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CONCORDANCE CROSSCAP NUMBERS OF KNOTS AND THE ALEXANDER POLYNOMIAL

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Abstract. For a knot K the concordance crosscap number, $c(K)$, is the minimum crosscap number among all knots concordant to K. Building on work of G. Zhang, which studied the determinants of knots with $c(K) < 2$, we apply the Alexander polynomial to construct new algebraic obstructions to $c(K) < 2$. With the exception of low crossing number knots previously known to have $c(K) < 2$, the obstruction applies to all but four prime knots of 11 or fewer crossings.

Every knot $K \subset S^3$ bounds an embedded surface $F \subset S^3$ with $F \cong \#_n P^2 - B^2$ for some $n \geq 0$, where P^2 denotes the real projective plane . The crosscap number of K, $\gamma(K)$, is defined to be the minimum such n. The careful study of this invariant began with the work of Clark in [Cl]; other references include [HT, MY1]. The study of the 4–dimensional crosscap number, $\gamma_4(K)$, defined similarly but in terms of $F \subset B^4$, appears in such articles as [MY2, Vi, Ya].

Gengyu Zhang [Zh] recently introduced a new knot invariant, the *concordance crosscap number*, $\gamma_c(K)$. This is defined to be the minimum crosscap number of any knot concordant to K. This invariant is the nonorientable version of the concordance genus, originally studied by Nakanishi [Na] and Casson [Ca], and later investigated in [Li].

In [Zh], Zhang presented an obstruction to $\gamma_c(K) \leq 1$ based on the homology of the 2–fold branched cover of the knot, or equivalently, $\det(K)$. Inspired by her work, in this note we will observe that the obstruction found in [Zh] extends to one based on the Alexander polynomial of K, $\Delta_K(t)$, and the signature of K, $\sigma(K)$.

Theorem 1. *Suppose $\gamma_c(K) = 1$ and set $q = |\sigma(K)| + 1$. For all odd prime power divisors p of q, the $2p$–cyclotomic polynomial $\phi_{2p}(t)$ has odd exponent in $\Delta_K(t)$. Furthermore, every other symmetric irreducible polynomial $\delta(t)$ with odd exponent in $\Delta_K(t)$ satisfies $\delta(-1) = \pm 1$.*

Proof. Any knot K' with $\gamma(K') = 1$ bounds a Mobius band and is thus a $(2, r)$–cable of some knot J for some odd r. If K is concordant to K', then $\sigma(K) = \sigma(K') = \pm(|r| - 1)$; the signature $\sigma(K')$ is given by a formula of Shinohara [Sh] for the signature of 2–stranded cable knots. It follows that $|\sigma(K)| = |r| - 1$, so $|r| = |\sigma(K)| + 1 = q$.

According to a result of Seifert [Se], the Alexander polynomial of K' is given by $\Delta_{2,q}(t)\Delta_J(t^2)$, where $\Delta_{2,q}(t)$ is the Alexander polynomial of the $(2, q)$–torus knot.

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A standard result states that $\Delta_{2,q}(t) = \frac{(t^{2q}-1)(t-1)}{(t^q-1)(t^2-1)} = \frac{t^q+1}{t+1}$. This can be written as the product of cyclotomic polynomials,

$$\Delta_{2,q}(t) = \prod_{p|q \ , \ p>1} \Phi_{2p}(t).$$

Since K is concordant to K', $K\#-K'$ is slice, and thus has Alexander polynomial of the form $g(t)g(t^{-1})$. That is, with $q = |\sigma(K)|+1$,

$$\Delta_K(t)\Delta_J(t^2)\Delta_{2,q}(t) = g(t)g(t^{-1}).$$

We now make two observations: **(1)** Any symmetric irreducible polynomial has even exponent in $g(t)g(t^{-1})$, and thus even exponent in $\Delta_K(t)\Delta_J(t^2)\Delta_{(2,q)}(t)$; **(2)** since $\Delta_J(t)$ is an Alexander polynomial, $\Delta_J(1) = \pm 1$, and thus $\Delta_J(t^2)|_{t=-1} = \pm 1$.

By Lemma 2, $\Phi_{2p}(-1) = p$ if p is an odd prime power, and $\Phi_{2p}(-1) = \pm 1$ if p is an odd composite. Thus, for p an odd prime power divisor of q, $\Phi_{2p}(t)$ has odd exponent in $\Delta_{2,q}(t)$ and does not divide $\Delta_J(t^2)$, so has odd exponent in $\Delta_K(t)$. Any other irreducible factor of $\Delta_K(t)$ with odd exponent is either a factor $\delta(t)$ of $\Delta_{2,q}(t)$, and thus of the form $\Phi_{2p}(t)$ with p an odd composite (and so $\delta(-1) = \pm 1$), or else is not a factor of $\Delta_{2,q}(t)$ and so has odd exponent in $\Delta_J(t^2)$, and again must satisfy $\delta(-1) = \pm 1$. This completes the argument. □

Lemma 2. *The cyclotomic polynomial $\Phi_{2p}(t)$ satisfies $\Phi_{2p}(-1) = p$ if p is an odd prime power and $\Phi_{2p}(-1) = \pm 1$ if p is an odd composite.*

Proof. For an odd r, $h_r(t) = \frac{t^r+1}{t+1}$ satisfies $h_r(-1) = r$ by l'Hôpital's rule. We have that $h_r(t)$ is the product

$$h_r(t) = \prod_{p|r \ , \ p>1} \Phi_{2p}(t).$$

For p a prime power, s^n, $\Phi_{2p}(t) = \frac{t^{s^n}+1}{t^{s^{n-1}}+1}$, and so, again by l'Hôpital's rule, $\Phi_{2s^n}(-1) = s$. Thus, the product

$$\prod_{p|r \ , \ p>1, \ p \text{ a prime power}} \Phi_{2p}(-1) = r.$$

It follows that all the other terms in the product expansion of $h_r(t)$ must equal ± 1 when evaluated at $t = -1$, as desired. □

Example. Theorem 1 is quite effective in ruling out $\gamma_c(K) = 1$. For instance, there are 801 prime knots with 11 or fewer crossings. Of these, 51 are known to be topologically slice, and 23 are known to be concordant to a $(2,q)$–torus knot for some q and thus have $\gamma_c = 1$. Of the remaining 727 knots, all but four can be shown to have $\gamma_c \geq 2$. These four are $11n_{45}$ and $11n_{145}$, both of which are possibly slice, and 9_{40} and $11n_{66}$, both of which are possibly concordant to the trefoil. Of the collection of 727 knots, Yasuhara's result [Ya] applies to show that 207 of them have 4–ball crosscap number $\gamma_4(K) \geq 2$. The 4–ball crosscap numbers of the rest are unknown.

As a second set of examples, consider knots K with $\Delta_K(t)$ of degree 2. It follows immediately from Theorem 1 that there are only two possibilities: either $\sigma(K) = 0$ and Δ_K is reducible (an irreducible symmetric quadratic $f(t)$ cannot satisfy $f(1) = \pm 1$ and $f(-1) = \pm 1$) or $\sigma(K) = \pm 2$ and $\Delta_K(t) = t^2 - t + 1$.

We conclude with the further special case consisting of (p, q, r)–pretzel knots, $P(p, q, r)$, with p, q, and r odd; some of these were studied in [Zh]. If we let $D = D(p, q, r) = pq + qr + rp$, then

$$\Delta_{P(p,q,r)}(t) = \frac{D+1}{4}t^2 - \frac{D-1}{2}t + \frac{D+1}{4},$$

which has discriminant $-D$. Thus, by the previous argument we have:

Corollary 3. *If $\gamma_c(P(p, q, r)) = 1$, then either $\sigma(P(p, q, r)) = 0$ and $D(p, q, r) = -l^2$ for some integer l or $\sigma(P(p, q, r)) = \pm 2$ and $D(p, q, r) = 3$.*

These pretzel knots include some shown by Zhang [Zh] to have 4–dimensional crosscap number $\gamma_4(K) = 1$.

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