THE FATOU SET FOR CRITICALLY FINITE MAPS

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Abstract. It is a classical result in complex dynamics of one variable that the Fatou set for a critically finite map on $\mathbb{P}^1$ consists of only basins of attraction for superattracting periodic points. In this paper, we deal with critically finite maps on $\mathbb{P}^k$. We show that the Fatou set for a critically finite map on $\mathbb{P}^2$ consists of only basins of attraction for superattracting periodic points. We also show that the Fatou set for a $k$—critically finite map on $\mathbb{P}^k$ is empty.

1. Introduction

A holomorphic map $f : \mathbb{P}^k \to \mathbb{P}^k$ is said to be critically finite if every component of the critical set for $f$ is periodic or preperiodic. In [5], Thurston has given a topological classification of critically finite maps on $\mathbb{P}^1$. And it is well known that the Fatou set for a critically finite map on $\mathbb{P}^1$ consists of only basins of attraction for superattracting periodic points, i.e. points $p$ with $f^n(p) = p$ and $(f^n)'(p) = 0$ for some $n \in \mathbb{N}$ (see [3]). In this paper, we show that the same is also true for critically finite maps on $\mathbb{P}^2$. More precisely, we have the following

Theorem 1.1. If $f : \mathbb{P}^2 \to \mathbb{P}^2$ is a critically finite holomorphic map, then the Fatou set for $f$ consists of only basins of attraction for superattracting periodic points.

With some extra assumptions, the above result has been obtained by Fornæss and Sibony ([2]).

We will also study critically finite maps on $\mathbb{P}^k$. In particular, we obtain the following theorem (see Section 2 for precise definitions).

Theorem 1.2. Let $f : \mathbb{P}^k \to \mathbb{P}^k$ be a holomorphic map. If $f$ is $k$—critically finite, then the Fatou set for $f$ is empty.

For 1—critically finite maps on $\mathbb{P}^1$ and 2—critically finite maps on $\mathbb{P}^2$, this was proved by Thurston ([5]) and Ueda ([6]), respectively.

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2. The Fatou set for critically finite maps

Let $f : \mathbb{P}^k \to \mathbb{P}^k$ be a holomorphic map of (algebraic) degree $d > 1$. 
Let \( C_1 \) be the critical set of \( f \) given by
\[
C_1 = \{ p \in \mathbb{P}^k \mid \text{rank}(df(p)) < k \},
\]
where \( df(p) \) denotes the differential of \( f \) at \( p \).

We define the post-critical set \( D_1 \) of \( f \) by
\[
D_1 = \bigcup_{j=1}^{\infty} f^j(C_1),
\]
and the \( \omega \)-limit set \( E_1 \) of \( f \) by
\[
E_1 = \bigcap_{j=1}^{\infty} f^j(D_1).
\]

By definition, a holomorphic map \( f \) on \( \mathbb{P}^k \) is critically finite if the post-critical set \( D_1 \) is an analytic (hence algebraic) set in \( \mathbb{P}^k \). This is equivalent to saying that there is an integer \( l \geq 1 \) such that \( D_1 = \bigcup_{j=1}^{l} f^j(C_1) \). Hence, in the critically finite case, the set \( D_1 \) is an algebraic set of pure codimension 1.

Let us take a closer look at the structure of the post-critical set \( D_1 \) and the \( \omega \)-limit set \( E_1 \). If \( f \) is critically finite, then \( f^{j-1}(D_1) = \bigcup_{j=1}^{\infty} f^j(C_1) \). Consequently \( E_1 = f^{l-1}(D_1) \) is an algebraic set of pure codimension 1. We can decompose \( E_1 \) into \( E_1' \cup F_1 \), where \( F_1 \) consists of those components in a critical cycle. (A periodic component \( L \) is said to be in a critical cycle if at least one of the forward images of \( L \) under \( f \) is contained in the critical set for \( f \).)

**Definition 2.1.** Let \( f : \mathbb{P}^k \to \mathbb{P}^k \) be a holomorphic map. The map \( f \) is said to be critically finite of order 1 if \( D_1 \), hence \( E_1 \), is algebraic. And \( f \) is said to be 1-critically finite if \( C_1 \) and \( E_1 \) have no common irreducible component, i.e. \( F_1 = \emptyset \).

We can now make the following inductive definition (cf. [3]).

**Definition 2.2.** Let \( f : \mathbb{P}^k \to \mathbb{P}^k \) be a holomorphic map. Suppose \( f \) is critically finite of order \( n-1 \), \( 1 < n \leq k \). Denote \( C_n = C_1 \cap E_{n-1} \), \( D_n = \bigcup_{j=1}^{\infty} f^j(C_n) \), and \( E_n = \bigcap_{j=1}^{\infty} f^j(D_n) \). We say that \( f \) is critically finite of order \( n \) if \( D_n \), hence \( E_n \), is algebraic. Let \( l_n \) be the least integer such that \( E_n = f^{l_n-1}(D_n) = \bigcup_{j=1}^{l_n} f^j(C_n) \). We can decompose \( E_n \) into \( E_n' \cup F_n \), where \( F_n \) consists of those components, of codimension less or equal to \( n \), in a critical cycle. If in addition \( f \) is \((n-1)\)-critically finite, then we say that \( f \) is \( n \)-critically finite if \( E_n \) has no irreducible component contained in \( C_1 \), i.e. \( F_n = \emptyset \).

**Remark 2.3.** A critically finite map is by definition critically finite of order 1. Jonsson ([4] Remark 2.10) noted that a critically finite map on \( \mathbb{P}^2 \) is always strictly critically finite in the sense of Fornæss and Sibony ([2]). Recall that a critically finite map \( f \) is said to be strictly critically finite if \( f \), when restricted to each irreducible periodic component of \( E_1 \), is also critically finite. We will use this remark implicitly in the proof of Theorem 1.1.

Before we go further, let us recall some definitions and results from [6].

**Definition 2.4.** Let \( f : \mathbb{P}^k \to \mathbb{P}^k \) be a holomorphic map and let \( U \) be a Fatou component, i.e. a connected component of the Fatou set for \( f \). A holomorphic
map \( \varphi : U \to \mathbb{P}^k \) is called a limit map on \( U \) if there is a sequence \( \{ f^n \} \) which converges to \( \varphi \) uniformly on compact sets in \( U \). A point \( q \in \mathbb{P}^k \) is called a Fatou limit point if there is a limit map \( \varphi \) on a Fatou component \( U \) such that \( q \in \varphi(U) \).

The set of all Fatou limit points is called the Fatou limit set.

**Definition 2.5.** A Fatou component \( U \) is called a rotation domain if the identity map \( id_U : U \to U \) is a limit map on \( U \).

**Definition 2.6.** A point \( q \in \mathbb{P}^k \) is said to be a point of bounded ramification with respect to \( f \) if the following conditions are satisfied:

(i) There is a neighborhood \( W \) of \( q \) such that \( D_1 \cap W \) is an analytic subset of \( W \).

(ii) There exists an integer \( l \) such that, for every integer \( j > 0 \) and every \( p \in f^{-j}(q) \), the cardinality \( \sharp(I) \) of the set

\[
I = \{ i | 0 \leq i \leq j - 1, f^i(p) \in C_1 \}
\]

is not greater than \( l \).

The following two theorems by Ueda are crucial.

**Theorem 2.7** \( \{ \text{[6, Theorem 4.8]} \} \). Suppose that \( q \in \mathbb{P}^k \) is a point of bounded ramification and also a Fatou limit point. Then \( q \) is contained in a rotation domain.

**Theorem 2.8** \( \{ \text{[6, Proposition 5.1, (1)]} \} \). If \( f : \mathbb{P}^k \to \mathbb{P}^k \) is critically finite, then there is no rotation domain.

**Remark 2.9.** Since the set \( D_1 \) is an analytic set in the critically finite case, condition (i) in Definition 2.6 is automatically true. So we only need to check condition (ii) in Definition 2.6 to see if a point \( p \in \mathbb{P}^k \) is of bounded ramification.

We need the following lemma, whose proof is an elaboration of the proof of [6, Lemma 5.7].

**Lemma 2.10.** Let \( f : \mathbb{P}^k \to \mathbb{P}^k \) be a holomorphic map. If \( f \) is critically finite of order \( n \), \( 1 \leq n < k \), then every point in \( \mathbb{P}^k \setminus E_n \) is a point of bounded ramification. If \( f \) is critically finite of order \( k \), then every point in \( \mathbb{P}^k \setminus F_k \) is a point of bounded ramification.

**Proof.** First assume that \( f \) is critically finite of order \( n \), \( 1 \leq n < k \). Let \( q \in \mathbb{P}^k \setminus E_n \) and let \( p \in f^{-j}(q) \) for some integer \( j > 0 \). By Remark 2.9 we only need to show that the cardinality \( \sharp(I) \) of the set

\[
I = \{ i | 0 \leq i \leq j - 1, f^i(p) \in C_1 \}
\]

is not greater than some integer \( l > 0 \). Let

\[
I_m = \{ i | 0 \leq i \leq j - 1, f^i(p) \in C_m \setminus C_{m+1} \}, \quad m = 1, \ldots, n - 1,
\]

\[
I_n = \{ i | 0 \leq i \leq j - 1, f^i(p) \in C_n \}.
\]

We claim that \( \sharp(I_m) \leq l_m, m = 1, \ldots, n \).

For each \( 1 \leq m < n \), suppose that \( I_m \) is non-empty and let \( i_m \) be the least index in \( I_m \). Then \( f^{i_m}(p) \in C_m \). For \( i \geq i_m + l_m \), we have \( f^i(p) \in E_m \) and hence \( f^i(p) \notin C_{m+1} \). Thus \( I_m \) is a subset of \( \{ i_m, \ldots, i_m + l_m - 1 \} \), and \( \sharp(I_m) \leq l_m \).

Now suppose that \( I_n \) is non-empty and let \( i_n \) be the least index in \( I_n \). Then \( f^{i_n}(p) \in C_n \). For \( i \geq i_n + l_n \), we have \( f^i(p) \in E_n \). Since \( f^j(p) = q \notin E_n \), we have \( i_n + l_n > j \). Thus \( I_n \) is a subset of \( \{ i_n, \ldots, i_n + l_n - 1 \} \) and \( \sharp(I_n) \leq l_n \).
Next assume that \( f \) is critically finite of order \( k \). Let \( q \in \mathbb{P}^k \setminus F_k \) and let \( p \in f^{-j}(q) \) for some integer \( j > 0 \). Let
\[
I_m = \{ i | 0 \leq i \leq j - 1, f^i(p) \in C_m \setminus C_{m+1} \}, \quad m = 1, \ldots, k - 1,
\]
\[
I_k = \{ i | 0 \leq i \leq j - 1, f^i(p) \in C_k \}.
\]

For the same reason as above we have that \( \sharp(I_m) \leq l_m \) for \( 1 \leq m < k \). Now suppose that \( I_k \) is non-empty and let \( i_k \) be the least index in \( I_k \). Then \( f^{i_k}(p) \in C_k \). For \( i \geq i_k + l_k \), we have \( f^i(p) \in E_k \). Note that \( f(F_k) = F_k \) and \( (E_k \setminus F_k) \cap C_k = \emptyset \). Since \( f^j(p) = q \notin F_k \), we have \( i_k + l_k > j \). Thus \( I_k \) is a subset of \( \{ i_k, \ldots, i_k + l_k - 1 \} \) and \( \sharp(I_k) \leq l_k \).

Combining this lemma with Theorems 2.7 and 2.8 we obtain the following result, which generalizes [6, Theorem 5.8].

**Theorem 2.11.** Let \( f : \mathbb{P}^k \to \mathbb{P}^k \) be a holomorphic map. If \( f \) is critically finite of order \( n \), \( 1 \leq n < k \), then the Fatou limit set is contained in \( E_n \). If \( f \) is critically finite of order \( k \), then the Fatou limit set is contained in \( F_k \).

By definition, a \( k \)-critically finite map on \( \mathbb{P}^k \) has \( F_k = \emptyset \). Therefore we obtain Theorem 1.1 as a corollary to the above theorem.

Now we turn to critically finite maps on \( \mathbb{P}^2 \). We say that a point \( p \) is a super-attracting periodic point for a holomorphic map \( f \) on \( \mathbb{P}^2 \) if there exists an \( n \in \mathbb{N} \) such that \( f^n(p) = p \) and both of the eigenvalues of the differential \( df^n(p) \) are equal to zero; i.e. \( df^n(p) \) is nilpotent.

We now prove Theorem 1.1.

**Proof of Theorem 1.1.** If \( f \) is not \( 1 \)-critically finite, we are done by [2, Theorem 7.8]. Therefore, we can assume that \( f \) is \( 1 \)-critically finite. Then by [6, Theorem 5.8], the Fatou limit set for \( f \) consists of finitely many periodic critical points in \( F_2 \). It is obvious from the definition of \( F_2 \) that these periodic points belong to the singular set of \( V = \bigcup_{j=0}^{\infty} f^j(C_1) \). Hence, arguing as in the first part of the proof of [2, Theorem 7.7], we are done. Note that the assumption of \( \mathbb{P}^2 \setminus V \) being hyperbolic in [2, Theorem 7.7] is only needed in the second part of its proof. \( \square \)

**Remark 2.12.** Bonifant and Dabija ([1, Theorem 4.1]) showed that an invariant critical component for a holomorphic map on \( \mathbb{P}^2 \) must be a rational curve. While most known examples of critically finite maps on \( \mathbb{P}^2 \) have only smooth rational curves as invariant critical components, here we give a family of critically finite maps on \( \mathbb{P}^2 \) with singular rational curves as invariant critical components:
\[
g_d : [z : w : t] \mapsto [z^d - w^{d-1}t : -w^d : -t^d], \quad d > 2.
\]

Note that \( g_d \) maps the critical component \( \{ z = 0 \} \) to the singular rational curve \( \{ z^d = w^{d-1}t \} \) and maps \( \{ z^d = w^{d-1}t \} \) back to \( \{ z = 0 \} \). So \( g_d^2 \) will have \( \{ z^d = w^{d-1}t \} \) as a fixed critical component and obviously \( g_d^2 \) is a critically finite map.

**References**


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