying exactly the same method in the reverse Hölder setting yields a proof of the characterization of Cruz-Uribe–Neugebauer that \( RH_\infty = e^{BUO} \), an approach that cleanly reveals how and why the asymmetrical relation between the limiting classes arises. First recall that \( RH_\infty(w) = \inf \{ \epsilon \mid \epsilon \geq w \text{ a.e.} \} \). It is easy to see that \( RH_\infty \subset \bigcap_s RH_s \) (Theorem 4.1, [3]), so that \( RH_\infty \subset A_\infty \).

**Theorem 3.2** (Characterization of \( RH_\infty \)). \( w \in RH_\infty \iff w \in e^{BUO} \). Precisely, \( RH_\infty(w) \leq e^{|| \log w ||_{BUO}} \leq A_\infty(w)RH_\infty(w) \).

**Proof.** \( \Leftarrow \) Say that \( w \in e^{BUO} \); \( \log w \in BUO \). By the characterization of \( BUO \) (Lemma 2.4),

\[
m^k \log w(x) \geq \log w(x) - || \log w ||_{BUO} \text{ a.e.}
\]

Now, although we have not assumed \( w \in A_\infty \), we do not need the full commutation lemma, Lemma 2.4 but only that part based on Jensen’s inequality, i.e.,

\[
\log m^k w(x) \geq m^k \log w(x) \text{ a.e.}
\]

Thus

\[
\log m^k w(x) \geq \log w(x) - || \log w ||_{BUO} \text{ a.e.},
\]

i.e.,

\[
w(x) \geq e^{-|| \log w ||_{BUO}} w(x) \text{ a.e.,}
\]

as desired. (Notice that this demonstrates \( e^{BUO} \subset A_\infty \), as \( RH_\infty \subset \bigcap_s RH_s \).)

\( \Rightarrow \) Say that \( w \in A_\infty \cap RH_\infty \). By the reverse Jensen inequality and the definition of \( RH_\infty \), we see that

\[
A_\infty(w)e^{\frac{1}{|Q|}\int_Q \log w(x)} \geq \frac{1}{|Q|}\int_Q w(x) \geq \frac{1}{RH_\infty(w)} w(x);
\]

i.e. \( \log A_\infty(w) + \log RH_\infty(w) \geq \log w(x) - \frac{1}{|Q|}\int_Q \log w \) for all \( x \in Q \). Taking the supremum over \( x \in Q \) shows the \( BUO \) norm bounded by \( \log A_\infty(w) + \log RH_\infty(w) \).

As in the previous theorem, considering the case of constant weights shows the bounds to be sharp. \( \Box \)

**Remark.** The characterization of \( A_1 \) (Theorem 3.1) is only “somewhat surprising” in that it can be realized as a simple consequence of \( A_1 = A_2 \cap e^{BLO} \) and Proposition 4.1 below. Given any \( w \in A_\infty \cap e^{BLO} \), by the John-Nirenberg inequality \( w^s \) (for any \( 0 < \epsilon < \frac{1}{\log || \log w ||_{BLO}} \)) will lie in \( A_2 \) (see [4], p. 409); thus \( w^s \in A_2 \cap e^{BLO} = A_1 \).

By Proposition 4.1 (which can be proven independently of Theorem 3.1 see [3]) below, \( w \in A_1 \). However, this approach, though simple, does not reveal the heart of the asymmetry.

4. **Consequences of \( A_1 = A_\infty \cap e^{BLO} \) and \( RH_\infty = e^{BUO} \)**

With the above characterizations in hand, various important properties of \( A_1 \) and \( RH_\infty \) now become transparent. For example, the following two propositions were used in [3] by Johnson and Neugebauer to characterize homeomorphisms preserving \( A_1 \) (i.e., \( h \) such that \( w \circ h \cdot h^{\alpha} \in A_1 \) for all \( w \in A_1 \), \( 0 < \alpha \leq 1 \)):

**Proposition 4.1.** If \( w \in A_\infty \) and \( w^s \in A_1 \) for any \( s > 0 \), then \( w \in A_1 \).

**Proof.** \( w^s \in A_1 \) implies \( \log w^s = s \log w \in BLO \); since \( BLO \) is closed under multiplication by positive scalars, \( \log w \in BLO \) also. \( \Box \)
Proposition 4.2. If \( w, w^{-1} \in A_1 \), then \( w \approx 1 \); i.e., \( w \) is bounded below away from zero and above.

Proof. \( w, w^{-1} \in A_1 \) implies that \( \log w \) and \( -\log w \) are in \( BLO \); thus \( \log w \in L^\infty \).

On the reverse Hölder side, one has elementary proofs of various properties (first given in [3]) of \( RH_\infty \) analogous to those of \( A_1 \), the first of which was originally used to demonstrate the characterization \( RH_\infty = e^{BUO} \).

Proposition 4.3. The following are equivalent:

1. \( w \in RH_\infty \).
2. \( w^{s_0} \in RH_\infty \) for some \( s_0 > 0 \),
3. \( w^s \in RH_\infty \) for all \( s > 0 \).

Proof. \( \Rightarrow \) (2): \( w^{s_0} \in RH_\infty = e^{BUO} \) implies \( \log w^{s_0} = s_0 \log w \in BUO \). Since \( BUO \) is closed under multiplication by positive scalars, \( s \log w \in BUO \) for all \( s > 0 \).

\( \Rightarrow \) (3): Suppose \( \phi \) is both in \( RH_\infty \) and \( e^{BUO} \). Since \( BUO \) is closed under vector addition, \( \phi w = e^{\log \phi + \log w} \in e^{BUO} \).

The characterization also cleanly yields Cruz-Uribe–Neugebauer’s characterization \((\text{[3]})\) of the multipliers of \( RH_\infty \), i.e., those functions \( \phi \) such that \( \phi w \in RH_\infty \) for all \( \phi \in RH_\infty \).

Theorem 4.5. \( \phi \) is a multiplier of \( RH_\infty \) \( \iff \) \( \phi \in RH_\infty \).

Proof. \( \Rightarrow \) Again obvious, since \( w = 1 \) is in \( RH_\infty \).

\( \Leftarrow \) Suppose \( \phi = e^f \) and \( w = e^g \) are both in \( RH_\infty = e^{BUO} \). Since \( BUO \) is closed under vector addition, \( \phi w = e^{f+g} \in e^{BUO} \).

The characterization of the multipliers of \( A_1 \) (by Johnson and Neugebauer [6]) as \( \bigcap_{b>1} A_p \cap e^{BLO} \), for its part, was simplified in [3] using the following result, which now becomes a consequence of the duality \( w \in A_p \iff w^{-1/p} \in A_p' \).

Theorem 4.6 (Cruz-Uribe–Neugebauer). \( (p > 1). w \in A_1 \iff w^{1-p} \in A_p \cap RH_\infty \).

Proof. \( \Rightarrow \) \( w \in A_1 \implies w \in A_p \implies w^{1-p} \in A_p' \); further, since \( A_1 \subset e^{BLO} \), \( w^{1-p} \in e^{BUO} = RH_\infty \), so \( w^{1-p} \in A_p \cap RH_\infty \).

\( \Leftarrow \) \( w^{1-p} \in A_p \implies w^{(1-p)(1-p')} = w \in A_p' \); and \( w^{1-p} \in RH_\infty = e^{BUO} \) (see below) implies \( w^{(1-p)(1-p')} \in e^{BLO} \); thus \( w \in A_1 \cap e^{BLO} = A_1 \).

In fact, the above is the now much-simplified crux of Cruz-Uribe and Neugebauer’s proof of their refined Jones factorization theorem, which we include briefly for completeness. We will need the following fact, a consequence of the Hölder and reverse Hölder inequalities; for convenience, we single out the case \( p = 1 \) as a corollary.

Theorem 4.7 (\([\text{3}]\)). \( w \in A_p \cap RH_s \iff w^s \in A_{s(p-1)+1} \).

Corollary 4.8. \( w \in A_1 \iff w^{1/s} \in A_1 \cap RH_s ; s > 1 \).
Theorem 4.9. \( w \in A_p \cap RH_s \) if and only if \( w = w_0 w_1 \) for some \( w_0 \in A_1 \cap RH_s \), \( w_1 \in A_p \cap RH_{\infty} \).

Proof. Let \( p > 1 \) and \( s < \infty \) (the case \( p = 1 \) or \( s = \infty \) is immediate; see [3]). By Theorem 4.7, \( w \in A_p \cap RH_s \) if and only if \( w^s \in A_{s(p-1)+1} \). In turn, by the (original) Jones factorization, this fact is equivalent to \( w^s = v_0 v_1^{1-s(p-1)} \) for some \( v_0, v_1 \in A_1 \); i.e., \( w = v_0^{1/s} v_1^{1-p} \) for some \( v_0, v_1 \in A_1 \). By Corollary 4.8 and Theorem 4.6 respectively, this fact in turn is true if and only if \( w = v_0^{1/s} v_1^{1-p} \), where \( v_0^{1/s} \in A_1 \cap RH_s \) and \( v_1^{1-p} \in A_p \cap RH_{\infty} \). Take \( w_0 = v_0^{1/s} \) and \( w_1 = v_1^{1-p} \). \( \square \)

5. Closing remarks

Given the near-reciprocity between \( A_1 = e^{BLO} \cap A_\infty \) and \( RH_{\infty} = e^{BUO} \), one might hope for a similar relation between the non-limiting \( A_p \) and \( RH_s \) classes. We give in closing an extension of Cruz-Uribe–Neugebauer’s result that \( A_1 = \frac{1}{RH_s} A_2 \), using the following version of the useful lemma of Strömberg and Wheeden [11].

Lemma 5.1. \( w^s \in A_\infty \iff w \in RH_s \), \( s > 1 \). Precisely, \( [RH_s(w)]^s \leq A_\infty(w^s) \leq [A_\infty(w)RH_s(w)]^s \).

Proof. \( w^s \in A_\infty \), so \( \frac{1}{|Q|} \int_Q w^s \leq A_\infty(w^s) e^{\frac{1}{|Q|} \int_Q \log w} \). By Jensen’s inequality, \( (\frac{1}{|Q|} \int_Q w^s)^{1/s} \leq [A_\infty(w^s)]^{1/s} \frac{1}{|Q|} \int_Q w \); thus \( RH_s(w) \leq A_\infty(w^s)^{1/s} \).

Conversely, if \( w \in RH_s \), then \( (\frac{1}{|Q|} \int_Q w^s)^{1/s} \leq RH_s(w) \frac{1}{|Q|} \int_Q w \); by the reverse Jensen inequality \( (\frac{1}{|Q|} \int_Q w^s)^{1/s} \leq RH_s(w) A_\infty(w) e^{\frac{1}{|Q|} \int_Q \log w} \). Thus \( (\frac{1}{|Q|} \int_Q w^s)^{1/s} \leq [RH_s(w) A_\infty(w)]^s e^{\frac{1}{|Q|} \int_Q \log w} \); i.e., \( A_\infty(w^s) \leq [RH_s(w) A_\infty(w)]^s \) \( \square \)

Theorem 5.2. \( A_{1+\frac{1}{s}} = \frac{1}{RH_s} A_{1+\frac{1}{s}} \) for all \( s > 1 \). Precisely, \( RH_s(\frac{1}{w}) \leq A_{1+\frac{1}{s}}(w) \leq A_\infty(w) A_{1+\frac{1}{s}}(\frac{1}{w}) \).

Proof. Suppose \( w \in A_{1+\frac{1}{s}} \), i.e., \( \frac{1}{|Q|} \int_Q w^{1+s} (\frac{1}{|Q|} \int_Q w^{-s})^{1/s} \leq A_{1+\frac{1}{s}}(w) \). Thus \( \frac{1}{|Q|} \int_Q (\frac{1}{w})^{1+s} \leq A_{1+\frac{1}{s}}(w) \frac{1}{|Q|} \int_Q (\frac{1}{w})^{-1} \leq A_{1+\frac{1}{s}}(w) \frac{1}{|Q|} \int_Q \frac{1}{w} \)

by negative Hölder’s inequality; therefore \( \frac{1}{w} \in RH_s \) with \( RH_s(\frac{1}{w}) \leq A_{1+\frac{1}{s}}(w) \).

Conversely, say \( \frac{1}{w} \in RH_s \cap A_2 \). By Lemma 5.1 \( \frac{1}{w} \in RH_s \iff (\frac{1}{w})^s \in A_\infty \), with \( A_\infty(\frac{1}{w})^s \leq [A_\infty(\frac{1}{w})RH_s(\frac{1}{w})]^s \).

So \( w, w^{-s} \in A_\infty \); thus (4), p. 408 \( w \in A_{1+\frac{1}{s}} \), with \( A_{1+\frac{1}{s}}(w) \leq A_\infty(w) A_{1+\frac{1}{s}}(\frac{1}{w}) \). \( \square \)

We note in passing that if we consider fractional reverse Hölder classes for \( 0 < s < 1 \), defined as \( w = e^\varphi \in RH_s \) if \( \sup_Q (\frac{1}{|Q|} \int_Q e^{s(\varphi(x) - \varphi(Q))})^{1/s} = RH_s(w) < \infty \), it is not difficult to extend the above statement to the full range \( 0 < s \leq \infty \).
References


Department of Mathematics, Scripps College, Claremont, California 91711
E-mail address: wcwou@scrippscollege.edu