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K-STABILITY ON TORIC MANIFOLDS

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Abstract. In this note, we prove that on polarized toric manifolds the relative K-stability with respect to Donaldson's toric degenerations is a necessary condition for the existence of Calabi's extremal metrics, and we also show that the modified K-energy is proper in the space of G_0-invariant Kähler potentials in the case of toric surfaces which admit the extremal metrics.

0. Introduction

Around the existence of Calabi's extremal metrics, there is a well-known conjecture (cf. [Ya], [Ti]):

Conjecture 0.1. *A polarized Kähler manifold M admits a Kähler metric with constant scalar curvature* (*or more generally an extremal metric*) *if and only if M is stable in a certain sense of GIT.*

For the "only if" part of this conjecture, the first breakthrough was made by Tian [Ti]. By introducing the concept of K-stability, he gave an answer to the "only if" part for Kähler-Einstein manifolds. Remarkable progress was made by Donaldson, who showed the Chow-Mumford stability for a polarized Kähler manifold with constant scalar curvature when the holomorphic automorphisms group $\mathrm{Aut}(M)$ of M is finite [D1]. Donaldson's result was later generalized by Mabuchi to any polarized Kähler manifold M which admits an extremal metric without any assumption for $\mathrm{Aut}(M)$ [M1], [M2]. However, it is still unknown whether there is a generalization of Tian's result for the K-stability on the Kähler-Einstein manifolds analogous to Donaldson-Mabuchi's result for the Chow-Mumford stability.

As we know, the definition of K-stability on a polarization Kähler manifold is associated to degenerations (or test configurations in the sense of Donaldson [D2]) on the underlying manifold. In order to study the relation between the K-stability and K-energy on a polarized toric manifold, Donaldson in [D2] introduced a class of special degenerations induced by rational, piecewise linear functions, called toric degenerations. He also proved that for the surfaces' case the K-energy is bounded from below in the space of G_0-invariant Kähler potentials under the assumption of K-stability for any toric degeneration, where G_0 is a maximal compact torus group. In this note we focus on polarized toric manifolds as in [D2], and we shall give an

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answer to the "only if" part in Conjecture 0.1 in the sense of relative K-stability with respect to toric degenerations for the extremal metrics. Furthermore, we show that the modified K-energy is proper in the space of G_0-invariant Kähler potentials on a toric surface which admits an extremal metric. The relative K-stability is a generalization of K-stability which was first introduced by Székelyhidi [Sz], and the modified K-energy is a generalization of the K-energy.

1. Statement of the main theorems

Let (M, L) be an n-dimensional polarized Kähler manifold. Recall the definition of relative K-stability.

Definition 1.1 ([D2], [Sz])**.** A test configuration for the polarized Kähler manifold (M, L) of exponent r consists of a $\mathbb{C}^*$-equivariant flat family of schemes $\pi : \mathcal{W} \longrightarrow \mathbb{C}$ (where $\mathbb{C}^*$ acts on $\mathbb{C}$ by multiplication) and a $\mathbb{C}^*$-equivariant ample line bundle $\mathcal{L}$ on $\mathcal{W}$. We require that the fibres $(\mathcal{W}_t, \mathcal{L}|_{\mathcal{W}_t})$ are isomorphic to (M, L^r) for any $t \neq 0$. A test configuration is called trivial if $\mathcal{W} = M \times \mathbb{C}$ is a product.

Let (M, L) be equipped with a $\mathbb{C}^*$-action β. We say that a test configuration $(\mathcal{W}, \mathcal{L})$ is compatible with β if there exists a $\mathbb{C}^*$-action $\tilde{\beta}$ on $(\mathcal{W}, \mathcal{L})$ such that $\pi : \mathcal{W} \longrightarrow \mathbb{C}$ is $\tilde{\beta}$-equivariant with the trivial $\mathbb{C}^*$-action on $\mathbb{C}$, and the restriction of $\tilde{\beta}$ to $(\mathcal{W}_t, \mathcal{L}|_{\mathcal{W}_t})$ for nonzero t coincides with that of β on (M, L^r) under the isomorphism.

Note that a $\mathbb{C}^*$-action on $\mathcal{W}$ induces a $\mathbb{C}^*$-action on the central fibre $M_0 = \pi^{-1}(0)$ and the restricted line bundle $\mathcal{L}|_{M_0}$. We denote by $\tilde{\alpha}$ and $\tilde{\beta}$ the induced $\mathbb{C}^*$-actions of α and β on $(M_0, \mathcal{L}|_{M_0})$, respectively. The relative K-stability is based on the modified Futaki invariant on the central fibre,

$$F_{\tilde{\beta}}(\tilde{\alpha}) = F(\tilde{\alpha}) - \frac{(\tilde{\alpha}, \tilde{\beta})}{(\tilde{\beta}, \tilde{\beta})} F(\tilde{\beta}), \tag{1.1}$$

where $F(\tilde{\alpha})$ and $F(\tilde{\beta})$ are generalized Futaki invariants of $\tilde{\alpha}$ and $\tilde{\beta}$ defined in [D2], respectively, and $(\tilde{\alpha}, \tilde{\beta})$ and $(\tilde{\beta}, \tilde{\beta})$ are inner products defined in [Sz].

Definition 1.2 ([Sz])**.** The polarized Kähler manifold (M, L) with a $\mathbb{C}^*$-action β is K-semistable relative to β if $F_{\tilde{\beta}}(\cdot) \leq 0$ for any test configuration compatible with β. In addition, it is called relative K-stable when the equality holds if and only if the test configuration is trivial.

An n-dimensional polarized toric manifold M corresponds to an integral polytope P in $\mathbb{R}^n$ which is described by a common intersection of some half-spaces,

$$\langle l_i, x \rangle < \lambda_i, \ i = 1, ..., d. \tag{1.2}$$

Here l_i are d vectors in $\mathbb{R}^n$ with all components in $\mathbb{Z}$ which satisfy the Delzant condition [Ab]. Without loss of generality, we may assume that the original point 0 lies in P, so all $\lambda_i > 0$. Recall that a piecewise linear (PL) function f on P is of the form

$$u = \max\{f^1, ..., f^r\},$$

where $f^\lambda = \sum a_i^\lambda x_i + c^\lambda, \ \lambda = 1, ..., r$, for some vectors $(a_i^\lambda) \in \mathbb{R}^n$ and some numbers $c^\lambda \in \mathbb{R}$. f is called a rational PL-function if components a_i^λ and numbers c^λ are all rational. According to [D2], a rational PL-function induces a test configuration, called a toric degeneration. Moreover, one can show that a toric degeneration is

always compatible to a $\mathbb{C}^*$-action induced by an extremal vector field on M (cf. [ZZ1]).

Now we can state our first main theorem in this paper.

Theorem 1.3. *Let M be a polarized toric manifold which admits an extremal metric in this polarized Kähler class. Then M is K-stable relative to a $\mathbb{C}^*$-action induced by an extremal vector field on M for any toric degeneration.*

We will give a proof of Theorem 1.3 in the next section. The proof can also be used to discuss the properness of the modified K-energy $\mu(\cdot)$. Similar to K-energy, $\mu(\cdot)$ is defined on Kähler potentials space of a Kähler class whose critical point is an extremal metric, while the critical point of K-energy is a Kähler metric with constant scalar curvature.

Let ω_g be a Kähler form of Kähler metric g on a compact Kähler manifold M and let $\omega_\phi = \omega_g + \sqrt{-1}\partial\overline{\partial}\phi$ be a Kähler form associated to a potential Kähler ϕ in Kähler class $[\omega_g]$. Let θ_X be a normalized potential of an extremal vector field X associated to the metric ω_g on M [ZZ1]. Then $\mu(\phi)$ is given by

$$\mu(\phi) = -\frac{1}{V}\int_0^1 \int_M \dot{\phi}_t[R(\omega_{\phi_t}) - \overline{R} - \theta_X(\omega_{\phi_t})]\omega_{\phi_t}^n \wedge dt,$$

where ϕ_t $(0 \le t \le 1)$ is a family of Kähler potentials connecting 0 to ϕ, $R(\omega_{\phi_t})$ denotes the scalar curvatures of ω_{ϕ_t}, $\theta_X(\omega_{\phi_t}) = \theta_X + X(\phi_t)$ are normalized potentials of X associated to ω_{ϕ_t}, and $\overline{R}$ is the average of the scalar curvature of ω_g. It can be shown that the functional $\mu(\phi)$ is well-defined; i.e., it is independent of the choice of path ϕ_t [Gu]. Thus one sees that ϕ is a critical point of $\mu(\cdot)$ iff the corresponding metric ω_ϕ is extremal.

Definition 1.4. Let

$$I(\phi) = \frac{1}{V}\int_M \phi(\omega_g^n - \omega_\phi^n)$$

be a functional on Kähler potential space $\mathcal{M}_g$ of $[\omega_g]$. Here $V = \int_M \omega_g^n$. $\mu(\phi)$ is called properly associated to a subgroup G of the automorphisms group $\mathrm{Aut}(M)$ in $[\omega_g]$ if there is a continuous function $p(t)$ in $\mathbb{R}$ with the property

$$\lim_{t\to+\infty} p(t) = +\infty,$$

such that

$$\mu(\phi) \ge \inf_{\sigma\in G} p(I(\phi_\sigma)),\ \forall\ \phi \in \mathcal{M}_g,$$

where ϕ_σ is defined by

$$\omega_g + \sqrt{-1}\partial\bar{\partial}\phi_\sigma = \sigma^*(\omega_g + \sqrt{-1}\partial\bar{\partial}\phi).$$

The above definition was first introduced by Tian for the K-energy [Ti]. He proved that the properness of K-energy is a sufficient and necessary condition for the existence of Kähler-Einstein metrics. In a very recent paper [ZZ2], the authors proved the existence of a minimizing weak solution of extremal metrics on toric manifolds under the assumption of properness of $\mu(\phi)$. This weak solution will be an extremal metric if one can further prove some regularities of the solution.

The following is our second main theorem in this note.

Theorem 1.5. *Let M be a toric surface which admits an extremal metric ω_E. Suppose that*

$$\bar{R} + \theta_X(\omega_E) > 0; \tag{1.3}$$

i.e., the scalar curvature of ω_E is positive. Then $\mu(\phi)$ is properly associated to T in the space of G_0-invariant Kähler potentials. Here T is a torus action group of M and G_0 is a maximal compact subgroup of T.

We note that in Theorem 1.5 we need not assume that the surface is polarized.

2. Proof of Theorem 1.3

Let P be an integral polytope in $\mathbb{R}^n$ associated to a polarized toric manifold M as in Section 1. We choose a G_0-invariant Kähler metric ω_g on M. Then there exists a convex function ψ_0 in $\mathbb{R}^n$ such that

$$\omega_g = \sqrt{-1}\partial\overline{\partial}\psi_0 \text{ in } (S^1)^n \times \mathbb{R}^n,$$

where $(S^1)^n \times \mathbb{R}^n$ is a holomorphic domain of affine coordinates on an open torus orbit of T. Let u_0 be a Legendre function of ϕ_0 which is a convex function on P. Set

$$\mathcal{C} = \{u|\ u \text{ is a convex function on } \ P \text{ with } u - u_0 \in C^\infty(\overline{P})\}.$$

Then one can show that functions in $\mathcal{C}$ correspond to G_0-invariant Kähler potentials on M by a one-to-one correspondence [Ab], [D2].

Let $d\sigma_0$ be the Lebesgue measure on the boundary ∂P and ν be the outer normal vector field on ∂P. Let $d\sigma$ be an induced measure on ∂P such that $d\sigma = \lambda_i^{-1}(\nu, x)d\sigma_0$ on each face $\langle l_i, x\rangle = \lambda_i$ of P. We define a linear functional on the space of continuous functions on $\overline{P}$ by

$$L(u) = \int_{\partial P} u d\sigma - \int_P (\bar{R} + \theta_X) u dx,$$

where θ_X is the normalized potential of the extremal vector field X as in Section 1, which is equal to $\langle A, x\rangle + a$ for some vector $A \in \mathbb{R}^n$ and constant a in the coordinates x in P [ZZ1]. The following result was proved in [ZZ1].

Lemma 2.1 ([ZZ1])**.** *Let $\tilde{\alpha}$ and $\tilde{\beta}$ be two C^*-actions induced by a rational PL-function f and the extremal vector field X, respectively. Then*

$$F_{\tilde{\beta}}(\tilde{\alpha}) = -\frac{1}{2Vol(P)} L(f). \tag{2.1}$$

By the above lemma, to prove Theorem 1.3 we only have to prove

Proposition 2.2. *Let M be a toric manifold which admits an extremal metric. Then for any PL-function f on P, we have*

$$L(f) \geq 0. \tag{2.2}$$

Moreover the equality holds if and only if f is an affine linear function on P.

Proof. By definition, we may assume that

$$f = \max\{f^1, ..., f^r\},$$

and each $f^\alpha = \sum c_i^\alpha x_i + c^\alpha$ is an affine linear function on P. Then P can be divided into $m(\geq r)$ small polytopes $P^1, ..., P^m$ such that for each P^λ there exists a f^{α_λ}

such that $f = f^{\alpha_\lambda}$ on P^λ. By the assumption of the existence of extremal metric, we see that there exists a $u \in \mathcal{C}$ which satisfies Abreu's equation [Ab],

$$-\sum_{i,j} u^{ij}_{ij} = \bar{R} + \theta_X, \tag{2.3}$$

where $(u^{ij}) = (u_{ij})^{-1}$ and $u^{ij}_{kl} = \frac{\partial^2 u^{ij}}{\partial x^k \partial x^l}$. Thus

$$L(f) = \int_{\partial P} f d\sigma + \int_P \sum_{i,j} u^{ij}_{ij} f dx. \tag{2.4}$$

For any small $\delta > 0$, we let P_δ be the interior polygon with faces parallel to those of P separated by distance δ. Taking integral by parts, one sees

$$\begin{aligned}
&\int_{P_\delta} u^{ij}_{ij} f dx \\
&= \sum_\lambda \int_{P^\lambda \cap P_\delta} \sum_{i,j} u^{ij}_{ij} f^{\alpha_\lambda} dx \\
&= \sum_\lambda \int_{\partial(P^\lambda \cap P_\delta)} \sum_{i,j} u^{ij}_i f^{\alpha_\lambda} n^{\alpha_\lambda}_j d\sigma_0 - \sum_\lambda \int_{P^\lambda \cap P_\delta} \sum_{i,j} u^{ij}_i c^{\alpha_\lambda}_j dx \\
&= \sum_\lambda \int_{\partial(P^\lambda \cap P_\delta)} \sum_{i,j} u^{ij}_i f^{\alpha_\lambda} n^{\alpha_\lambda}_j d\sigma_0 - \sum_\lambda \int_{\partial(P^\lambda \cap P_\delta)} \sum_{i,j} u^{ij} c^{\alpha_\lambda}_j n^{\alpha_\lambda}_i d\sigma_0 \\
&= \sum_\lambda \int_{\partial P_\delta} \sum_{i,j} u^{ij}_i f^{\alpha_\lambda} n^{\alpha_\lambda}_j d\sigma_0 - \sum_\lambda \int_{\partial P_\delta} \sum_{i,j} u^{ij} c^{\alpha_\lambda}_j n^{\alpha_\lambda}_i d\sigma_0 \\
&- \sum_{\lambda<\mu} \int_{\partial P^\lambda \cap \partial P^\mu \cap P_\delta} \sum_{i,j} (u^{ij} c^{\alpha_\lambda}_j n^{\alpha_\lambda}_i + u^{ij} c^{\alpha_\mu}_j n^{\alpha_\mu}_i) d\sigma_0,
\end{aligned} \tag{2.5}$$

where $(n^{\alpha_\lambda}_1, ..., n^{\alpha_\lambda}_n)$ is the unit outer normal vector on ∂P^λ. By [D2] we know that the first term tends to $-\int_{\partial P} f d\sigma$ and the second term tends to 0 at the last equality as δ tends to zero. On the other hand, if $\partial P^\lambda \bigcap \partial P^\mu$ is a common $(n-1)$-dimensional face of P^λ and P^μ, then $(c^{\alpha_\lambda}_1 - c^{\alpha_\mu}_1, ..., c^{\alpha_\lambda}_n - c^{\alpha_\mu}_n)$ is a nonzero vector, and

$$n^{\alpha_\lambda}_i = -\frac{1}{\sqrt{\sum_l (c^{\alpha_\lambda}_l - c^{\alpha_\mu}_l)^2}} (c^{\alpha_\lambda}_i - c^{\alpha_\mu}_i), n^{\alpha_\mu}_i = -\frac{1}{\sqrt{\sum_l (c^{\alpha_\mu}_l - c^{\alpha_\lambda}_l)^2}} (c^{\alpha_\mu}_i - c^{\alpha_\lambda}_i).$$

Substituting them into the third term of the last equality in (2.5) and letting δ tend to 0, we derive

$$\begin{aligned}
&\int_P \sum_{i,j} u^{ij}_{ij} f dx \\
&= -\int_{\partial P} f d\sigma + \sum_{\lambda<\mu} \int_{\partial P^\lambda \cap \partial P^\mu} \frac{\sum_{i,j} u^{ij} (c^{\alpha_\lambda}_i - c^{\alpha_\mu}_i)(c^{\alpha_\lambda}_j - c^{\alpha_\mu}_j)}{\sqrt{\sum_i (c^{\alpha_\lambda}_i - c^{\alpha_\mu}_i)^2}} d\sigma_0.
\end{aligned} \tag{2.6}$$

Hence, combining (2.4) and (2.6),

$$L(f) = \sum_{\lambda<\mu} \int_{\partial P^\lambda \cap \partial P^\mu} \frac{\sum_{i,j} u^{ij} (c^{\alpha_\lambda}_i - c^{\alpha_\mu}_i)(c^{\alpha_\lambda}_j - c^{\alpha_\mu}_j)}{\sqrt{\sum_l (c^{\alpha_\lambda}_l - c^{\alpha_\mu}_l)^2}} d\sigma_0 \geq 0.$$

Note that the equality holds if only if there is no common $(n-1)$-dimensional face for any P^λ and P^μ, which implies that f is just an affine linear function. $\square$

3. Proof of Theorem 1.5

To prove Theorem 1.5, we need to use Donaldson's version of the modified K-energy obtained in [ZZ1].

Lemma 3.1. *Let ω_ϕ be a G_0-invariant Kähler metric on M and u be a Legendre function of ϕ in $\mathcal{C}$. Then*

$$\mu(\phi) = \frac{2^n n!(2\pi)^n}{Vol(M)}\mathcal{F}(u),$$

where

$$\mathcal{F}(u) = -\int_P \log(\det(D^2u))dx + L(u). \tag{3.1}$$

Let $p \in P$. We set

$$\mathcal{C}_\infty = \{u \in C^0(\overline{P}) \cup C^\infty(P) |\ u \text{ is a convex function on } P \text{ with } \inf_P u = u(p) = 0\}.$$

The following proposition was also proved in [ZZ1].

Proposition 3.2. *Suppose that there exists a $\lambda > 0$ such that*

$$L(u) \geq \lambda \int_{\partial P} u d\sigma \tag{3.2}$$

for any $u \in \mathcal{C}_\infty$. Then there exist two uniform constants $\delta, C > 0$ such that for any G_0-invariant Kähler potential ϕ it holds that

$$\mu(\phi) \geq \delta \inf_{\sigma \in T} I(\phi_\sigma) - C.$$

In particular, $\mu(\phi)$ is properly associated to subgroup T in the space of G_0-invariant Kähler potentials.

Proof of Theorem 1.5. According to Proposition 3.2, we suffice to verify the condition (3.2). We use an argument by the contradiction. First note that by the convexity of functional $\mathcal{F}(u)$ on $\mathcal{C}$, one sees that $\mathcal{F}(u)$ is bounded from below by the existence of extremal metrics. Then one can show (cf. [D2], [ZZ1])

$$L(u) \geq 0, \forall\ u \in \mathcal{C}_\infty. \tag{3.3}$$

Furthermore one concludes that

$$L(u) \geq 0,\ \forall\ u \in \mathcal{C}_1, \tag{3.4}$$

where $\mathcal{C}_1$ is a set of positive convex functions u on $P \cup \partial P$ such that

$$\int_{\partial P} u d\sigma_0 dx < \infty.$$

On the other hand, suppose that (3.2) is not true. Then by (3.4) it is easy to see that there exists a $u_\infty \in \mathcal{C}_1$, which is not affine linear, such that

$$L(u_\infty) = 0. \tag{3.5}$$

Recall that a simple PL-function is a form of

$$u = \max\{0, \sum a_i x_i + c\}$$

for some vector $(a_i) \in \mathbb{R}^n$ and number $c \in \mathbb{R}$. We call the hyperplane $\sum a_i x_i + c = 0$ a crease of u. Then applying Proposition 5.3.1 in [D2], we see that under the relations (3.4), (3.5) and the assumption (1.3) in Theorem 1.5 for the case of toric surfaces there exists a simple PL-function v_0 with a crease intersecting the interior of P such that $L(v_0) = 0$. But the latter is a contradiction to Proposition 2.2. Thus Proposition 3.2 is true, and so is Theorem 1.5. □

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