THE SERRE DUALITY THEOREM
FOR A NON-COMPACT WEIGHTED CR MANIFOLD

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Abstract. It is proved that the Hodge decomposition and Serre duality hold on a non-compact weighted CR manifold with negligible boundary. A complete CR manifold has negligible boundary. Some examples of complete CR manifolds are presented.

1. Introduction

Let $M$ be a strongly pseudo-convex CR manifold, an s.p.c. CR manifold for short, without boundary. A weighted CR manifold is an s.p.c. CR manifold endowed with a measure $\mu$, which has a smooth positive density $\eta$ with respect to the volume form of the CR structure. Then the space $(M, \mu)$ has a natural weighted Kohn Laplacian $\Box_\mu$, which we call the Witten-Kohn Laplacian.

In this article, we are interested in Serre duality and Hodge decomposition on a non-compact weighted CR manifold. The Serre duality of a compact s.p.c. CR manifold was proved by Tanaka [15] for the case of a trivial line bundle, and recently, the first and the third named authors generalized it to any holomorphic vector bundle $E$ [8]. On the other hand, Kohn’s Hodge decomposition for a compact s.p.c. CR manifold was extended to a general s.p.c. CR manifold with negligible boundary (Definition 2.4) when $E$ is a trivial line bundle by the second author [13].

The aim of the present article is to extend these results to an arbitrary holomorphic vector bundle $E$ over a general weighted CR manifold with negligible boundary, and to relate them to each other. Namely, by denoting $\mathbb{H}^{p,q}(E)$ the space of $E$-valued $L^2$-harmonic forms of $(p, q)$-type, we will show

Main Theorem. Let $M$ be a $(2n - 1)$-dimensional weighted CR manifold with negligible boundary, and let $E$ be a holomorphic vector bundle over $M$. Then the $L^2$-Hodge decomposition

$$L^2(\Omega^{\bullet, q}(E)) = \mathbb{H}^{\bullet, q} \oplus \text{range } (1 - 1) \oplus \text{range } (1 + 1)$$

holds for $0 < q < n - 1$, and the Serre duality

$$\#_\mu : \mathbb{H}^{p, q}(E) \cong \mathbb{H}^{n - p, n - (q + 1)}(E^*)$$

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holds for every $0 \leq p, q \leq n - 1$, where $E^*$ is the dual bundle of $E$ and $\bar{\mu}_\nu$ is the complex-conjugate weighted Hodge star operator. In addition, it follows that

$$\ker \left( \overline{\partial} ^* \right) \bigg/ \text{range} \left( \overline{\partial} ^* \right) \cong H^p,q(E) \cong H^{n-p,n-\left(q+1\right)}(E^*), \text{ for } 0 < q < n - 1.$$ 

We say that $M$ is complete and $M$ is Riemannian complete if it is complete with respect to the Carnot-Carathéodory distance $d_{CC}$ and Riemannian distance $d_\theta$ associated to the CR structure, respectively (see Section 2). Then we have

**Theorem (Proposition 4.1).**

(i) If $M$ is Riemannian complete, then it is complete

(ii) If $M$ is complete, then $M$ has negligible boundary.

Therefore, the Main Theorem is applicable, for example, to the Heisenberg group, Sasakian space forms, spherical orbits, and unbranched covering over a compact s.p.c. CR manifold with any weight. These are very important s.p.c. CR manifolds, but they have been excluded from the literature because of their non-compactness (see Section 2). Two main points in the proof are: identification of the weak solution of the Laplace equation $\square_\alpha = 0$ with the harmonic form (Corollary 2.7); explicit expressions for $\overline{\partial}$ and $\delta_\mu$ in terms of $\overline{\partial}_s$ and $\delta_{\mu_s}$ via $\bar{\mu}_\nu$ (Lemma 3.3).

We organize the article in the following manner: In Section 2, we recall some necessary notions which we will use in the article. Some new results are given, including the self-adjointness of the $E$-valued Witten-Kohn Laplacian. In Section 3, we will prove the Serre duality, and finally, in Section 4, we present the examples.

2. Strongly pseudo-convex CR manifolds

This section contains preliminary results. We recall some definitions related to a strongly pseudo-convex CR manifold $M$, focusing on the construction of the space $\Omega^{p,q}(E)$ of $E$-valued $(p,q)$-forms, its holomorphic structure $\overline{\partial}^*$, and the Witten-Kohn Laplacian $\square_\mu$. For a thorough discussion on a geometrical analysis of an s.p.c. CR manifold, we refer the reader to [5] and [15]. We also establish the essential self-adjointness of $\square_\mu$ and characterize the harmonic forms as the solutions of the Laplace equation with respect to $\square_\mu$. These results are important steps when we extend our knowledge of a compact manifold to a non-compact one.

A $(2n - 1)$-dimensional strongly pseudo-convex CR manifold $M$, we call it simply an s.p.c. CR manifold, is an oriented smooth manifold which carries a structure $(P, J, \theta)$, that is: $P = \ker(\theta) \subset TM$ is an $(n - 1)$-dimensional real subbundle with an almost complex structure $J : P \to P$ satisfying:

$$[X, Y] - J[X, JY] - J[JX, Y] - J[X, JY] \in \Gamma(TM/P), \text{ for } X, Y \in \Gamma(P),$$

and a contact form $\theta \in \Gamma((TM/P)^*)$ whose Levi-form $L(X, Y) = -d\theta(JX, Y)$, for $X, Y \in P$, is positive definite.

Consider the complexification of $J$ and its eigenspace $S = \{X - \sqrt{-1} JX : X \in P\} \subset \mathbb{C}TM$. Then $S \cap \overline{S} = \{0\}$ and $\Gamma(S), \Gamma(S) \subset \Gamma(S)$, where $\overline{S}$ is the complex conjugation of $S$. With the assumption of the strong convexity of $M$, there exist the following implications:

- a Riemannian metric $g = -d\theta + \theta \otimes \theta$;
- a volume form $dv = (n - 1)! \theta \wedge (d\theta)^{n-1};$
• a distance $d_{CC}$ on $M$.

Indeed, since the Levi form $L$ is positive definite, it follows for $0 \neq X \in P_x$ that

$$0 \neq 2L(X, X) = -(JX(\theta(X))) - X(\theta(JX)) - \theta([JX, X]) = \theta([JX, X]).$$

This shows that $[JX, X] \notin P_x$, and hence, $P$ satisfies the Hörmander condition \cite{7}. Due to the Chow theorem \cite{4}, $P$ implies a non-degenerate distance $d_{CC}$ on $M$ defined as

\begin{equation}
(2.1) \quad d_{CC}(x, y) := \sup\{u(x) - u(y) : u \in C^\infty(M), \|\pi\nabla u\|_L^\infty \leq 1\},
\end{equation}

where $\pi : TM \to P$ is the projection with respect to $g$. We say $M$ is complete if the distance space $(M, d_{CC})$ is complete.

**Example 2.2** (E.g. \cite{15}). Let $M$ be a boundary of a strongly pseudoconvex complex manifold and $E$ be a holomorphic vector bundle on the neighbourhood of $M$. Then $E|_M$ is holomorphic in the above sense.

Hereafter, $E$ stands for a holomorphic vector bundle over $M$. Consider the vector bundle $\tilde{T}M = \mathbb{C}TM/\mathcal{S}$, which is holomorphic (e.g. p. 15 in \cite{15}) together with the operator $\overline{\partial} = \overline{\partial}_M$:

$$\overline{\partial}_M u = \varpi(\overline{X}, Z),$$

for $u \in \Gamma(\tilde{T}M)$ with $Z \in \Gamma(\mathbb{C}TM)$ such that $\varpi(Z) = u$ and $X \in \Gamma(S)$. Here $\varpi : \mathbb{C}TM \to \tilde{T}M$ is the canonical projection. The distinguished vector bundle $E \otimes \bigwedge^p(\tilde{T}M)^*$ with $0 \leq p \leq n - 1$ carries a holomorphic structure:

\begin{equation}
(2.2) \quad \overline{\partial} E \otimes \text{id}_{\Lambda^p} + \text{id}_E \otimes \overline{\partial}_{A^p},
\end{equation}

where $\text{id}$ is the identity operator on the indicated space, and $\bigwedge^p = \bigwedge^p \tilde{T}M$. Hereafter we assume additionally that $(E, \overline{\partial}_E)$ is furnished with a smooth Hermitian fiber metric $(\cdot, \cdot)_E$. The bundle which we will study is

$$\Omega^{p,q}(E) = \Omega^{p,q}(M; E) = \Gamma(M; E \otimes \bigwedge^p \tilde{T}M^* \otimes \bigwedge^q \mathcal{S}),$$

with the holomorphic structure $\overline{\partial} : \Omega^{p,q}(E) \to \Omega^{p,q+1}(E)$ defined as

$$(\overline{\partial}^p)\alpha(X_1, \cdots, X_{q+1}) := \sum (-1)^i \overline{\partial}_{X_i} \left( \alpha \begin{pmatrix} \overline{X}_1, & \cdots, \overline{X}_i, & \cdots, \overline{X}_{q+1} \end{pmatrix} \right)$$

$$+ \sum_{i<j} (-1)^{i+j} \alpha \begin{pmatrix} \overline{X}_i, \overline{X}_j, & \overline{X}_1, & \cdots, \hat{X}_i, & \cdots, \hat{X}_j, & \cdots, \overline{X}_{q+1} \end{pmatrix},$$

where $\overline{\partial}$ is the holomorphic structure of $E \otimes \bigwedge^p \tilde{T}M^*$, $\alpha \in \Omega^{p,q}(E)$, and $X_1, \cdots, X_{q+1}$ belong to $\Gamma(S)$. If $E$ is the trivial line bundle, we simply denote $\Omega^{p,q}(M) = \Omega^{p,q}(M; \mathbb{C})$. Set

$$\Omega^{p,q}(E) = \bigoplus_q \Omega^{p,q}(E), \quad \Omega(E) := \bigoplus_q \Omega^{p,q}(E);$$

$$\Omega^p_0(E) = \{ \alpha \in \Omega^{p,q}(E) \mid \alpha \text{ has compact support} \}.$$
Let $\eta$ be the weight, which is a positive smooth function on $M$, and consider the measure $d\mu = \eta dv$. The associated inner product $(\alpha, \beta)$ of $\alpha, \beta \in \Omega^0_{\delta} (E)$ is

$$(\alpha, \beta) = \int_M \langle \alpha, \beta \rangle (x) d\mu (x),$$

where $\langle \alpha, \beta \rangle (x)$ is the pointwise inner product of $\alpha$ and $\beta$ at $x \in M$. Denote by $\| \alpha \|_2$ the norm $\sqrt{\langle \alpha, \alpha \rangle}$, and by $L^2 (\Omega^{p, q} (E)) = L^2 (\Omega^{p, q} (E), \mu)$ the set of square integrable $E$-valued measurable $(p, q)$-forms, which coincides with the completion of $\Omega^{p, q}_0 (E)$ with respect to $\| \cdot \|_2$.

Let $\delta^\mu_\gamma : \Omega^{*, q+1} (E) \to \Omega^{*, q} (E)$ be the formal adjoint of $\overline{\partial}^\mu$ in $L^2 (\Omega (E))$. The Witten-Kohn Laplacian $\square^\mu_\gamma : \Omega^{*, q} (E) \to \Omega^{*, q} (E)$ is defined by

$$(\square^\mu_\gamma \alpha, \alpha) = \langle \overline{\partial}^\mu \delta^\mu_{\gamma-1} \alpha + \delta^\mu_{\gamma+1} \overline{\partial}^\mu, \alpha \rangle.$$ 

In abbreviation, we remove the super index $\gamma$ when the operator is acting on the space of mixed degree forms. The operator $\square^\mu_\gamma$ is called subelliptic (e.g. [5, 15]) if there are positive numbers $\sigma$ and $C_\sigma$ such that

$$\| \alpha \|^2_2 (\sigma) \leq C_\sigma (\| \square^\mu_\gamma \alpha \| + \| \alpha \|_2^2),$$

for every $\alpha \in \Omega^0_\sigma (E)$, where $\| \cdot \|_{(\sigma)}$ is the Sobolev norm of order $\sigma$.

**Proposition 2.3.** The Witten-Kohn Laplacian $\square^\mu_\gamma$ is subelliptic for $0 < q < n - 1$.

**Proof.** By Lemma 3.3 the Kohn Laplacian $\square = \square_{dv}$, which is the Witten-Kohn Laplacian with $\eta = 1$, has the form:

$$(2.3) \quad \square = \sharp^* \overline{\partial} \sharp \delta \sharp + \sharp^* \delta \overline{\partial} \sharp,$$

where $\sharp = \sharp_{dv}$ (see Definition 3.1), $\overline{\partial}$ is the holomorphic structure of $\Omega (E^*)$ and $\delta$ is its adjoint operator in $L^2 (\Omega (E^*), dv)$. Again by Lemma 3.3 and (2.3),

$$\square_\mu \alpha = \sharp^* \overline{\partial}_\mu \sharp \delta_\mu \sharp + \sharp^* \delta_\mu \overline{\partial}_\mu \sharp \alpha + \eta^{-1} \sharp^* \left( \overline{\partial}_\mu \eta \wedge \sharp \delta_\mu \sharp \alpha \right)$$

$$+ \eta \delta_\mu \sharp \overline{\partial}_\mu \sharp \alpha + \eta \delta_\mu \left( \eta^{-1} \sharp^* \left( \overline{\partial}_\mu \eta \wedge \sharp \alpha \right) \right)$$

$$= \square \alpha + \eta^{-1} \sharp^* \left( \overline{\partial}_\mu \eta \wedge \delta \sharp \alpha \right) + \sharp^* \delta_\mu \left( \eta^{-1} \overline{\partial}_\mu \eta \wedge \sharp \alpha \right),$$

for $\alpha \in \Omega (E)$. Therefore, $\square_\mu$ and $\square$ have the same principal symbols. Since $\square$ is subelliptic [10], we can draw this conclusion. \hfill \Box

We consider the following domains:

$$D(\overline{\partial}^\mu) = \{ \alpha \in \Omega^{*, q} : \alpha \text{ and } \overline{\partial}^\mu \alpha \text{ are square integrable} \};$$

$$D(\delta^\mu_\gamma) = \{ \alpha \in \Omega^{*, q+1} : \alpha \text{ and } \delta^\mu_{\gamma-1} \alpha \text{ are square integrable} \};$$

$$D(\square^\mu_\gamma) = \{ \alpha \in D(\overline{\partial}^\mu) \cap D(\delta^\mu_{\gamma+1}) : \overline{\partial}^\mu \alpha \in D(\delta^\mu_{\gamma}) \text{ and } \delta^\mu_{\gamma-1} \alpha \in D(\overline{\partial}^{\mu-1}) \}.$$

We need the following assumption so that $\square_\mu$ is symmetric:

**Definition 2.4.** We say $M$ has negligible boundary if

$$(\overline{\partial} \alpha, \beta) = (\alpha, \delta_\mu \beta), \text{ for every } \alpha \in D(\overline{\partial}) \text{ and } \beta \in D(\delta_\mu).$$

We say $\square^\mu_\gamma$ is essentially self-adjoint if its $L^2$-closure is self-adjoint, and $\square^\mu_\gamma$ is hypoelliptic if, whenever the distribution $\square^\mu_\gamma \alpha$ is smooth, then $\alpha$ is smooth.

It is proved in [11] that

**Lemma 2.5.** A subelliptic operator is hypoelliptic.
The assumption such that $M$ has negligible boundary implies a stronger property to $\square_\mu$:

**Proposition 2.6** (e.g. [13]). If $M$ has negligible boundary, then $\square_\mu^q$ is essentially self-adjoint in $L^2(\Omega^q(\mathcal{E}))$ with $0 < q < n - 1$.

**Outline of the proof.** Set $\alpha_\epsilon = e^{-\square_\mu \epsilon} \alpha$ for $\alpha \in D(\delta_\mu^{-1})$. By Proposition 2.3 and Lemma 2.5, $\alpha_\epsilon$ is smooth for every $\epsilon > 0$ (here we need the assumption: $0 < q < n - 1$). Therefore, since $\delta_\mu^{-1} \alpha_\epsilon = \overline{\partial} \alpha_\epsilon \rightarrow \overline{\partial}^* \alpha_\epsilon$ as $\epsilon \rightarrow 0$, we deduce that $\delta_\mu^{-1} \subset \overline{\partial}^*$.

Since $M$ has negligible boundary, $\delta_\mu^{-1} \subset \overline{\partial}^*$ is self-adjoint. Moreover, it follows that

$$\square_\mu^{-1} = \overline{\partial}^* \delta_\mu^* + \delta_\mu^* \overline{\partial},$$

where the right-hand side is self-adjoint.

We say $\alpha$ is harmonic if $\overline{\partial} \alpha = 0$ and $\delta_\mu \alpha = 0$ in the weak sense. A harmonic form always solves the Laplace equation $\square_\mu \alpha = 0$, but in general, the converse does not need to be true. However, it follows that

**Corollary 2.7.** If $M$ has negligible boundary, then the following conditions are equivalent:

(i) $\overline{\partial} \alpha = 0$ and $\delta_\mu \alpha = 0$ pointwise;

(ii) $\alpha$ is harmonic;

(iii) $\alpha$ solves the Laplace equation;

here, $\alpha \in L^2(C^{\bullet,q}(E))$ and $0 < q < n - 1$.

**Proof.** (i) $\Rightarrow$ (ii) is obvious. (ii) $\Rightarrow$ (iii) If $\alpha$ is harmonic, then $(\alpha, \delta_\mu \beta) = 0$ and $\langle \alpha, \overline{\partial} \gamma \rangle = 0$ for every $\beta \in D(\delta_\mu)$ and $\gamma \in D(\overline{\partial})$. This implies $\langle \alpha, \square_\mu \beta \rangle = 0$ for every $\beta \in D(\square_\mu)$; that is, $\alpha$ is the solution of the Laplace equation.

(iii) $\Rightarrow$ (i) Let $\alpha$ be a solution of the Laplace equation. By Proposition 2.6 there exists a sequence $\alpha_l \in D(\square_\mu)$ such that $\alpha_l \rightharpoonup \alpha$ and $\square_\mu \alpha_l \rightarrow 0$, as $l \rightarrow \infty$.

Due to the fact that $M$ has negligible boundary,

$$\|\overline{\partial} \alpha_l\|^2_2 + \|\delta_\mu \alpha_l\|^2_2 = (\square_\mu \alpha_l, \alpha_l) \rightarrow 0, \text{ as } l \rightarrow \infty.$$  

This shows that $\alpha \in D(\overline{\partial}) \cap D(\delta_\mu)$, and $\overline{\partial} \alpha = \delta_\mu \alpha = 0 \mu$-a.e. Due to the hypoellipticity of $\square_\mu$, $\alpha$ is smooth, and hence, $\overline{\partial} \alpha = \delta_\mu \alpha = 0$ pointwise.

A consequence of the celebrated Kohn’s harmonic theory [10] is the Hodge decomposition of a vector bundle over a compact s.p.c. CR manifold. The corresponding result on a non-compact manifold, which is a consequence of Corollary 2.7 is the $L^2$-Hodge decomposition in the Main Theorem. Since the proof is similar to the case where $\eta \equiv 1$ and $E$ is trivial (e.g. [13]), we will omit the proof here.

### 3. Serre Duality

In this section, we study Serre duality and complete the proof of the Main Theorem. Our method is to relate the operators on $E$ to those on $E^*$ via the weighted complex-conjugate Hodge star operator $\ast_\mu$ (see e.g. [8], [6]). Together with results from the previous section, we obtain the Main Theorem.
We start from the construction of $\sharp_\mu$. Let $\ast : \bigwedge^k T^*M \to \bigwedge^{2n-1-k} T^*M$ be the Hodge star operator of $M$ with respect to $g$, which is uniquely determined by $g(\ast \alpha, \beta) dv = (n-1)! \alpha \wedge \beta$, for $\alpha \in \bigwedge^k T^*M$ and $\beta \in \bigwedge^{2n-1-k} T^*M$. $\ast$ is isometric and involutive, i.e. $g(\ast \alpha, \ast \beta) = g(\alpha, \beta)$ and $\ast^2 = id$, because $M$ is odd-dimensional. As the complexification of $\ast$ exchanges the set of holomorphic forms and the set of anti-holomorphic forms, the linear map $\sharp = \sharp_M := - \circ \ast$ satisfies (e.g. Lemma 7.1 [E5]):

\begin{equation}
\sharp (\Omega^{p,q}(M)) = \Omega^{n-p,n-(q+1)}(M).
\end{equation}

We extend (3.1) to

**Definition 3.1.** Define

$$\sharp_\mu : \Omega^{p,q}(E) \to \Omega^{n-p,n-(q+1)}(E^*)$$

by

$$\sharp_\mu \alpha := \sum_{1 \leq i,j \leq r} \eta_{ij} (\sharp \alpha^i) \otimes s^j,$$

for $\alpha \in \Omega^{p,q}(E)$,

where $\alpha = \sum \alpha^i \otimes s_i$, $\{s_i\}_{1 \leq i \leq r}$ is a local frame of $E$, $\{s^j\}$ is its dual frame of $E^*$, and $a_{ij} = (s_i, s_j)_E$. Moreover, define $\sharp_\mu^* : \Omega^{n-p,n-(q+1)}(E^*) \to \Omega^{p,q}(E)$ by

$$\sharp_\mu^* \phi := \sum \eta^{-1} \bar{\alpha}^j (\sharp \phi_j) \otimes s_i,$$

where $\phi = \sum \phi_j \otimes s^j$ and $\alpha^j = (s^j, s^i)_E$, which is the entry of the inverse matrix of $(a_{ij})$. Here $E^*$ is furnished with the Hermitian fiber metric induced from $E$:

$$\langle \phi, \psi \rangle_E = \langle \phi, \psi \rangle_{(E^*, d\mu^\ast)} = \frac{1}{(n-1)!} \int \sum \phi_i \overline{\psi}_j a^{ij} d\mu^\ast,$$

where $\phi = \sum \phi_j \otimes s^j$, $\psi = \sum \psi_i \otimes s^i$, and $d\mu^\ast = \eta^{-1} dv$. For $\alpha \in \Omega^{p,q}(E)$ and $\phi \in \Omega^{p,q}(E^*)$, the product $\alpha \wedge \phi$ is defined by

$$\alpha \wedge \phi := \alpha^i \wedge \phi_i \in \Omega^{p+s,q+t}(M),$$

where $\alpha = \sum \alpha^i \otimes s_i$ and $\phi = \sum \phi_j \otimes s^j$. The definition is well defined; i.e. it is independent of the choice of the frames. Similar to the Hodge star operator, it follows that

\begin{equation}
(\alpha, \beta)_E = \frac{1}{(n-1)!} \int \alpha \wedge \sharp_\mu \beta, \quad \text{for } \alpha, \beta \in \Omega(E),
\end{equation}

and

\begin{equation}
(\phi, \psi)_E = \frac{1}{(n-1)!} \int \phi \wedge \sharp_\mu^* \psi, \quad \text{for } \phi, \psi \in \Omega(E^*).
\end{equation}

The operators $\sharp_\mu$ and $\sharp_\mu^*$ satisfy the following properties.

**Proposition 3.2.** It follows that

(i) $\sharp_\mu \sharp_\mu = id_{\Omega(E)}$ and $\sharp_\mu^* \sharp_\mu^* = id_{\Omega(E^*)}$;

(ii) $(\alpha, \sharp_\mu^* \phi)_E = (\sharp_\mu \alpha, \phi)_{E^*}$, for every $\alpha \in \Omega(E)$ and $\phi \in \Omega(E^*)$.

**Proof.** (i) $\sharp_\mu^* \sharp_\mu \alpha = \sharp_\mu^* \left( \overline{\eta_{ij}} \alpha^i \otimes s^j \right) = \overline{\alpha}^{kj} \sharp \left( a_{ji} \ast \overline{\alpha}^i \right) \otimes s_k = \alpha$.

(ii) $(\alpha, \sharp_\mu \phi)_E = (\alpha, \sharp_\mu \phi)_{(E, dv)} = (\sharp_\mu \alpha, \phi)_{(E^*, dv)} = (\sharp_\mu \alpha, \phi)_{E^*}. \qed$

We denote by $\overline{\partial}$ and $\delta_\mu^*$ the holomorphic structure of $L^2(\Omega(E^*))$ and its formal adjoint, respectively.
Lemma 3.3. It follows that
\begin{equation}
(3.4) \quad (\alpha, \varphi_{\mu}^* \overline{\partial} s_{\mu} \beta)_{E} = (\overline{\partial} \alpha, (-1)^{p+q+1} \beta)_{E},
\end{equation}
for \( \alpha \in \Omega^{p,q}(E) \) and \( \beta \in \Omega^{p,q+1}_{0}(E) \), and
\begin{equation}
(3.3) \quad (-1)^{(p+q+1)} (\psi, \delta_{\mu} \phi)_{E} = (\psi, \varphi_{\mu}^* \overline{\partial} s_{\mu} \phi)_{E},
\end{equation}
for \( \phi \in \Omega^{p,q}(E^*) \) and \( \psi \in \Omega^{p,q-1}_{0}(E^*) \).

Proof. Recall that the holomorphic structure \( \overline{\partial} \lambda_{\cap} \) coincides with the tangential Cauchy-Riemann operator \((-1)^{n}d^\prime\prime\) (e.g. Proposition 1.1 in [13]) defined as:
\[ d^\prime f := df|_{\overline{\partial}} \text{ for } f \in C(M). \]
Therefore, since \( \varphi_{\mu}^* \beta \in \Omega^{n-p,n-(q+2)}(E^*) \) and \( \alpha \wedge \varphi_{\mu}^* \beta \in \Omega^{n,n-2}(M), \)
\[ 0 = (-1)^{n} \int d(\alpha \wedge \varphi_{\mu}^* \beta) = (-1)^{n} \int d^\prime(\alpha \wedge \varphi_{\mu}^* \beta) \]
\[ = \int \overline{\partial} \lambda_{\cap} (\alpha \wedge \varphi_{\mu}^* \beta) \]
\[ = \int (\overline{\partial} \alpha \wedge \varphi_{\mu}^* \beta + (-1)^{p+q} \alpha \wedge \overline{\partial} s_{\mu} \beta) \]
\[ = (\overline{\partial} \alpha, \beta)_{E} + (\alpha, (-1)^{p+q} \varphi_{\mu}^* \overline{\partial} s_{\mu} \beta)_{E}. \]
We used (3.3) and Proposition 3.2 for the last step. We have (3.4).

Next, by taking into account that \( \int d(\phi \wedge \varphi_{\mu}^* \psi) = 0, \) we can prove (3.4) in a similar way. \( \square \)

We are now in a position to show:

Theorem 3.4 (Serre duality). It follows for every \( 0 \leq p, q \leq n-1 \) that
\[ \varphi_{\mu}^* : \mathbb{H}^{p,q}(E) \cong \mathbb{H}^{n-p,n-(q+1)}(E^*). \]

Proof. Take \( \alpha \in \mathbb{H}^{p,q}(E) \). Then \( \alpha \in D(\overline{\partial}) \cap D(\delta), \overline{\partial} \alpha = 0 \) and \( \delta_{\mu} \alpha = 0. \) By Proposition 3.2 and (3.4), for \( \beta \in \Omega_{0}(E) \) we have that
\[ \left( \delta_{\mu} \alpha, \varphi_{\mu}^* \beta \right)_{E} = \left( \varphi_{\mu}^* \alpha, \overline{\partial} s_{\mu} \beta \right)_{E} = \left( \alpha, \varphi_{\mu}^* \overline{\partial} s_{\mu} \beta \right)_{E} = (\overline{\partial} \alpha, (-1)^{p+q} \beta)_{E} = 0. \]
Since \( \{ \varphi_{\mu}^* \beta : \beta \in \Omega_{0}(E) \} \) is dense in \( L^{2}(E^*) \) by Proposition 3.2 we have \( \delta_{\mu} \varphi_{\mu}^* \alpha = 0. \) On the other hand, for \( \phi \in \Omega_{0}(E^*) \), we have by Proposition 3.2 and (3.3) that
\[ (-1)^{p+q+1} \left( \overline{\partial} \varphi_{\mu}^* \alpha, \phi \right)_{E} = (-1)^{p+q+1} \left( \varphi_{\mu}^* \alpha, \delta_{\mu} \phi \right)_{E} = \left( \varphi_{\mu}^* \alpha, \overline{\partial} s_{\mu} \phi \right)_{E} = \left( \alpha, \overline{\partial} s_{\mu} \phi \right)_{E} = \left( \delta_{\mu} \alpha, \varphi_{\mu}^* \phi \right)_{E} = 0. \]
Therefore, \( \overline{\partial} \varphi_{\mu}^* \alpha = 0 \), and we deduce that \( \varphi_{\mu}^* \alpha \in \mathbb{H}^{n-p,n-(q+1)}(E^*). \)

The reverse implication can be shown by running the argumentation above from the bottom to the top. Now we obtain
\[ \varphi_{\mu}^* : \mathbb{H}^{p,q}(E) \cong \mathbb{H}^{n-p,n-(q+1)}(E^*), \]
where \( \varphi_{\mu}^* \) is a complex conjugate linear isomorphism. \( \square \)
4. Examples

In this section, we present examples of non-compact s.p.c. CR manifolds with negligible boundary. Our argumentation relies on the following proposition.

**Proposition 4.1.**

(i) If $M$ is Riemannian complete, then it is complete with respect to $d_{CC}$.

(ii) If $M$ is complete, then $M$ has negligible boundary.

**Proof.** (i) Assume that $M$ is Riemannian complete, and let $\{x_n\}_{n>0} \subset M$ be a Cauchy sequence with respect to $d_{CC}$. Since $d_g$ has the alternative expression \[ d_g(x, y) = \sup \{ u(x) - u(y) : u \in C^\infty(M), \| \nabla u \|_{L^\infty} \leq 1 \}, \]

it follows that $d_g \leq d_{CC}$ by Equation (2.1). Hence, $\{x_n\}_{n>0}$ is a Cauchy sequence with respect to $d_g$, and thus, the limit belongs to $M$, by the assumption. Since the topologies generated by $d_g$ and $d_{CC}$ are the same, we have the assertion.

(ii) Due to the fact that $M$ is complete, there exists a sequence $\{\chi_l\}_{l>0}$ of smooth functions with compact support such that $0 \leq \chi_l \leq 1$, $\chi_l \to 1$, and $\partial \chi_l \to 0$ as $l \to \infty$ (\cite{1}, \cite{2}). For $\alpha \in D(\partial)$, set $\alpha_l = \chi_l \alpha$. Since $\alpha_l$ has compact support for every $l > 0$,

\[ (\partial \alpha_l, \beta) = (\alpha_l, \delta \beta) \to (\alpha, \delta \beta), \]

as $l \to \infty$ for every $\beta \in D(\delta)$, where the left-hand side tends to $(\partial \alpha, \beta)$ as $l \to \infty$. \qed

The most fundamental example is

**Example 4.2 (Heisenberg group).** The Heisenberg group $H(n)$ is

\[ H(n) = \{(w, z) \in \mathbb{C}^n \times \mathbb{C} : \text{Im} z = \|w\|^2\} \]

with the group structure

\[ (w, z) \circ (w', z') = (w + w', z + z' + 2\sqrt{-1}w \cdot w'). \]

It is a quadratic CR submanifold (see e.g. \cite{2}), whose defining function is

\[ f(w, z) = \text{Im} z - \|w\|^2. \]

Consider the following CR manifold which is CR-equivalent to $H(n) : \mathbb{C}^n \times \mathbb{R}$ with the contact form

\[ \theta = dt + 2 \sum (x_i dy_i - y_i dx_i). \]

Then since the orthonormal frame $\{X_i, Y_i\}$ of $P$ and the characteristic direction $\xi$ are given by

\[ X_i = \frac{1}{2} \frac{\partial}{\partial x_i} - y_i \frac{\partial}{\partial t}, \quad Y_i = \frac{1}{2} \frac{\partial}{\partial y_i} + x_i \frac{\partial}{\partial t}, \quad \text{and} \quad \xi = \frac{\partial}{\partial t}, \]

no geodesics with respect to $g = -d\theta + \theta \otimes \theta$ reach $\infty$ in finite time. Due to the Hopf-Rinow theorem, $H(n)$ is Riemannian complete, and we conclude by Proposition 5.1 that $H(n)$ is complete. Moreover, it was proved in \cite{13} that $\mathbb{H}^{p,q}(E) = 0$ for $0 < q < n - 1$ when $E$ is the trivial bundle over $H(n)$.

**Remark 4.3 (\cite{13}).** If $M$ has negligible boundary and additionally either

1. the Ricci operator is positive on $\Omega^{p,q}$ or

2. the Ricci operator is non-negative on $\Omega^{p,q}$ and $M$ has infinite volume,

then $\mathbb{H}^{p,q}(E) = 0$ for $0 < q < n - 1$ when $E$ is the trivial bundle over $M$. 

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Example 4.4 (Sasakian space forms). There exist exactly three types of Riemannian complete simply connected Sasakian space forms: $S^{2n+1}$, $\mathbb{R}^{2n+1}$, and $D^n \times \mathbb{R}$, where $D \subset \mathbb{C}^n$ is a simply connected bounded domain with Kähler form $dw$ (e.g. [3]). The latter two space forms $\mathbb{R}^{2n+1}$ and $D^n \times \mathbb{R}$ are non-compact and they have the contact form $dt - \sum_i y_idx_i$ and $\omega + dt$, respectively. They have negligible boundary.

Example 4.5 (Spherical orbits). Let $O$ be the orbit of an $n^2$-dimensional automorphism in an $n$-dimensional non-homogeneous hyperbolic manifold. If $O$ is spherical, i.e. each point of $O$ has a neighbourhood which is CR-equivalent to an open set of $S^{2n-1}$, then $O$ is CR-equivalent to one of the following hypersurfaces (e.g. [4]):

1. A lens space $S^{2n-1}/\mathbb{Z}_m$;
2. $\sigma = \{(w, z) \in \mathbb{C}^{n-1} \times \mathbb{C} : \text{Re}z = \|w\|^2\}$;
3. $\sigma' = \{(w, z) \in \mathbb{C}^{n-1} \times \mathbb{C} : |z| = \exp \|w\|^2\}$;
4. $\omega = \{(w, z) \in \mathbb{C}^{n-1} \times \mathbb{C} : \|w\|^2 + \exp(\text{Re}z) = 1\}$;
5. $\omega_\alpha = \{(w, z) \in \mathbb{C}^{n-1} \times \mathbb{C} : \|w\|^2 + |z|^\alpha = 1, z \neq 0\}$, for some $\alpha > 0$.

We are interested in $\sigma$ and $\sigma'$ because they are non-compact. First we show that $\sigma$ is complete. Consider the map $\Phi : \mathcal{H}(n) \rightarrow \sigma$ defined as

$\Phi(w, z) := (w, -\sqrt{-1}z)$.

Clearly, $\Phi$ preserves the holomorphic structure. Moreover, since

$f_\Phi(w, z) = f_H(w, z),

$where $f_\Phi = \|w\|^2$ is the defining function of $\sigma$, the contact structure is also preserved. Since $\mathcal{H}(n)$ is complete, so is $\sigma$.

Next we proceed to show that $\sigma'$ is complete. The differential $df_{\sigma'}$ of the $\sigma'$’s defining function $f_{\sigma'} = |z|^2 - \exp \|w\|^2$ is

$df_{\sigma'} = \frac{1}{2} \left( \frac{\overline{z}}{|z|} dz + \frac{z}{|z|} d\overline{z} \right) - \exp \|w\|^2 \left( \sum (\overline{w}_i dw_i + w_i d\overline{w}_i) \right),$

and its pull-back $\Psi^*df_{\sigma'}$ by the covering map $\Psi : \sigma \rightarrow \sigma'$, defined as $\Psi(w, z) = (w, \exp z)$, is

$\frac{1}{2} \exp(\text{Re}z)^{-1} \left( \exp \sum w^* dz + \exp z \Psi^* d\overline{z} \right) - \exp \|w\|^2 \left( \sum (\overline{w}_i dw_i + w_i d\overline{w}_i) \right)

= \frac{1}{2} \exp(\text{Re}z)^{-1} \exp(2\text{Re}z)(dz + d\overline{z}) - \exp \|w\|^2 \left( \sum (\overline{w}_i dw_i + w_i d\overline{w}_i) \right)

= \exp \|w\|^2 d\theta_{\sigma'},$

where we have used the fact that $\text{Re}z = \|w\|^2$ for the last step. This shows that $\ker(\theta_{\sigma'}) = \ker(\theta_{\sigma'})$ via $\Psi$. Moreover, since

$\Psi^*d\theta_{\sigma'} = d\exp \|w\|^2 \wedge \theta_{\sigma'} + \exp \|w\|^2 d\theta_{\sigma'},$

where the first term on the right-hand side vanishes on $P$ and $\exp \|w\|^2 \geq 1$, it follows that the distance associated to $\Psi^*d\theta_{\sigma'}$ is not less than the one associated to $d\theta_{\sigma'}$.

Thus, since these two distances generate the same topology and $\sigma$ is complete with respect to $d\theta_{\sigma}$, we may conclude that the distance associated to $\Psi^*d\theta_{\sigma'}$ is complete by the same reason as in the proof of Proposition [11].

We can also show that $\omega$ is complete, where the proof will appear in a forthcoming paper.
Example 4.6. If $M$ is a compact s.p.c. CR manifold and $M' \to M$ is an unbranched covering, then $M'$ has negligible boundary.

Remark 4.7. Since the distance structure of $M$ is independent of the choice of the weight, all of those examples have negligible boundary with an arbitrary weight.

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