REPRESENTATION OF MEASURES WITH POLYNOMIAL DENSENESS IN $L_p(\mathbb{R}, d\mu)$, $0 < p < \infty$, AND ITS APPLICATION TO DETERMINATE MOMENT PROBLEMS

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Abstract. It has been proved that algebraic polynomials $P$ are dense in the space $L_p(\mathbb{R}, d\mu)$, $0 < p < \infty$, if the measure $\mu$ is representable as $d\mu = wdf$ with a finite non-negative Borel measure $\nu$ and an upper semi-continuous function $w : \mathbb{R} \to \mathbb{R}^+ := [0, \infty)$ such that $P$ is a dense subset of the space $C^0_w := \{ f \in C(\mathbb{R}) : w(x)f(x) \to 0 as |x| \to \infty \}$ equipped with the seminorm $\|f\|_w := \sup_{x \in \mathbb{R}} w(x)|f(x)|$. The similar representation $(1 + x^2)d\mu = w^2d\nu$ (1 + $x^2)d\mu = w^2d\nu$) with the same $\nu$ and $w$ ($w(x) = 0, x < 0$, and $P$ is also a dense subset of $C^0_{\sqrt{w}}$) corresponds to all those measures (supported by $\mathbb{R}^+$) that are uniquely determined by their moments on $\mathbb{R} (\mathbb{R}^+)$. The proof is based on de Branges’ theorem (1959) on weighted polynomial approximation. A more general question on polynomial denseness in a separable Fréchet space in the sense of Banach $L^p(\mathbb{R}, d\mu)$ has also been examined.

1. Introduction and main results

Let $\mathcal{M}^+(\mathbb{R})$ denote the set of non-zero finite non-negative Borel measures on $\mathbb{R}$. Each space $L_p(\mathbb{R}, d\mu)$, $0 < p < \infty$, of complex-valued functions contains the family $\mathcal{P}[C]$ of all algebraic polynomials with complex coefficients if $\mu$ belongs to the class $\mathcal{M}^+(\mathbb{R})$ of all measures in $\mathcal{M}^+(\mathbb{R})$ with all moments $s_n(\mu) := \int_{\mathbb{R}} x^n d\mu(x)$, $n \in \mathbb{N}_0 := \{0, 1, 2, \ldots\}$, finite. In the sequel, we write $\mathcal{M}^+(\mathbb{R}^+)$ for the set of all measures $\mu$ in $\mathcal{M}^+(\mathbb{R})$ which are supported by $\mathbb{R}^+ := [0, +\infty)$, i.e. $\sup \mu := \{ x \in \mathbb{R} | \mu((x - \varepsilon, x + \varepsilon)) > 0 \forall \varepsilon > 0 \} \subset \mathbb{R}^+$.

The Hamburger (Stieltjes) moment problem consists in finding for a sequence of real numbers $\{\gamma_n\}_{n \in \mathbb{N}_0}$ a measure $\mu \in \mathcal{M}^+(\mathbb{R}) (\mathcal{M}^+(\mathbb{R}^+))$ with moments $s_n(\mu) = \gamma_n$, $n \in \mathbb{N}_0$. If the solution is unique it is said that the corresponding moment problem is determinate. Measures $\mu$ solving such problems are also called determinate. In other words, a measure $\mu \in \mathcal{M}^+(\mathbb{R}) (\mathcal{M}^+(\mathbb{R}^+))$ is said to be determinate in the sense of Hamburger (Stieltjes) (in short: $\mu \in \text{det}\mathcal{H}(\det\mathcal{S})$) if $\mu$ is the only measure in $\mathcal{M}^+(\mathbb{R}) (\mathcal{M}^+(\mathbb{R}^+))$ with the same moments as $\mu$ (see [11]).

In 1923 M. Riesz [25] established a direct connection between determinate Hamburger moment problems and the problem of polynomial denseness in the space

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exist a finite non-negative Borel measure \( \mu \). He proved that (see [11] Prop. 1.3)

\[
(1.1) \quad \mu \in \det \mathcal{H} \iff \mathcal{P}[\mathbb{C}] \text{ is dense in } L_2(\mathbb{R}, (1+x^2) \, d\mu) .
\]

In 1991 Ch. Berg and M. Thill [11] Th. 3.8 supplemented (1.1) with

\[
(1.2) \quad \mu \in \det \mathcal{S} \iff \mathcal{P}[\mathbb{C}] \text{ is dense in } L_2(\mathbb{R}, (1 + x) \, d\mu) \text{ and } L_2(\mathbb{R}, x \cdot (1 + x) \, d\mu) .
\]

Let \( \mathcal{W}^+(\mathbb{R}) \) be the set of upper semi-continuous and uniformly bounded functions \( w : \mathbb{R} \to \mathbb{R}^+ \). For \( w \in \mathcal{W}^+(\mathbb{R}) \) denote by \( C^0_w \) the semi-normed space composed of the linear set of all \( f \in C(\mathbb{R}) \) with \( \lim_{|x| \to \infty} w(x)f(x) = 0 \) and the semi-norm \( ||f||_w := \sup_{x \in \mathbb{R}} w(x)|f(x)| \) and \( C(\mathbb{R}) \) denotes the linear space of all continuous complex-valued functions on \( \mathbb{R} \). In this context a function \( w \in \mathcal{W}^+(\mathbb{R}) \) is also called a weight and \( \mathcal{P}[\mathbb{C}] \subseteq C^0_w \) if \( w \) belongs to the set \( \mathcal{W}^*(\mathbb{R}) \) of all weights \( w \) in \( \mathcal{W}^+(\mathbb{R}) \) satisfying \( \lim_{|x| \to \infty} |x|^n w(x) = 0 \) for all \( n \in \mathbb{N}_0 \).

In 1924 S. Bernstein [13] asked for conditions on \( w \in \mathcal{W}^*(\mathbb{R}) \) to have the denseness of \( \mathcal{P}[\mathbb{C}] \) in \( C^0_w \). In 1959 L. de Branges [15] obtained a solution to this problem (see also the survey papers of N. Akhiezer [1], S. Mergelyan [23], B. Ya. Levin [21] and M. Sodin [27]). A slightly improved version (see [28]) of his result is as follows. Let \( \mathcal{E}_0(\mathbb{R}) \) be the family of entire functions \( F \) of minimal exponential type having real and simple zeros only, and let \( \Lambda_B \) denote the set of these zeros and \( S_w := \{ x \in \mathbb{R} : w(x) > 0 \} \).

**Theorem A** ([8] de Branges, 1959]). For \( w \in \mathcal{W}^*(\mathbb{R}) \) assume that \( S_w \) is unbounded. Then \( \mathcal{P}[\mathbb{C}] \) fails to be dense in \( C^0_w \) if and only if there exists a transcendental function \( B \in \mathcal{E}_0(\mathbb{R}) \) with \( \Lambda_B \subseteq S_w \) such that

\[
\sum_{\lambda \in \Lambda_B} \frac{1}{w(\lambda)B'(\lambda)} < \infty.
\]

The problem of finding conditions on \( \mu \in \mathcal{M}^*(\mathbb{R}) \) to have a polynomial denseness in \( L_p(\mathbb{R}, d\mu) \) for a given \( p \in [1, \infty) \) was treated by various authors (see [1] [11] [24] [10] [15] [21] [12] [22], etc.). The first successful attempt to apply Theorem A for solving this problem was made in 1998 by A. Borichev and M. Sodin [14]. However, a complete analogue of Theorem A for \( L_p(\mathbb{R}, d\mu) \) was found only for discrete measures \( \mu \in \mathcal{M}^*(\mathbb{R}) \) with sufficiently thin support: \( \sum_{\lambda \in \supp \mu} (1 + |\lambda|)^{-a} < \infty \) for some \( a > 0 \). The final result on the description of all \( \mu \in \mathcal{M}^*(\mathbb{R}) \) such that \( \mathcal{P}[\mathbb{C}] \) is dense in \( L_p(\mathbb{R}, d\mu) \) for some \( p \in [1, \infty) \) was obtained in 1998 [9]; the sketch of its proof and an application to the Hamburger moment problem was published in 2001 [6].

Our main results are the following two theorems. Let \( B(\mathbb{R}) \) (\( B(\mathbb{R}^+) \)) denote the family of all Borel subsets of \( \mathbb{R} \) (\( \mathbb{R}^+ \)) and \( \mathcal{W}^*(\mathbb{R}) \) the set of all weights \( w \in \mathcal{W}^*(\mathbb{R}) \) with \( w(x) = 0, x < 0 \).

**Theorem 1.1.** Let \( 0 < p < \infty \) and \( \mu \in \mathcal{M}^*(\mathbb{R}) \) have unbounded support. Then \( \mathcal{P}[\mathbb{C}] \) is dense in \( L_p(\mathbb{R}, d\mu) \) if and only if the measure \( \mu \) can be represented in the following form:

\[
(1.3) \quad \mu(A) = \int_A w(x)^p \, d\nu(x) , \quad A \in B(\mathbb{R}) ,
\]

(in short: \( d\mu(x) = w(x)^p \, d\nu(x) \)) for some \( \nu \in \mathcal{M}^+(\mathbb{R}) \) and \( w \in \mathcal{W}^*(\mathbb{R}) \) such that \( \mathcal{P}[\mathbb{C}] \) is dense in \( C^0_w \).

**Theorem 1.2.** Let \( \mu \in \mathcal{M}^*(\mathbb{R}) \) (\( \mathcal{M}^*(\mathbb{R}^+) \)). There is no other measure in \( \mathcal{M}^*(\mathbb{R}) \) (\( \mathcal{M}^*(\mathbb{R}^+) \)) with the same moments as \( \mu \), i.e. \( \mu \in \det \mathcal{H}(\det \mathcal{S}) \), if and only if there exist a finite non-negative Borel measure \( \nu \) on \( \mathbb{R} \) (\( \mathbb{R}^+ \)) and a weight \( w \in \mathcal{W}^*(\mathbb{R}) \)
Let \( \nu \in \mathcal{M}^+(\mathbb{R}) \) have unbounded support. The set of algebraic polynomials \( \mathcal{P}[\mathbb{C}] \) is dense in the space \( L^p(\mathbb{R}, d\mu) \) iff \( d\mu(x) = w(x) \, d\nu(x) \), where \( \nu \in \mathcal{M}^+(\mathbb{R}) \) and \( w \in W^*_\Phi(\mathbb{R}) \) are such that \( \mathcal{P}[\mathbb{C}] \) is dense in all seminormed spaces \( C_{w_n} \), \( n \in \mathbb{N} \), where
\[
\begin{align*}
w_n(x) &:= 1/\varphi\left( \frac{1}{n \cdot w(x)} \right), \quad x \in \mathbb{R}, \quad n \in \mathbb{N}, \\
\varphi \text{ is the inverse function of } \Phi \text{ and it is assumed that } 1/0 := +\infty, 1/ +\infty := 0.
\end{align*}
\]

Additional restriction to \( \Phi \) essentially simplifies the conditions of Theorem 1.3 in the following corollary.

**Corollary 1.1.** Let \( \mu \in \mathcal{M}^*_\Phi(\mathbb{R}) \) have unbounded support, and there exists a constant \( \lambda > 1 \) satisfying
\[
\lim_{x \to +\infty} \frac{\Phi(\lambda \cdot x)}{\Phi(x)} > 1.
\]

Then \( \mathcal{P}[\mathbb{C}] \) is dense in \( L^p(\mathbb{R}, d\mu) \) if and only if there exist a measure \( \nu \in \mathcal{M}^+(\mathbb{R}) \) and a weight \( w \in W^*_\Phi(\mathbb{R}) \) such that
\[
d\mu(x) = \frac{d\nu(x)}{\Phi\left( \frac{1}{w(x)} \right)}
\]
and \( \mathcal{P}[\mathbb{C}] \) is dense in \( C_{w_n} \).
2. Preliminary results

It follows from the definition of $L^\Phi(\mathbb{R}, d\mu)$ that $f \in L^\Phi(\mathbb{R}, d\mu)$ belongs to the closure $\overline{C_0^\Phi} A$ of $A \subset L^\Phi(\mathbb{R}, d\mu)$ in the space $L^\Phi(\mathbb{R}, d\mu)$ iff

\begin{equation}
\inf_{a \in A} \int_{\mathbb{R}} \Phi(n \cdot |f(x) - a(x)|) \, d\mu(x) = 0, \quad n = 1, 2, \ldots.
\end{equation}

**Lemma 2.1.** For arbitrary $\mu \in \mathcal{M}^+(\mathbb{R})$ the space $L^\Phi(\mathbb{R}, d\mu)$ is a complete metrizable Hausdorff topological linear space (see [20, Ch. I, 6.1]); i.e. it is a Fréchet space in the sense of Banach (see [16, 6.1.1, Rem. 4]).

**Proof.** For arbitrary $f \in L^\Phi(\mathbb{R}, d\mu)$, $R > 0$, $n, M \in \mathbb{N}$, denote $J_R := \mathbb{R} \setminus [-R, R]$, $J_R^f(M) := \{x : |x| \leq R, |f(x)| > M\}$, $I_R^f(M) := \{x : |x| \leq R, |f(x)| \leq M\}$, and $L_n(A; f) := \int_{J_R^f(M)} \Phi(nf(x)) \, d\mu(x)$, $A \in \mathcal{B}(\mathbb{R})$.

It is evident that $U_{n+1}^\Phi \subset U_n^\Phi \subset U_n^\Phi$ for any $n \in \mathbb{N}$ and $|\theta| \leq 1$, $\theta \in \mathbb{C}$. Obvious inequality $\Phi(|x_1 + x_2|) \leq \Phi(2|x_1|) + \Phi(2|x_2|)$, $x_1, x_2 \in \mathbb{C}$, yields $L_n(\mathbb{R}, f + g) \leq L_{2n}(\mathbb{R}, f) + L_{2n}(\mathbb{R}, g)$, $f, g \in L^\Phi(\mathbb{R}, d\mu)$, and $U_{2n}^\Phi \subset U_{2n}^\Phi \subset U_{2n}^\Phi$, $n \in \mathbb{N}$. This, combined with $L_n(\mathbb{R}; f) \geq L_1(\mathbb{R}; f) > 0$, $f \neq 0$ a.e. $\mu$, $n \in \mathbb{N}$, allows for arbitrary $g_1, g_2 \in L^\Phi(\mathbb{R}, d\mu)$, $g_1 \neq g_2$ a.e. $\mu$, to choose $p \in \mathbb{N}$ such that $1/p < L_1(\mathbb{R}; g_1 - g_2)$ and to get $g_1 + U_{2p}^\Phi \cap g_2 + U_{2p}^\Phi = \emptyset$.

We claim in addition that $\lim_{\varepsilon \to 0} L_n(\mathbb{R}, \varepsilon f) = 0$ for every $f \in L^\Phi(\mathbb{R}, d\mu)$ and every positive integer $n$. Indeed, for every $\delta > 0$ due to the definition of a Lebesgue integral, one can find $R = R(\delta) > 0$ and $M = M(\delta) \in \mathbb{N}$ such that $L_n(J_R; f) + L_n(J_R^f(M); f) \leq \delta/2$. Choosing $\varepsilon > 0$ so that $\Phi(\varepsilon M) \cdot \mu(\mathbb{R}) < \delta/2$, we get $L_n(\mathbb{R}; \varepsilon f) = L_n(J_R; \varepsilon f) + L_n(J_R^f(M); \varepsilon f) + L_n(I_R^f(M); \varepsilon f) \leq \delta/2 + \Phi(\varepsilon M) \cdot \mu(\mathbb{R}) < \delta$, which was to be proved.

That’s why $\bigcup_{\lambda > 0} \lambda \cdot U_n^\Phi = L^\Phi(\mathbb{R}, d\mu)$, $n \in \mathbb{N}$, and in view of [20, Ch. I, 1.2] (see also [3, Ch. III, 1.1, Th. 1]) and [20, Ch. I, 6.1] the properties above imply that $L^\Phi(\mathbb{R}, d\mu)$ is a metrizable Hausdorff topological linear space.

To prove the completeness of $L^\Phi(\mathbb{R}, d\mu)$ observe that if $\{f_k\}_{k \in \mathbb{N}}$ is a Cauchy sequence in $L^\Phi(\mathbb{R}, d\mu)$, then for any $n \in \mathbb{N}$ it is possible to find $N_n \in \mathbb{N}$ such that $f_k - f_r \in U_n^\Phi \mu$, $k, r \geq N_n$, and so for these indices and every $\sigma > 0$ we have:

\[ \frac{1}{n} > L_n(\mathbb{R}; f_k - f_r) \geq L_n(\{x \in \mathbb{R}: |f_k(x) - f_r(x)| > \sigma\}; f_k - f_r) \geq \Phi(\sigma) \cdot \mu(\mathbb{R}) \geq \sigma \]

whence it follows that $\{f_k\}_{k \in \mathbb{N}}$ is a Cauchy sequence with respect to the convergence in measure $\mu$ [19, §22]. Then by theorem 5 in [19, §22] there exists a Borel measurable function $f$ such that the sequence \{f_k\}_{k \in \mathbb{N}} converges to $f$ in measure $\mu$ and in view of F. Riesz’s theorem [9, II, Th. 4.3] there exists a subsequence $\{f_{r_j}\}_{j \in \mathbb{N}}$ converging to $f$ a.e. $\mu$. For any given $n, m \in \mathbb{N}$ the sequence of non-negative functions $\Phi(n|f_{r_j+m}(x) - f_{r_m}(x)|)$, $j \in \mathbb{N}$, converges to $\Phi(n|f(x) - f(x)|)$ a.e. $\mu$ as $j$ tends to infinity, and by Fatou’s lemma [9, III, Th. 6.2] it follows from $L_{n+1}(\mathbb{R}; f_{r_{j+m}} - f_{r_m}) < 1/(n+1)$, $r_m > N_{n+1}$, that $f - f_{r_m} \in U_n^\Phi \mu$. That’s why $f \in L^\Phi(\mathbb{R}, d\mu)$, and furthermore [27] and inequalities $L_n(\mathbb{R}; f - f_k) = L_n(\mathbb{R}; f - f_r + f_r - f_k) \leq L_{2n}(\mathbb{R}; f - f_r) + L_{2n}(\mathbb{R}; f_r - f_k)$, $j, k \in \mathbb{N}$, prove that the initial sequence $\{f_k\}_{k \in \mathbb{N}}$ converges to $f$ in the space $L^\Phi(\mathbb{R}, d\mu)$. The lemma follows.

\[ \square \]

Observe that if $M(x)$ is an even convex function on $\mathbb{R}$ strictly increasing on $\mathbb{R}^+$ and satisfying $\lim_{x \to 0} M(x)/x = 0$ and $\lim_{x \to +\infty} M(x)/x = +\infty$, then $L^M(\mathbb{R}, d\mu)$
coincides with the space $E_M(\mathbb{R})$ which is the closure of all bounded Borel measurable functions in the Orlicz space $L_M(\mathbb{R})$ (see [3 Ch. IV, 3.6]).

Let $\mathcal{K}(\mathbb{R})$ denote the linear space of all $f \in C(\mathbb{R})$ with compact support (see [10 Ch. IV, 4.1]) and $\text{lin} S$ the linear hull of $S$.

The following lemma implies that the space $L^\Psi(\mathbb{R}, d\mu)$ is separable.

**Lemma 2.2.** Let $\mu \in \mathcal{M}^+(\mathbb{R})$. Then

$$
\text{Cl}^\mu_{\Psi} \left\{ \frac{1}{(x \pm i)^n} \right\}_{n \in \mathbb{N}} = \text{Cl}^\mu_{\Psi} \mathcal{K}(\mathbb{R}) = L^\Psi(\mathbb{R}, d\mu).
$$

**Proof.** To prove the second equality of (2.2) a standard scheme of approximation can be used. We first prove that $\text{Cl}^\mu_{\Psi} \left\{ \chi_A \right\}_{A \in B(\mathbb{R})} = L^\Psi(\mathbb{R}, d\mu)$ (see [9 Ch. VI, Th. 8.4]), where $\chi_A$ denotes the characteristic function of $A \subset \mathbb{R}$. In addition to the notation introduced in the proof of Lemma 2.1 we use for a given $f \in L^\Psi(\mathbb{R}, d\mu)$ and arbitrary $R > 0$, $n, N, M \in \mathbb{N}$, the following: $I_0 := [-R, R] \cap f^{-1}(\{0\}), I_k := [-R, R](f^{-1}(\pm(1)/(\Phi(n/N)))$, $\omega_k(x) := (\pm(M + \frac{1}{N})\chi_{I_k}(x), 1 \leq k \leq NM$, $\omega_0(x) := 0$, $\psi_N(x) := \sum_{k=-NM}^{NM} \omega_k(x)$, $x \in \mathbb{R}$.

The definition of the Lebesgue integral implies for a given $n \in \mathbb{N}$ an existence of $R = R(n) > 0$ and $M = M(n) \in \mathbb{N}$ such that $L_n(J_R; f) + L_n(J_R^*(M); f) \leq 1/(2n)$. Since the sets $J_R, J_R^*(M), I_0$ and $I_k$, $1 \leq k \leq NM$, are disjoint, $|f(x) - \psi_N(x)| \leq \frac{1}{n}$ for every $x \in I_k^0(M)$ and there exist $N = N(n) \in \mathbb{N}$ such that $\Phi(n/N) \cdot \mu(\mathbb{R}) < 1/(2n)$; then $L_n(\mathbb{R}; f - \psi_N) = L_n(J_R; f) + L_n(J_R^*(M); f) + \sum_{k=-NM}^{NM} L_n(I_k, f - \omega_k) \leq 1/(2n) + \Phi(n/N) \cdot \mu(\mathbb{R}) < 1/n$, i.e. $f - \psi_n \in U_{n^4}^n$. Equality $\text{Cl}^\mu_{\Psi} \left\{ \chi_A \right\}_{A \in B(\mathbb{R})} = L^\Psi(\mathbb{R}, d\mu)$ is proved.

It remains to prove that for arbitrary bounded $A \in B(\mathbb{R})$ it is possible to approximate $\chi_A$ by functions in $\mathcal{K}(\mathbb{R})$. Due the regularity of any $\mu \in \mathcal{M}^+(\mathbb{R})$ (see [19 §52, Th. 7]) for every $\varepsilon > 0$ one can find a compact set $F_\varepsilon$ and an open set $G_\varepsilon$ such that $F_\varepsilon \subset A \subset G_\varepsilon$ and $\mu(G_\varepsilon \setminus F_\varepsilon) < \varepsilon$. Then the known function (see [9 Ch. VI, Th. 8.6]) $\rho_\varepsilon(x) := \rho(x, \mathbb{R} \setminus G_\varepsilon)/[\rho(x, \mathbb{R} \setminus G_\varepsilon) + \rho(x, F_\varepsilon)] \in \mathcal{K}(\mathbb{R})$, where $\rho(x, B) := \inf_{y \in B} |x - y|$, $B \subset \mathbb{R}$, equals $\chi_A$ when $x \in F_\varepsilon \cup (\mathbb{R} \setminus G_\varepsilon)$ and has its values in $[0, 1]$ when $x \in G_\varepsilon \setminus F_\varepsilon$. Therefore for every $n \in \mathbb{N}$: $L_n(\mathbb{R}, \chi_A - \rho_\varepsilon) = L_n(G_\varepsilon \setminus F_\varepsilon, \chi_A - \rho_\varepsilon) \leq \Phi(n) \cdot \mu(G_\varepsilon \setminus F_\varepsilon) \leq \Phi(n) \varepsilon$, and so $\chi_A - \rho_\varepsilon \in U_{n^4}^n$, which finishes the proof of the right-hand side equality of (2.2).

The left-hand side equality of (2.2) follows from the denseness of $(x \pm i)^{-n}$, $n \in \mathbb{N}$, in the normed linear space $C^0(\mathbb{R})$ defined as $C^0_w$ with $w \equiv 1$ and the inclusion $\mathcal{K}(\mathbb{R}) \subset C^0(\mathbb{R})$. The proof of this denseness can be achieved by the method given in the proof of Theorem 2.3.2 in [2]. Arguing by contradiction we get the existence of a linear continuous functional on $C^0(\mathbb{R})$ (see [16 Ex. 4.45], [7 Th. 1]) vanishing on all $(x \pm i)^{-n}$, $n \in \mathbb{N}$, i.e. (see [26 Ch. 1, 7.1]) there exist two real-valued finite Borel measures $\sigma_1$ and $\sigma_2$ such that $F_{\pm}^{(n-1)}(z) = 0$, $n \in \mathbb{N}$, where $F_{\pm}(z) := \int_\mathbb{R} (x - z)^{-1}d\sigma_{\pm}(x)$ and $\sigma_{\pm} := \sigma_1 \pm i\sigma_2$. Since functions $F_{\pm}$ are analytic in the upper half-plane, $F_{\pm}(z) = 0$ for any $\text{Im} z > 0$, and by the Stieltjes-Perron inversion formula [2 Ch. III, 1a] we get a contradiction. The lemma is proved.

A suitable criterion for polynomial denseness in the space $L^\Psi(\mathbb{R}, d\mu)$ is established in the following lemma.
Lemma 2.3. Let $\mu \in M_+^0(\mathbb{R})$ have unbounded support. Then $\text{Cl}_A^\mu \mathcal{P}[C] = L^\Phi(\mathbb{R}, d\mu)$ iff there exists a sequence \{\(P_n\)\}$_{n \in \mathbb{N}} \subset \mathcal{P}[C]$ such that
\begin{equation}
(2.3) \quad \lim_{n \to \infty} \int_{\mathbb{R}} \Phi(n \cdot |\frac{1}{x+i} - P_n(x)|) \, d\mu(x) = 0 .
\end{equation}

Proof. It follows from (2.3) and (2.2) that to prove the polynomial denseness in $L^\Phi(\mathbb{R}, d\mu)$ it is sufficient to verify that $(x+i)^{-n}$ belongs to $\text{Cl}_A^\mu \text{lin}\{ (x+i)^{-1} \cup \mathcal{P}[C] \}$ for every integer $n \geq 2$. But this is a corollary of the evident inequalities (see [2, Th. 2.3.2])
\[
\left| \frac{1}{(x+i)^{n+1}} - \frac{A}{x+i} - P(x) \right| \leq \left| \frac{1}{(x+i)^n} - A - (x+i)P(x) \right| , \quad n \in \mathbb{N} , \quad x \in \mathbb{R} ,
\]
where $A \in \mathbb{C}$ and $P \in \mathcal{P}[C]$. Triviality of the converse implication finishes the proof.

Let $W^\Phi(\mathbb{R})$ be the set of $w \in W^+(\mathbb{R})$ satisfying $\lim_{|x| \to +\infty} w(x)\Phi(|x|^n) = 0$ for all $n \in \mathbb{N}_0$. It is assumed that $|x|^0 = 1$ for every $x \in \mathbb{R}$. For $w \in W^+(\mathbb{R})$ we introduce a pseudometrizable topological linear space $C^0_{w,\Phi}(\mathbb{R})$ composed of the linear space \{ $f \in C^0(\mathbb{R})$ | $\Phi(nf) \in C^0_w$, $n \in \mathbb{N}$ \} and the local base at zero $W^\Phi_{w} := \{ f \in C^0_{w,\Phi}(\mathbb{R}) | \|\Phi(nf)\|_w \leq 1/n \}$, $n \in \mathbb{N}$ (see [17, 4.1], [26, Ch. I, §6]). In the case where $\Phi(x) = x$, we have $C^0_{w,\Phi}(\mathbb{R}) = C^0_w$.

**Lemma 2.4.** For $w \in W^+(\mathbb{R})$ and $n \in \mathbb{N}$ define $w_n(x) = 1/\varphi(\frac{1}{n \cdot w(x)})$, $x \in \mathbb{R}$. Then $C^0_{w,\Phi}(\mathbb{R})$ coincides with the linear space $\bigcap_{n \in \mathbb{N}} C^0_{w_n}$ equipped with the locally convex topology determined by the semi-norms $\| \cdot \|_{w_n}$, $n \in \mathbb{N}$ (see [26, Ch. II, §4]).

**Proof.** For arbitrary $\omega \in W^+(\mathbb{R})$ observe that $f \in C^0_{w,\Phi}$ iff $f \in C^0_w$ and $\lim_{M \to +\infty} \|f\|_{w,\Phi} = 0$, where $\|f\|_{w,\Phi} := \sup_{|x| \geq M} \omega(x)\|f(x)\|$. It is easy to see that for every $n, m \in \mathbb{N}$, $f \in C(\mathbb{R})$ and $x \in \mathbb{R}$ the inequality $w(x)\Phi(n|f(x)|) \leq 1/m$ holds iff $w_n(x)|f(x)| \leq 1/n$. Therefore
\begin{equation}
(2.4) \quad \|\Phi(nf)\|_{w,\Phi}^M = \frac{1}{m} \Leftrightarrow |f|_{w_n}^M \leq \frac{1}{n} , \quad M \geq 0 , \quad n, m \in \mathbb{N} , \quad f \in C(\mathbb{R}) .
\end{equation}

Properties (2.4) and $w_n \leq w_{n+1}$, $n \in \mathbb{N}$, imply that
\begin{align}
(2.5a) & \quad W^\Phi_{w_n} = \frac{1}{n} \cdot U_{w_n} , \quad U_{w_n} := \{ f \in C(\mathbb{R}) | \|f\|_{w_n} \leq 1 \} , \quad n \in \mathbb{N} , \\
(2.5b) & \quad W^\Phi_{w_n} = \frac{1}{n+m} \cdot U_{w_{n+m}} \subset \frac{1}{n+m} \cdot U_{w_n} , \quad n \in \mathbb{N} , \quad m \in \mathbb{N}_0 ,
\end{align}

and for $f \in C(\mathbb{R})$ the following assertions are equivalent: 1) $\Phi(nf) \in C^0_w \forall n \in \mathbb{N}$; 2) $\forall n \in \mathbb{N}$ : $\lim_{M \to +\infty} \|\Phi(nf)\|_{w,\Phi}^M = 0$; 3) $\forall n, m \in \mathbb{N}$ $\exists M_{n,m} > 0 \forall M \geq M_{n,m}$ : $\|\Phi(nf)\|_{w,\Phi}^M \leq 1/m$; 4) $\forall m \in \mathbb{N}$ $\forall n \in \mathbb{N}$ $\exists M_{n,m} > 0 \forall M \geq M_{n,m}$ : $\|f\|_{w_n}^M \leq 1/n$; 5) $\forall m \in \mathbb{N}$ : $\lim_{M \to +\infty} \|f\|_{w,n}^M = 0$; 6) $f \in C^0_w \forall m \in \mathbb{N}$. The proof of Lemma 2.4 is complete.

The equality $C^0_{w,\Phi}(\mathbb{R}) = \bigcap_{n \in \mathbb{N}} C^0_{w_n}$, proved in Lemma 2.4, implies, in particular, that $w$ belongs to $W^\Phi_{w_n}$ iff $w_n \in W^*(\mathbb{R})$ for every $n \in \mathbb{N}$. Furthermore, if $\text{Cl}_A^\mu \text{A}$ denotes the closure of $A \subset C^0_{w,\Phi}(\mathbb{R})$ in the space $C^0_{w,\Phi}(\mathbb{R})$, then by (2.5):
\begin{equation}
(2.6) \quad f \in \text{Cl}_A^\mu \text{A} \Leftrightarrow f \in C(\mathbb{R}) \text{ and } \inf_{a \in A} \|f-a\|_{w_n} = 0 , \quad n = 1, 2, \ldots ;
\end{equation}
whence it follows that $\mathcal{P}[C]$ is dense in $C^0_{w,\Phi}(\mathbb{R})$ iff $\mathcal{P}[C]$ is dense in $C^0_{w_n}$, $n \in \mathbb{N}$. Since $1/(x+i)$ is in $C^0_{w,\Phi}(\mathbb{R})$ for any $w \in W^+(\mathbb{R})$, the known equivalence of polynomial denseness in $C^0_w$, $\omega \in W^*(\mathbb{R})$, and the possibility of approximating the
single function $1/(x+i)$ by polynomials in $C_0^w$ (see [8] p. 238) gives for arbitrary $w \in W^*_0(\mathbb{R})$:

\begin{equation}
(2.7) \quad C^w_\Phi \mathcal{P}[\mathbb{C}] = C^0_{w_\Phi}(\mathbb{R}) \iff \mathcal{P}[\mathbb{C}] \text{ is dense in all } C^0_{w_n}, n \in \mathbb{N} \quad \text{where} \quad \frac{1}{x+i} \in C^w_\Phi \mathcal{P}[\mathbb{C}].
\end{equation}

We recall the definition of the so-called upper Baire function $M_F$ of $F : \mathbb{R} \to \mathbb{R}$ as $M_F(x):= \lim_{\delta \downarrow 0} \sup_{y \in (x-\delta,x+\delta)} F(y)$. If $F$ is locally bounded from above, then $M_F$ is an upper semi-continuous function.

Denote by $\mathcal{F}$ the set of topological linear spaces of complex-valued functions on $\mathbb{R}$ which have $\mathcal{P}[\mathbb{C}]$ as a dense subset. The crucial tool in the proof of Theorem 1.3 will be the following theorem contained in [8] as Lemma 6 with a proof based on the result of Theorem A.

**Theorem B** ([8] Lemma 6, p. 238]). For $w \in W^*(\mathbb{R})$ with unbounded $S_w$ let $S \subseteq S_w$ be such that $w(x) = M_{w, x}(x), x \in \mathbb{R}$. If for any countable set $G \subseteq S$ without finite accumulation points we have $C^0_{w, x \in G} \in \mathcal{F}$, then also $C^0_w \in \mathcal{F}$.

To prove Theorem 1.3 we need the following auxiliary result.

**Lemma 2.5.** Let $\mu \in \mathcal{M}^+(\mathbb{R})$ have unbounded support and $a \in L_1(\mathbb{R}, d\mu)$. Then there exists an upper semi-continuous function $\theta : \mathbb{R} \to (0, 1]$ such that $\lim_{|t| \to \infty} \theta(t) = 0$ and $a/\theta \in L_1(\mathbb{R}, d\mu)$.

**Proof.** Since $\alpha(x):= 1 + |a(x)| \in L_1(\mathbb{R}, d\mu)$, the sequence of positive numbers

\[ t_n := \int_{|x| > n} \alpha(x) \, d\mu(x), \quad n \in \mathbb{N}_0, \]

tends to zero as $n \to \infty$ and it is possible to find a subsequence $\{n_k\}_{k \in \mathbb{N}_0}$ such that $n_0 := 0, \sum_{k \in \mathbb{N}_0} t_{n_k} < \infty$ and $t_{n+1} < t_n, k \in \mathbb{N}_0$. Introduce the function $\theta$ defined by

\[ 1/\theta(x) := \chi_{\{0\}}(x) + \sqrt{t_0} \cdot \sum_{k \in \mathbb{N}_0} \chi_{(n_k, n_{k+1})}(|x|)/\sqrt{t_{n_k}}, \quad x \in \mathbb{R}. \]

It is evident that $\theta$ is an even and upper semi-continuous function on $\mathbb{R}$ which does not increase on $\mathbb{R}^+$, $\theta(x) \in (0, 1]$ for every $x \in \mathbb{R}$, and $\theta(x) \to 0$ as $|x| \to +\infty$. Inequalities

\[ \int_{\mathbb{R}} \frac{\alpha(x)}{\theta(x)} \, d\mu(x) = a(0) \cdot \mu(\{0\}) + \sqrt{t_0} \cdot \sum_{k \in \mathbb{N}_0} (t_{n_k} - t_{n_{k+1}}) / \sqrt{t_{n_k}} < \infty \]

and $|a(x)| < \alpha(x), x \in \mathbb{R}$, complete the proof of Lemma 2.5.

3. **Proof of Theorem 1.3**

**3.1. Necessity.** For every $n \in \mathbb{N}$ it is possible to approximate the function $c(x) := 1/(x+i) \in C^0_{w_n}$ by a sequence $(P_{n,k})_{k \in \mathbb{N}} \subseteq \mathcal{P}[\mathbb{C}]$ such that $c - P_{n,k} \in \frac{1}{n} U_{w_n}$. Therefore by (2.5) for $Q_n := P_{n,n}, c - Q_n \in \frac{1}{n} U_{w_n} = W^*_0 \Phi, w$, and so $(Q_n)_{n \in \mathbb{N}}$ converges to $c$ in the space $C^0_{w_\Phi}(\mathbb{R})$. That’s why $d\mu_c = w(x) \, dv(x)$ yields

\[ \int_{\mathbb{R}} \Phi(n \mid c(x) - Q_n(x) \mid) \, d\mu(x) \leq \|\Phi(n \mid c - Q_n)\|_{w} \cdot \nu(\mathbb{R}) \leq \frac{\nu(\mathbb{R})}{n}. \]

This proves $C^w_\Phi \mathcal{P}[\mathbb{C}] = L^w(\mathbb{R}, d\mu)$ by virtue of (2.3).
3.2. **Sufficiency.** Due to \([23]\) for every \(n \in \mathbb{N}\) there exists \(P_n \in \mathcal{P}[\mathbb{C}]\) such that
\[
\Phi(n \cdot |c(x) - P_n(x)|) = \frac{1}{2^{0^n}} U_{L_1(\mu)} , \quad U_{L_1(\mu)} := \{ f \in L_1(\mathbb{R}, d\mu) \mid \parallel f \parallel_{L_1(\mathbb{R}, d\mu)} \leq 1 \} .
\]
Then the non-decreasing sequence of non-negative continuous functions
\[
\alpha_N(x) := \sum_{n=1}^{N} \Phi(n \cdot |c(x) - P_n(x)|) \in U_{L_1(\mu)} , \quad N \in \mathbb{N} ,
\]
has a limit \(\alpha_0 \in U_{L_1(\mu)}\) by the Beppo Levi theorem \([9, \text{Ch. III, Th. 6.3}]\). It follows from Theorem 1.4 in \([20, \text{Ch. I}]\) that the function \(\alpha_0\) is lower semi-continuous on \(\mathbb{R}\). The unboundedness of \(\text{supp } \mu\) implies that the numbers \(s_n^\Phi(\mu)\) grow to infinity faster than any function \(\Phi(Q^n)\), \(Q > 1\), as \(n \to \infty\), and therefore \((0^\Phi := 1)\)
\[
(3.1) \quad h(x) := 2\mu(\mathbb{R}) \cdot \sum_{n \in \mathbb{N}_0} \frac{\Phi(|x|^n)}{2^{\nu + 1} s_n^\Phi(\mu)} \in C(\mathbb{R}) \cap U_{L_1(\mu)} , \quad h(x) \geq 1 , \quad x \in \mathbb{R} .
\]
In addition, by F. Riesz's theorem (see \([19, \text{§25, Th. 1}]\)) there exists a subsequence \(\{n_k\}_{k \in \mathbb{N}} \subset \mathbb{N}\) such that the sequence \(\{\Phi(n_k \cdot |c(x) - P_{n_k}(x)|)\}_{k \in \mathbb{N}}\) converges to zero a.e. \(\mu\), i.e.
\[
(3.2) \quad \lim_{k \to \infty} \Phi(n_k \cdot |c(x) - P_{n_k}(x)|) = 0 , \quad x \in \mathbb{R} \setminus D , \quad \mu(D) = 0 , \quad D \in \mathcal{B}(\mathbb{R}) .
\]
When applied to the function \(\alpha_0 + h \in 2U_{L_1(\mu)}\), Lemma \(25\) gives a function \(\theta\) with values in \((0, 1]\) such that: \(1/w_* := (\alpha_0 + h)/\theta \in L_1(\mathbb{R}, d\mu)\); \(w_*\) is an upper semi-continuous function on \(\mathbb{R}\); \(0 \leq w_* = \theta/(\alpha_0 + h) \leq 1\) in view of \((3.1)\); and \(w_* \in \mathcal{W}_\Phi(\mathbb{R})\) due to \(w_* \Phi(|x|^n) \leq \theta(x) \Phi(|x|^n)/h(x) \leq 2^n s_n^\Phi(\mu) \theta(x)/\mu(\mathbb{R}) \to 0\), \(|x| \to \infty\), \(n \in \mathbb{N}\). We introduce the function
\[
(3.3) \quad w(x) := M_w \cdot \chi_{\mathbb{R} \setminus D}(x) , \quad x \in \mathbb{R} .
\]
Then \(0 \leq w \leq w_* \leq 1\) and \(w = w_*\) a.e. \(\mu\). Therefore \(1/w \in L_1(\mathbb{R}, d\mu)\) and \(w \in \mathcal{W}_\Phi(\mathbb{R})\). Defining the measure \(\nu\) by \(\nu(A) := \int_A 1/w(x) \, d\mu(x)\), \(A \in \mathcal{B}(\mathbb{R})\), we get the representation of \(\mu\) in the form \(d\mu(x) = w(x) \, dw(x)\), and due to \((2.7)\) it only remains for us to prove that \(\mathcal{P}[\mathbb{C}]\) is dense in the space \(C_{w, \Phi}(\mathbb{R})\).

Let \(S := \mathbb{R} \setminus D, q \in \mathbb{N}, a \vee b := \max\{a, b\}, a, b \in \mathbb{R}\), and \(G\) be any countable set \(G \subset S\) without infinite accumulation points. Definition \((3.3)\) implies \(w(x) = M_w \cdot \chi_S(x), x \in \mathbb{R}\), and taking into account \((1.4)\) we get \(w_n = M_{w_n} \cdot \chi_S\) for every \(n \in \mathbb{N}\). It is possible to find \(K \in \mathbb{N}\) such that \(n_k > q \quad \forall \; k \geq K\) and \(\theta(x) < 1/q\) \(\forall \; |x| \geq K\). Since the set \(G \cap [-K, K]\) is finite, \((3.2)\) implies an existence of \(k_n > K\) such that
\[
\Phi\left(q \cdot |c(x) - P_{n_k}(x)|\right) < \frac{1}{q} , \quad k \geq k_n , \quad x \in G \cap [-K, K] ;
\]
whence for every integer \(k \geq k_n\),
\[
\sup_{x \in G} w(x) \Phi\left(q \cdot |c(x) - P_{n_k}(x)|\right) \leq \sup_{x \in G \cap [-K, K]} w(x) \Phi\left(q \cdot |c(x) - P_{n_k}(x)|\right)
\]
\[
\vee \sup_{|x| > K} \frac{\theta(x)}{\alpha_0(x) + h(x)} \Phi\left(q \cdot |c(x) - P_{n_k}(x)|\right) \leq \frac{1}{q} \vee \sup_{|x| > K} \theta(x) \leq \frac{1}{q} ,
\]
i.e., \(P_{n_k} \in c + W_{\Phi}^{q, w, \chi_S}, k \geq k_n\). By this it is meant that \(c \in C_{w_n}^{0, \chi_S} \mathcal{P}[\mathbb{C}]\), and due to \((1.4), (2.7)\), \(\mathcal{P}[\mathbb{C}]\) is dense in all \(C_{w_n}^{0, \chi_S}, n \in \mathbb{N}\). The application of Theorem B gives the denseness of \(\mathcal{P}[\mathbb{C}]\) in all \(C_{w_n}^{0, \chi_S}, n \in \mathbb{N}\). The theorem is proved.
4. Proof of Corollary 1.1 and Theorem 1.1

Condition (1.3) implies an existence of two finite positive numbers $q$ and $L$ such that $\Phi(\lambda \cdot x) \geq (1 + q)\Phi(x)$, $x \geq L$, and after the change of variables $x = \varphi(y)$ we get $\varphi((1 + q)y) \leq \lambda \cdot \varphi(y)$, $y \geq \Phi(L)$. Successive application of this inequality gives $\varphi((1 + q)^k y) \leq \lambda^k \cdot \varphi(y)$, $y \geq \Phi(L)$, $k \in \mathbb{N}$, and hence, for every $n \in \mathbb{N}$ one can find finite positive constants $C_n, d_n$ such that $\varphi(x) \leq C_n \cdot \varphi(x/n), x \geq d_n \Phi(L)$. Since for arbitrary $d > 0$ the function $\varphi(x)/\varphi(x/n)$ is continuous on $[1/d, d \cdot \Phi(L)]$, it follows that $\varphi(x) \leq C_n(d) \cdot \varphi(x/n), x \geq 1/d, n \in \mathbb{N}$, with finite positive numbers $C_n(d)$. Therefore for $d := ||1||$ the formula (1.4) yields $w_n \leq C_n(d) \cdot w_1$, and also $S_{w_n} = S_w, w_n \leq w_{n+1}$, for any $n \in \mathbb{N}$, so that the denseness of $\mathcal{P}[C]$ in all $C_{w_n}^0, n \in \mathbb{N}$, is equivalent to the denseness of $\mathcal{P}[C]$ in $C_{w_1}$, and the formula $w = 1/\Phi(1/w_1)$ finishes the proof of Corollary 1.1.

Theorem 1.1 follows readily from Corollary 1.1 when $\Phi(x) = x^p$, $p > 0$.

5. Proof of Theorem 1.2

Sufficiency of both assertions of Theorem 1.2 follows easily from Theorem 1.1 with $p = 2$. The necessity of Theorem 1.2 for det $\mathcal{H}$ is an obvious consequence of (1.4). It remains to prove the necessity of Theorem 1.2 for det $\mathcal{S}$.

If $\mu \in \text{det } \mathcal{S}$, then $\mu \in \mathcal{M}^*(\mathbb{R}^+)$, and in any of its representation of type (1.3) we may assume (see [8, Rem. 1, p. 223]) that $S_w, \text{supp } \mu \subset \mathbb{R}^+$. Let $\mathcal{M}^*_w(\mathbb{R}^+)$ denote the set of all $\mu \in \mathcal{M}^*(\mathbb{R}^+)$ such that $L_2(\mathbb{R}, d\mu) \in \mathcal{P}$ and $W^*_\mathcal{P}(\mathbb{R}^+)$ the subset of weights $w \in W^*(\mathbb{R}^+)$ with $C_{\mathcal{P}}^0 \subset \mathcal{P}$. It follows from (1.2) that $\mu^* \in \mathcal{M}^*_w(\mathbb{R}^+)$, where $d\mu^*(x) := (1+x)d\mu(x)$, and so by Theorem 1.1

\begin{equation}
(1+x) d\mu(x) = w(x)^2 d\nu(x), \ \nu \in \mathcal{M}^+(-\mathbb{R}), \ w \in W^*_\mathcal{P}(\mathbb{R}^+), \ \text{supp } \nu \subset \mathbb{R}^+.
\end{equation}

If the weight $w$ is regular (see [14, p. 249]), i.e. $(1+x)^n w \in W^*_\mathcal{P}(\mathbb{R}^+)$ for every $n \in \mathbb{N}_0$, then $(1+x)^n x, x \in \mathbb{R}^+$, implies $\sqrt{x} \cdot w(x) \in W^*_\mathcal{P}(\mathbb{R}^+)$, as requested. Otherwise, $w$ is called singular (see [14, p. 249]), and by Proposition A1.1 in [14, p. 249] there exists $F \in C_0(\mathbb{R})$ such that $S_w = \Lambda_F$, and in (5.1) we can assume that $\text{supp } \mu = \text{supp } \nu = S_w$. Denote $\lambda_0 := 0, \{\lambda_k\}_{k \in \mathbb{N}} := \Lambda_F \setminus \{0\}$, and $\omega_k := \omega(\lambda_k)$, $\sigma_k := \sigma(\{\lambda_k\})$, $k \in \mathbb{N}_0$, for any given $\omega \in W^*(\mathbb{R})$ and $\sigma \in \mathcal{M}^*(\mathbb{R})$. According to (1.2) and Theorem 1.1 we also have

\begin{equation}
x(1+x) d\mu(x) = v(x)^2 d\eta(x), \ \eta \in \mathcal{M}^+(-\mathbb{R}), \ v \in W^*_\mathcal{P}(\mathbb{R}^+),
\end{equation}

where $\text{supp } \eta = S_v = \Lambda_F \setminus \{0\}$. Comparison of (5.1) and (5.2) gives

\begin{equation}
\lambda_k w_k^2 = v_k^2 \eta_k, \ k \in \mathbb{N}.
\end{equation}

We introduce the discrete weight $\Omega \in W^*(\mathbb{R})$ and the discrete measure $\rho \in \mathcal{M}^*(\mathbb{R})$ defined by

\begin{equation}
S_\Omega = \text{supp } \rho \subset \Lambda_F, \ \Omega_0 := w_0, \ \Omega_k := w_k \cdot \sqrt{\frac{v_k}{\nu_k + \eta_k}}, \ \rho_0 := \nu_0, \ \rho_k := \nu_k + \eta_k, \ k \in \mathbb{N}.
\end{equation}

Then, obviously, $(1+x) d\mu(x) = \Omega^2(x)d\rho(x)$ and $\Omega \in W^*_\mathcal{P}(\mathbb{R}^+)$ in view of $\Omega(x) \leq w(x), x \in \mathbb{R}$. In addition, (5.3) yields

\begin{equation}
\lambda_k \Omega_k^2 = \frac{\lambda_k w_k^2}{\nu_k + \eta_k} = \frac{v_k^2}{\nu_k + \eta_k}, \ k \in \mathbb{N},
\end{equation}

i.e. $\sqrt{x} \cdot \Omega(x) \leq v(x)$, whence $\sqrt{x} \cdot \Omega(x) \in W^*_\mathcal{P}(\mathbb{R}^+)$, which was to be proved.
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