CUTOFF RESOLVENT ESTIMATES
AND THE SEMILINEAR SCHröDINGER EQUATION

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Abstract. This paper shows how abstract resolvent estimates imply local smoothing for solutions to the Schrödinger equation. If the resolvent estimate has a loss when compared to the optimal, non-trapping estimate, there is a corresponding loss in regularity in the local smoothing estimate. As an application, we apply well-known techniques to obtain well-posedness results for the semi-linear Schrödinger equation.

1. Introduction

In this short note we show how cutoff semiclassical resolvent estimates for the Laplacian on a non-compact manifold, with spectral parameter on the real axis, lead to well-posedness results for the semi-linear Schrödinger equation. Motivated by the requirements of [Chr3] and [BGT2], and the microlocal inverse estimates of [Chr1] [Chr2], we first prove a general theorem for a large class of resolvents. Following the recent work of Nonnenmacher-Zworski [NoZw], we apply the general theorem in the case there is a hyperbolic fractal trapped set.

Let \((M, g)\) be a Riemannian manifold of dimension \(n\) without boundary, with (non-negative) Laplace-Beltrami operator \(-\Delta\) acting on functions. The Laplace-Beltrami operator is an unbounded, essentially self-adjoint operator on \(L^2(M)\) with domain \(H^2(M)\). We assume \((M, g)\) is asymptotically Euclidean in the sense of [NoZw] §3.2 and that the classical resolvent \((-\Delta - (\lambda^2 + i\epsilon))^{-1}\) obeys a limiting absorption principle as \(\epsilon \to 0^+\), \(\lambda \neq 0\).

Our first result is that if we have cutoff semiclassical resolvent estimates with a sufficiently small loss, then we have weighted smoothing for the Schrödinger propagator with a loss. Let \(\rho_s\) be a smooth, non-vanishing weight function satisfying

\[
\rho_s(x) \equiv \langle d_g(x, x_0) \rangle^{-s},
\]

for some fixed \(x_0\) and \(x\) outside a compact set.
Theorem 1. Suppose for each compactly supported function $\chi \in C_c^\infty(M)$ with sufficiently small support, there is $h_0 > 0$ such that the semi-classical Laplace-Beltrami operator satisfies

\begin{equation}
\|\chi(-h^2\Delta - E)^{-1}\chi u\|_{L^2(M)} \leq \frac{g(h)}{h}\|u\|_{L^2(M)}, \quad E > 0,
\end{equation}

uniformly in $0 < h \leq h_0$, where $g(h) \geq c_0 > 0$, $g(h) = o(h^{-1})$. Then for each $T > 0$ and $s > 1/2$, there is a constant $C = C_{T,s} > 0$ such that

\begin{equation}
\int_0^T \|\rho_s e^{it\Delta}u_0\|^2_{H^{1/2-\eta}(M)} \, dt \leq C\|u_0\|^2_{L^2(M)},
\end{equation}

where $\eta \geq 0$ satisfies $g(h)h^{2\eta} = O(1)$, and $\rho_s$ is given by (1.1).

The assumption that $(M, g)$ is asymptotically Euclidean is that there exists $R_0 > 0$ sufficiently large that each infinite branch of $M \setminus B(0, R_0)$ agrees with $\mathbb{R}^n$ and on each branch the semi-classical Laplacian $-h^2\Delta$ takes the form

\[-h^2\Delta|_{M \setminus B(0, R_0)} = \sum_{|\alpha| \leq 2} a_\alpha(x, h)(hD_x)^\alpha,
\]

with $a_\alpha(x, h) \in C_c^\infty(\mathbb{R}^n)$ and independent of $h$ for $|\alpha| = 2$,

\[
\sum_{|\alpha| = 2} a_\alpha(x, h)(hD_x)^\alpha \geq C^{-1}|\xi|^2, \quad 0 < C < \infty, \quad \text{and}
\]

\[
\sum_{|\alpha| = 2} a_\alpha(x, h)(hD_x)^\alpha \to |\xi|^2, \quad \text{as } |x| \to \infty \text{ uniformly in } h.
\]

In order to quote the results of [NoZw] we also need the following analyticity assumption: $\exists \varepsilon > 0$ such that the $a_\alpha(x, h)$ extend holomorphically to

\[
\{r\omega : \omega \in \mathbb{C}^n, \text{dist}(\omega, \mathbb{R}^n) < \varepsilon, \ r \in \mathbb{C}, \ |r| \geq R_0, \ \arg r \in (-\varepsilon, \varepsilon)\}
\]

and satisfy the same estimates in this extended region. As in [NoZw], the analyticity assumption immediately implies

\[
\partial^2_x \left( \sum_{|\alpha| \leq 2} a_\alpha(x, h)\xi^\alpha - |\xi|^2 \right) = o(|x|^{-|\beta|})|\xi|^2, \quad |x| \to \infty.
\]

Recall the free Laplacian $(-\Delta_0 - \lambda^2)^{-1}$ on $\mathbb{R}^n$ has a holomorphic continuation from $\Im \lambda > 0$ to $\lambda \in \mathbb{C}$ for $n \geq 3$ odd, and to the logarithmic covering space for $n$ even. This motivates the limiting absorption assumption that

\[
\lim_{\varepsilon \to 0^+} \rho_s(-\Delta - (\lambda^2 + i\varepsilon))^{-1}\rho_s
\]

exists as a bounded operator

\[L^2(M, d\vol_g) \to L^2(M, d\vol_g),\]

provided $s > 1/2$. As in the free case, we allow a possible logarithmic singularity at $\lambda = 0$. 
The problem of “local smoothing” estimates for the Schrödinger equation has a long history. The sharpest results to date are those of Doi [Doi] and Burq [Bur]. Doi proved if $M$ is asymptotically Euclidean, then one has the estimate

$$
\int_0^T \| \chi e^{it\Delta} u_0 \|_{H^{1/2}(M)}^2 \, dt \leq C \| u_0 \|_{L^2(M)}^2
$$

for $\chi \in C_c^\infty(M)$ if and only if there are no trapped sets. Burq’s paper showed if there is trapping due to the presence of several convex obstacles in $\mathbb{R}^n$ satisfying certain assumptions, then one has the estimate (1.5) with the $H^{1/2}$ norm replaced by $H^{1/2-\eta}$ for $\eta > 0$. In [Chr3], the author considered an arbitrary, single trapped hyperbolic orbit. One of the goals of this paper is to use estimates obtained by Nonnenmacher-Zworski [NoZw] for fractal hyperbolic trapped sets to obtain similar results to [Chr3] for the semilinear Schrödinger equation. To that end we have the following corollary to Theorem 1.

**Corollary 1.1.** Assume $(M, g)$ admits a hyperbolic fractal trapped set, $K_E$, in the energy level $E > 0$ and that the topological pressure $P_E(1/2) < 0$. Then $-\hbar^2 \Delta - E$ satisfies (1.2) for some $E > 0$ with $g(h) = C \log(1/h)$, and for every $\eta > 0$, $T > 0$, and $s > 1/2$, there exists a constant $C = C_{P_E, \eta, T, s} > 0$ such that

$$
\int_0^T \| \rho e^{it\Delta} u_0 \|_{H^{1/2-s, \eta}(M)}^2 \, dt \leq C \| u_0 \|_{L^2(M)}^2.
$$

We remark that the assumption $P_E(1/2) < 0$ implies the trapped set $K_E$ is filamentary or “thin” (see [NoZw] for definitions).

We consider the following semilinear Schrödinger equation problem:

$$
\begin{align*}
\{ & i \partial_t u + \Delta u = F(u) \text{ on } I \times M, \\
& u(0, x) = u_0(x),
\end{align*}
$$

where $I \subset \mathbb{R}$ is an interval containing 0. Here the non-linearity $F$ satisfies

$$
F(u) = G(u)|u|^2u,
$$

and $G : \mathbb{R} \to \mathbb{R}$ is at least $C^3$ and satisfies

$$
|G^{(k)}(r)| \leq C_k |r|^{\beta-k},
$$

for some $\beta \geq \frac{1}{2}$.

In [Chr] we prove a family of Strichartz-type estimates which will result in the following well-posedness theorem.

**Theorem 2.** Suppose $(M, g)$ satisfies the assumptions of the introduction, and set

$$
\delta = \frac{4\eta}{2\eta + 1} \geq 0.
$$

Then for each

$$
\frac{n}{2} + \frac{2}{\max\{2\beta - 2, 2\}} + \delta
$$

and each $u_0 \in H^s(M)$ there exists $p > \max\{2\beta - 2, 2\}$ and $0 < T \leq 1$ such that (1.0) has a unique solution

$$
u \in C([-T, T]; H^s(M)) \cap L^p([-T, T]; L^\infty(M)).$$

Moreover, the map $u_0(x) \mapsto u(t, x) \in C([-T, T]; H^s(M))$ is Lipschitz continuous on bounded sets of $H^s(M)$, and if $\|u_0\|_{H^s}$ is bounded, $T$ is bounded from below.
If, in addition, \((M, g)\) satisfies the assumptions of Corollary 1.1 \(n \leq 3, \beta < 3,\) and \(G(r) \to +\infty\) as \(r \to +\infty,\) then \(u\) in (1.9) extends to a solution \(u \in C((-\infty, \infty); H^1(M)) \cap L^p((-\infty, \infty); L^\infty(M)).\)

Remark 1.2. In particular, the cubic defocusing non-linear Schrödinger equation is globally \(H^1\)-well-posed in three dimensions with a fractal trapped hyperbolic set which is sufficiently filamentary. Of course other non-linearities can be considered, but for simplicity we consider only those in this work.

2. Proof of Theorem 1

Since we are assuming \((-\Delta - z)^{-1}\) obeys a limiting absorption principle, we have
\[
\|\rho_s(-\Delta - (\tau - i\epsilon))^{-1}\rho_s\|_{L^2 \to L^2} \leq C_{\epsilon}
\]
for \(0 < \epsilon_0 \leq |\tau| \leq C.\) For \(|\sigma| \geq C\) for some \(C > 0, \sigma \in \mathbb{C}\) in a neighbourhood of the real axis, write
\[
-\Delta - \sigma = -\Delta - \frac{z}{h^2} = h^{-2}(-h^2\Delta - z)
\]
for
\[
z \in [E - \alpha, E + \alpha] + i[-c_0h, c_0h].
\]
Now
\((-h^2\Delta - z)\)
is a Fredholm operator for \(z\) in the specified range, and hence the "gluing" techniques from \[Vod\] and \[Chr3 \S2\] can be used to conclude for \(s > 1/2,\)
\[
\rho_s(-h^2\Delta - z)^{-1}\rho_s
\]
has a holomorphic extension to a slightly smaller neighbourhood in \(z,\) and in particular,
\[
\|\rho_s(-h^2\Delta - E)^{-1}\rho_s\|_{L^2 \to L^2} \leq C g(h) h.
\]
Rescaling, we have
\[
\|\rho_s(-\Delta - \tau)^{-1}\rho_s\|_{L^2 \to L^2} \leq C g((\tau)^{1/2})^{1/2}, \tau \in \mathcal{C}_{\pm \epsilon},
\]
where (see Figure 1)
\[
\mathcal{C}_{\pm \epsilon} = \{\tau \in \mathbb{R} : |\tau| \geq \epsilon\} \cup \{\tau \in \mathbb{C} : |\tau| = \epsilon, \pm \text{Im}\ \tau \geq 0\}.
\]
As in \[Chr3\] and \[Bur\], the following lemma follows from integration by parts and interpolation, together with the condition on \(\eta, (1.4).\)

**Lemma 2.1.** With the notation and assumptions above, we have
\[
\|\rho_s(-\Delta - \tau)^{-1}\rho_s\|_{L^2 \to H^1} \leq C g((\tau)^{1/2}), \tau \in \mathcal{C}_{\pm \epsilon},
\]
and for every \(r \in [-1, 1],\)
\[
\|\rho_s(-\Delta - \tau)^{-1}\rho_s\|_{H^r \to H^{1+r-n/2}} \leq C, \tau \in \mathcal{C}_{\pm \epsilon}.
\]
Figure 1. The curve $C_{-\epsilon}$ in the complex plane.

Theorem 1 now follows from the standard "TT∗" argument, letting $\epsilon \to 0$ in (2.1) (see [BGT2], the references cited therein, and [Chr3]). □

The following corollary uses interpolation with an $H^2$ estimate to replace the $H^{1/2-\eta}$ norm on the left hand side of (1.3) with $H^{1/2}$, and will be of use in [3]. See [Chr3] for the details of the proof.

Corollary 2.2. Suppose $(M,g)$ satisfies the assumptions of Theorem 1. For each $T > 0$ and $s > 1/2$, there is a constant $C > 0$ such that

$$\int_0^T \|\rho_s e^{it\Delta} u_0\|^2_{H^{1/2}(M)} dt \leq C \|u_0\|^2_{H^s(M)},$$

where $\delta \geq 0$ is given by (1.7).

In particular, if $(M,g)$ satisfies the assumptions of Corollary 1.1 then for any $\delta > 0$, there is $C = C_\delta > 0$ such that (2.2) holds.

3. Strichartz-type inequalities

In this section we give several families of Strichartz-type inequalities and prove Theorem 2. The statements and proofs are mostly adaptations of similar inequalities in [BGT2], so we leave out the proofs of these in the interest of space.

If we view $M \setminus U$, where $U$ is a neighbourhood of $K_E$, as a manifold with non-trapping geometry, we may apply the results of [HTW] or [BoTz] to a solution of the Schrödinger equation away from the trapping region, resulting in perfect Strichartz estimates. For this section we need (1.3) only with a compact cutoff $\chi$ instead of with the more general weight $\rho_s$.

Proposition 3.1. For every $0 < T \leq 1$ and each $\chi \in C_c^\infty(M)$ satisfying $\chi \equiv 1$ near $U$, there is a constant $C > 0$ such that

$$\|(1 - \chi)u\|_{L^p([0,T])W^{s,q}(M)} \leq C \|u_0\|_{H^s(M)},$$

where $u = e^{it\Delta} u_0$, $s \in [0, 1]$, and $(p, q)$, $p > 2$ satisfy

$$\frac{2}{p} + \frac{n}{q} = \frac{n}{2}.$$

Remark 3.2. In the sequel, wherever unambiguous, we will write

$L^p_T W^{s,q} := L^p([0,T])W^{s,q}(M)$

and

$H^s := H^s(M)$. 

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Proposition 3.3. Suppose \((M, g)\) satisfies the assumptions of the Introduction, \(u = e^{i t \Delta} u_0\), and
\[
v = \int_0^t e^{i(t-\tau)\Delta} f(\tau) d\tau.
\]
Then for each \(0 < T \leq 1\) and \(\delta \geq 0\) satisfying \((1.7)\), we have the estimates
\[
\|u\|_{L^p_T W^{s-\delta,q}} \leq C \|u_0\|_{H^s},
\]
and
\[
\|v\|_{L^p_T W^{s-\delta,q}} \leq C \|f\|_{L^1_T H^s},
\]
where \(s \in [0, 1]\) and \((p, q), p > 2\) satisfy the Euclidean scaling
\[
\frac{2}{p} + \frac{n}{q} = \frac{n}{2}.
\]

The proof uses a local WKB expansion also localized in time to the scale of inverse frequency, followed by summing over frequency bands (see [Chr3] and [BGT1]). The only difference here is the explicit dependence of \(\delta\) on \(\eta\), which is related to the growth of the function \(g(h)\).

Proof of Theorem 2. The proof of Theorem 2 is a slight modification of the proof of Proposition 3.1 in [BGT1], but we include it here in the interest of completeness. Fix \(s\) satisfying \((1.8)\) and choose \(p > \max\{2\beta - 2, 2\}\) satisfying
\[
s > \frac{n}{2} - \frac{2}{p} + \delta \geq \frac{n}{2} - \frac{1}{\max\{2\beta - 2, 2\}},
\]
where \(\delta \geq 0\) satisfies \((1.7)\). Set \(\sigma = s - \delta\) and
\[
Y_T = C([-T, T]; H^s(M)) \cap L^p([-T, T]; W^{\sigma,q}(M))
\]
for
\[
\frac{2}{p} + \frac{n}{q} = \frac{n}{2},
\]
equipped with the norm
\[
\|u\|_{Y_T} = \max_{|t| \leq T} \|u(t)\|_{H^s(M)} + \|u\|_{L^p_T W^{\sigma,q}}.
\]
Let \(\Phi\) be the non-linear functional
\[
\Phi(u) = e^{i t \Delta} u_0 - i \int_0^t e^{i(t-\tau)\Delta} F(u(\tau)) d\tau.
\]
If we can show that \(\varphi: Y_T \to Y_T\) and is a contraction on a ball in \(Y_T\) centered at 0 for sufficiently small \(T > 0\), this will prove the first assertion of the proposition, along with the Sobolev embedding
\[
W^{\sigma,q}(M) \subset L^\infty(M),
\]
since \(\sigma > n/q\). From Proposition 3.3 we bound the \(W^\sigma\) part of the \(Y_T\) norm by the \(H^s\) norm, giving
\[
\|\Phi(u)\|_{Y_T} \leq C \left( \|u_0\|_{H^s} + \int_{-T}^T \|F(u(\tau))\|_{H^s} d\tau \right)
\]
\[
\leq C \left( \|u_0\|_{H^s} + \int_{-T}^T \|(1 + |u(\tau)|)^{\frac{2\beta-2}{L^\infty}}\|_{H^s} d\tau \right),
\]
where the last inequality follows by our assumptions on the structure of $F$. Applying Hölder’s inequality in time with $\tilde{p} = p/(2\beta - 2)$ and $\tilde{q}$ satisfying

$$\frac{1}{\tilde{q}} + \frac{1}{\tilde{p}} = 1$$

gives

$$\|\varphi(u)\|_{Y_T} \leq C \left( \|u_0\|_{H^s} + T^\gamma \|u\|_{L^p_{\infty} L^\infty} \|(1 + |u|)\|_{L^p_{\infty} L^\infty}^{2\beta-2} \right)$$

where $\gamma = 1/\tilde{q} > 0$. Thus

$$\|\varphi(u)\|_{Y_T} \leq C \left( \|u_0\|_{H^s} + T^\gamma \|u\|_{Y_T} + \|u\|_{Y_T}^{2\beta} \right).$$

Similarly, we have for $u, v \in Y_T$,

(3.6) $\|\Phi(u) - \Phi(v)\|_{Y_T}$
(3.7) $\leq CT^\gamma \|u - v\|_{L^p_{\infty} L^\infty} \|(1 + |u|)\|_{L^p_{\infty} L^\infty}^{2\beta-2} + \|(1 + |v|)\|_{L^p_{\infty} L^\infty}^{2\beta-2}$,

which is a contraction for sufficiently small $T$. This concludes the proof of the first assertion in the proposition.

To get the second assertion, we observe from (3.6) and the definition of $Y_T$, if $u$ and $v$ are two solutions to (1.6) with initial data $u_0$ and $u_1$ respectively, so

$$\Phi(v) = e^{it\Delta} u_1 - i \int_0^t e^{i(t-\tau)\Delta} F(v(\tau)) d\tau,$$

we have

$$\max_{|t| \leq T} \|u(t) - v(t)\|_{H^s} = \max_{|t| \leq T} \|\Phi(u)(t) - \Phi(v)(t)\|_{H^s} \leq C \left( \|u_0 - u_1\|_{H^s} + T^\gamma \max_{|t| \leq T} \|u(t) - v(t)\|_{H^s} \|(1 + |u|)\|_{L^p_{\infty} L^\infty}^{2\beta-2} + \|(1 + |v|)\|_{L^p_{\infty} L^\infty}^{2\beta-2} \right),$$

which, for $T > 0$ sufficiently small gives the Lipschitz continuity.

If $(M, g)$ satisfies the assumptions of Corollary 1.1, $n \leq 3$, $\beta < 3$, and $G(r) \to +\infty$ as $r \to +\infty$, we can take $s$ and $p$ satisfying $p > \max\{2\beta - 2, 2\}$ and

$$s > \frac{n}{2} - \frac{2}{p} + \delta \geq \frac{n}{2} - \frac{2}{\max\{2\beta - 2, 2\}}$$

for any $\delta > 0$. Then $\sigma = s - \delta > q/n$ and the preceding argument holds. Finally, the proof of the global well-posedness now follows from the standard global well-posedness arguments from, for example, [Caz, Chapter 6]. □
References


