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RADEMACHER MULTIPLICATOR SPACES EQUAL TO L^∞

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Abstract. Let X be a rearrangement invariant function space on [0,1]. We consider the Rademacher multiplicator space $\Lambda(\mathcal{R},X)$ of measurable functions x such that $x\cdot h \in X$ for every a.e. converging series $h=\sum a_n r_n \in X$, where (r_n) are the Rademacher functions. We characterize the situation when $\Lambda(\mathcal{R},X)=L^\infty$. We also discuss the behaviour of partial sums and tails of Rademacher series in function spaces.

1. Introduction

In this paper we study the behaviour in function spaces of the Rademacher functions, $r_n(t) := \operatorname{sign}\sin(2^n\pi t)$, $t\in[0,1]$, $n\ge 1$. For a rearrangement invariant (r.i.) space X on [0,1], let $\mathcal{R}(X)$ be the closed linear subspace of X given by $\mathcal{R}(X) := \mathcal{R}\cap X$ where $\mathcal{R}$ is the set of all a.e. converging series $\sum a_n r_n$, that is, $(a_n)\in \ell^2$ [13, Theorem V.8.2]. The *Rademacher multiplicator space* of X is the space $\Lambda(\mathcal{R},X)$ of all measurable functions $x\colon[0,1]\to\mathbb{R}$ such that $x\cdot\sum a_n r_n\in X$, for every $\sum a_n r_n \in \mathcal{R}(X)$. It is a Banach function space on [0,1] when endowed with the norm

$$\|x\|_{\Lambda(\mathcal{R},X)} := \sup\Big\{\Big\|x\sum a_n r_n\Big\|_X : \sum a_n r_n\in X, \Big\|\sum a_n r_n\Big\|_X\le 1\Big\}.$$

The space $\Lambda(\mathcal{R},X)$ can be viewed as the space of operators from $\mathcal{R}(X)$ into the whole space X given by multiplication by a measurable function.

The nature of the Rademacher multiplicator space $\Lambda(\mathcal{R},X)$ was first considered in [7], where it was shown that for a broad class of classical r.i. spaces X (including, for example, the Lorentz $L^{p,q}$ spaces and the Orlicz spaces satisfying the Δ' condition globally; see [9] for the definition) the space $\Lambda(\mathcal{R},X)$ is not r.i. [7, Theorem]. The simplest case of the opposite situation is $\Lambda(\mathcal{R},X)=L^\infty$. An example of this situation was exhibited in [7]; namely $X=L_N$, where L_N is the Orlicz space with $N(t)=\exp(t^2)-1$ [7, Example 1, and also Example 3]. The situation $\Lambda(\mathcal{R},X)=L^\infty$ was studied in [8], where it was shown that a sufficient condition is the boundedness on X of a certain quasilinear operator (which implies that $L^\infty\subset X\subsetneq L_N$) [8, Theorem 2]. In particular, this holds for the Lorentz spaces $\Lambda(\varphi)$ with $\varphi(t):=\log_2^{-1/\beta}(2/t)$, for $\beta>2$ [8, Corollary]. In [2], the previous

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results were extended showing that $\Lambda(\mathcal{R},X)=L^\infty$ holds for all r.i. spaces X which are interpolation spaces for the couple (L^∞,L_N) [2, Theorem 1].

An important tool for the study of $\Lambda(\mathcal{R},X)$ is the symmetric kernel $\mathrm{Sym}\,(\mathcal{R},X)$ of $\Lambda(\mathcal{R},X)$ which is the largest r.i. space embedded into $\Lambda(\mathcal{R},X)$. In [4] it was shown that if X is an r.i. space satisfying the Fatou property and $X\supset L_N$, then $\mathrm{Sym}\,(\mathcal{R},X)=X_{\log^{1/2}}$, where $X_{\log^{1/2}}$ is the r.i. space with norm equivalent to $\|x\|_{\log^{1/2}}=\|x^*(t)\log_2^{1/2}(2/t)\|_X$ [4, Corollary 2.11]. The role of the space L_N in these problems is not surprising, since Rodin and Semenov, in a result extending the Khintchine inequality, showed that $\mathcal{R}(X)$ is isomorphic to ℓ^2 if and only if $(L_N)_0\subset X$ ([12]), where, if Z is any r.i. space, Z_0 denotes the closure of L^∞ in Z.

The main aim of this note is to give the following necessary and sufficient condition which guarantees the equality $\Lambda(\mathcal{R},X)=L^\infty$.

Theorem 1. *Let X be an r.i. space on $[0,1]$. Then, $\Lambda(\mathcal{R},X)=L^\infty$ if and only if $\log_2^{1/2}(2/t)\notin X_0$.*

The proof of Theorem 1 is presented in Section 3 after some preliminaries in Section 2. In Section 4 we provide some remarks and examples concerning the behaviour of Rademacher tails and partial sums in r.i. spaces X satisfying $\Lambda(\mathcal{R},X)=L^\infty$.

2. Preliminaries

A rearrangement invariant space X is a Banach space of classes of measurable functions on [0,1] such that if $y^*\le x^*$ and $x\in X$, then $y\in X$ and $\|y\|_X\le\|x\|_X$. Here x^* is the decreasing rearrangement of x, that is, the right continuous inverse of its distribution function $n_x(\lambda):=m\{t\in[0,1]:\,|x(t)|>\lambda\}$, $\lambda>0$, where m is the Lebesgue measure on [0,1]. Functions x and y are said to be equimeasurable if $n_x(\lambda)=n_y(\lambda)$, for all $\lambda>0$. The characteristic function of the set $A\subset[0,1]$ will be denoted by χ_A. The fundamental function of X is the function $\varphi_X(t):=\|\chi_{[0,t]}\|_X$.

Important examples of r.i. spaces are the Lorentz and Orlicz spaces. Let $\varphi\colon[0,1]\to[0,+\infty)$ be an increasing concave function; the Lorentz space $\Lambda(\varphi)$ consists of all measurable functions x on [0,1] such that

$$\|x\|_{\Lambda(\varphi)}=\int_0^1 x^*(s)\,d\varphi(s)\,<\,\infty.$$

Let M be an Orlicz function, that is, an increasing convex function on $[0,\infty)$ with $M(0)=0$. The norm of the Orlicz space L_M is defined as follows:

$$\|x\|_{L_M}=\inf\left\{\lambda>0:\,\int_0^1 M\left(\frac{|x(s)|}{\lambda}\right)ds\,\le\,1\right\}.$$

Given Banach spaces X_0 and X_1 continuously embedded in a common Hausdorff topological vector space, a Banach space X is an interpolation space with respect to the couple (X_0,X_1) if $X_0\cap X_1\subset X\subset X_0+X_1$ and for every linear operator T with $T\colon X_i\to X_i$ continuously, $i=0,1$, we have $T\colon X\to X$. We denote by $\mathcal{I}\,(X_0,X_1)$ the set of all interpolation spaces with respect to (X_0,X_1). The K–functional of $x\in X_0+X_1$ is defined, for $t>0$, as

$$K(t,x;X_0,X_1)=\inf\{\|x_0\|_{X_0}+t\|x_1\|_{X_1}:x=x_0+x_1,x_i\in X_i\}.$$

Throughout this paper $A\asymp B$ means that there exist constants $C>0$ and $c>0$ such that $c\cdot A\le B\le C\cdot A$.

For any undefined notion regarding r.i. spaces and interpolation of linear operators, we refer the reader to the monographs [5], [6] and [10].

3. Proofs

Denote by Σ the set of all dyadic intervals of [0,1], that is, intervals of the form $\Delta = [(k-1)2^{-n}, k2^{-n}]$, where $n = 0, 1, \ldots,\ k = 1, \ldots, 2^n$; in this case we say that Δ has *rank* n.

Lemma 2. *Let X be an r.i. space on $[0,1]$ such that $\log_2^{1/2}(2/t) \notin X_0$. Then there exists a constant $C_1 > 0$, depending only on X, such that for every $\eta \in (0,1]$ there exists $\delta_0 \in (0,\eta)$ such that*

$$\frac{\|\log_2^{1/2}(2/t)\chi_{[\delta,\eta]}\|_X}{\|\log_2^{1/2}(2/t)\chi_{[\delta,1]}\|_X} \geq C_1, \quad \textit{for all } \delta \in (0,\delta_0).$$

Proof. We first show that

$$\varepsilon := \inf_{0<\eta\leq 1} \lim_{\delta\to 0^+} \|\log_2^{1/2}(2/t)\chi_{[\delta,\eta]}\|_X > 0. \tag{1}$$

Indeed, if $\varepsilon = 0$ we may construct a sequence $\{\eta_n\}$ strictly decreasing to zero such that

$$\|\log_2^{1/2}(2/t)\chi_{[\eta_{n+1},\eta_n]}\|_X \leq \frac{1}{2^n}, \quad n = 1, 2, \ldots.$$

Since X is a Banach space, this implies that $\log_2^{1/2}(2/t) \in X$ and also that $\|\log_2^{1/2}(2/t)\chi_{[0,\eta_n]}\|_X \to 0$. Therefore, $\log_2^{1/2}(2/t) \in X_0$. This contradicts our hypothesis, so (1) is established.

Set $\alpha := \lim_{\delta\to 0^+} \|\log_2^{1/2}(2/t)\chi_{[\delta,1]}\|_X$. Suppose $\alpha < \infty$. Then, given $\eta \in (0,1]$ by (1) we have $\|\log_2^{1/2}(2/t)\chi_{[\delta,\eta]}\|_X \geq \varepsilon/2$, for all sufficiently small $\delta > 0$. Hence, for such δ

$$\frac{\|\log_2^{1/2}(2/t)\chi_{[\delta,\eta]}\|_X}{\|\log_2^{1/2}(2/t)\chi_{[\delta,1]}\|_X} \geq \frac{\varepsilon}{2\alpha}.$$

In the case when $\alpha = \infty$, for $0 < \delta < \eta \leq 1$, we have

$$\begin{aligned}\frac{\|\log_2^{1/2}(2/t)\chi_{[\delta,\eta]}\|_X}{\|\log_2^{1/2}(2/t)\chi_{[\delta,1]}\|_X} &\geq \frac{\|\log_2^{1/2}(2/t)\chi_{[\delta,1]}\|_X - \|\log_2^{1/2}(2/t)\chi_{[\eta,1]}\|_X}{\|\log_2^{1/2}(2/t)\chi_{[\delta,1]}\|_X} \\ &= 1 - \frac{\|\log_2^{1/2}(2/t)\chi_{[\eta,1]}\|_X}{\|\log_2^{1/2}(2/t)\chi_{[\delta,1]}\|_X} \\ &\geq \frac{1}{2},\end{aligned}$$

if $\eta \in (0,1]$ is fixed and $\delta > 0$ is sufficiently small. Thus, the result holds. □

Proof of Theorem 1. *Step 1.* We first prove that there exists a constant $C_2 > 0$, depending only on X, such that for every $m \geq 0$ there exists $n_0 \geq 1$ such that if $n \geq n_0$ and Δ is an arbitrary dyadic interval of rank m, we have

$$\frac{\|\chi_\Delta \sum_{m+1}^{m+n} r_i\|_X}{\|\sum_{m+1}^{m+n} r_i\|_X} \geq C_2. \tag{2}$$

Note that since the functions $\chi_\Delta \sum_{m+1}^{m+n} r_i$ and $\chi_{[0,2^{-m}]} \sum_{m+1}^{m+n} r_i$ are equimeasurable, it suffices to prove (2) for $\Delta = [0, 2^{-m}]$.

For arbitrary $m \geq 0, n \geq 1$, let

$$x_{m,n}(t) := \frac{1}{n} \sum_{m+1}^{m+n} r_i(t) \quad \text{and} \quad y_{m,n}(t) := x_{m,n}(t) \cdot \chi_{[0,2^{-m}]}(t).$$

By the definition of the Rademacher functions we have

$$y^*_{m,n}(t) = x^*_{m,n}(2^m t) \cdot \chi_{[0,2^{-m}]}(t), \quad 0 < t \leq 1. \tag{3}$$

Results of Montgomery–Smith, [11], imply that there exist universal constants $\beta \in (0,1]$, $C_3 > 0$ and $C_4 > 0$ such that for an arbitrary Rademacher series $f_b := \sum_1^\infty b_k r_k$, the following inequalities hold:

$$f_b^*(t) \leq C_3 K(\log_2^{1/2}(2/t), b; \ell_1, \ell_2)$$

and

$$f_b^*(\beta t) \geq C_4^{-1} K(\log_2^{1/2}(2/t), b; \ell_1, \ell_2).$$

So, if $t \in (0,1]$, then

$$x^*_{m,n}(t) \leq C_3 K(\log_2^{1/2}(2/t), b_n; \ell_1, \ell_2) \tag{4}$$

and

$$x^*_{m,n}(\beta t) \geq C_4^{-1} K(\log_2^{1/2}(2/t), b_n; \ell_1, \ell_2), \tag{5}$$

where $b_n = \frac{1}{n} \sum_{m+1}^{m+n} e_k$ and $(e_k)_1^\infty$ are the canonical unit vectors in sequence spaces. Using Holmstedt's formula (see, for example, [5, Theorem 5.2.1]) it can be shown that, for b_n as above,

$$K(t, b_n; \ell_1, \ell_2) \asymp \min\{1, \frac{t}{\sqrt{n}}\}, \quad t > 0,$$

with constants independent of $t > 0$ and $n \geq 1$; see [2, Lemma 2]. Therefore, relations (3)–(5) imply that we have

$$x^*_{m,n}(t) \leq C_5 G_n(t), \tag{6}$$

where

$$G_n(t) := \min\{1, \frac{\log_2^{1/2}(2/t)}{\sqrt{n}}\}, \quad 0 < t \leq 1,$$

and

$$y^*_{m,n}(t) \geq C_6^{-1} F_{m,n}(t), \tag{7}$$

where

$$F_{m,n}(t) := \min\{1, \frac{\log_2^{1/2}(2^{1-m}\beta/t)}{\sqrt{n}}\}, \quad 0 < t \leq 2^{-m}\beta.$$

We will prove that there exists a constant $C_7 > 0$ such that for every $m \geq 0$ the inequality

$$\|F_{m,n}\|_X \geq C_7 \|G_n\|_X \tag{8}$$

holds for all sufficiently large $n \in \mathbb{N}$.

For this, we first show that for every $m \geq 0$ we have

$$F_{m,n}(t) \geq \frac{1}{2}\, G_n(t) \quad \text{if } n \geq \frac{4m}{3} + \frac{4}{3}\log_2(1/\beta) \text{ and } 0 < t < 2^{1-4m/3}\beta^{4/3}. \tag{9}$$

In the case $0 < t \le 2^{-m-n+1}\beta$, the last inequality is obvious because $F_{m,n}(t) = G_n(t) = 1$. If $2^{-m-n+1}\beta < t \le 2^{-n+1}$, then (9) turns into the inequality

$$\log^{1/2}(2^{1-m}\beta/t) \ge \frac{1}{2}\sqrt{n},$$

which holds when $n \ge \frac{4m}{3} + \frac{4}{3}\log_2(1/\beta)$. Finally, in the case $t > 2^{-n+1}$ the inequality (9) is equivalent to

$$\log^{1/2}(2^{1-m}\beta/t) \ge \frac{1}{2}\log_2^{1/2}(2/t),$$

which holds if $t < 2^{1-4m/3}\beta^{4/3}$. Thus, (9) is proved and therefore, for $n \ge \frac{4m}{3} + \frac{4}{3}\log_2(1/\beta)$, we have

$$\|F_{m,n}\|_X \ge \frac{1}{2}\|G_n\chi_{[0,c2^{-4m/3}]}\|_X, \tag{10}$$

where $c = 2\beta^{4/3}$.

Taking into account the definition of G_n and (10), we have, for $n \ge \frac{4m}{3} + \frac{4}{3}\log_2(1/\beta)$,

$$\|F_{m,n}\|_X \ge \frac{1}{2\sqrt{n}}\|\log_2^{1/2}(2/t)\chi_{[2^{-n+1},2^{-4m/3}]}\|_X. \tag{11}$$

From Lemma 2, there is a constant $C_1 > 0$ such that

$$\|\log_2^{1/2}(2/t)\chi_{[2^{-n+1},c2^{-4m/3}]}\|_X \ge C_1\|\log_2^{1/2}(2/t)\chi_{[2^{-n+1},1]}\|_X$$

holds for all $n \ge n_1(m)$. This last inequality and (11) imply that, for all $n \ge n_2(m) := \max\{\frac{4m}{3} + \frac{4}{3}\log_2(1/\beta), n_1(m)\}$, we have

$$\|F_{m,n}\|_X \ge \frac{C_1}{2}\|G_n\chi_{[2^{-n+1},1]}\|_X.$$

Combining this with (10), we conclude that (8) holds for all $n \ge n_2(m)$.

From (8), (6) and (7) it follows that for every $m \ge 0$ and $n \ge n_2(m)$, we have

$$\frac{\|\chi_{[0,2^{-m}]}\sum_{m+1}^{m+n} r_i\|_X}{\|\sum_{m+1}^{m+n} r_i\|_X} = \frac{\|y_{m,n}\|_X}{\|x_{m,n}\|_X} \ge \frac{C_6^{-1}\|F_{m,n}\|_X}{C_5\|G_n\|_X} \ge \frac{C_7C_6^{-1}}{C_5} = C_2.$$

Hence, (2) is proved.

Step 2. Let $D \subset [0,1]$ be any measurable set with positive measure. By Lebesgue's density theorem, for sufficiently large $m \in \mathbb{N}$, we can find a dyadic interval $\Delta := \Delta_m^{k_0} = [(k_0-1)2^{-m}, k_0 2^{-m}]$ such that

$$2^{-m} = m(\Delta) \ge m(\Delta \cap D) > 2^{-m-1}.$$

Let us consider the set $E = \bigcup_{k=1}^{2^m} E_m^k$, where E_m^k is obtained by translating the set $\Delta \cap D$ to the interval Δ_m^k, $k = 1, 2, \ldots, 2^m$ (in particular, $E_m^{k_0} = \Delta \cap D$). Denote $f_i = r_i \cdot \chi_E$, $i \in \mathbb{N}$. It follows easily that $|f_i(t)| \le 1$, $t \in [0,1]$, $\|f_i\|_{L_2} \ge 1/\sqrt{2}$, and $f_i \to 0$ weakly in $L_2([0,1])$ when $i \to \infty$. Therefore, by [3, Theorem 5], the sequence $\{f_i\}_{i=1}^\infty$ contains a subsequence $\{f_{i_j}\}$, which is equivalent in distribution to the Rademacher system. The last means that there exists a constant $C>0$ such that

$$\begin{aligned} C^{-1}m\Big\{t\in[0,1]:\Big|\sum_{j=1}^{l}a_jr_j(t)\Big|>Cz\Big\} &\le m\Big\{t\in[0,1]:\Big|\sum_{j=1}^{l}a_jf_{i_j}(t)\Big|>z\Big\}\\ &\le Cm\Big\{t\in[0,1]:\Big|\sum_{j=1}^{l}a_jr_j(t)\Big|>C^{-1}z\Big\}\end{aligned}$$

for all $l\in\mathbb{N}$, $a_j\in\mathbb{R}$, $j=1,2,\dots,l$, and $z>0$. Hence, by the definition of r_j and f_j, for every $n\in\mathbb{N}$ we have

$$\begin{aligned} C^{-1}m\Big\{t\in[0,1]:\Big|\sum_{j=m+1}^{m+n}r_j(t)\chi_{[0,2^{-m}]}(t)\Big|>Cz\Big\}&\\ \le m\Big\{t\in[0,1]:\Big|\sum_{j=m+1}^{m+n}f_{i_j}(t)\chi_{\Delta}(t)\Big|>z\Big\}&\\ \le Cm\Big\{t\in[0,1]:\Big|\sum_{j=m+1}^{m+n}r_j(t)\chi_{[0,2^{-m}]}(t)\Big|>C^{-1}z\Big\},&\end{aligned}$$

whence

$$\Big\|\sum_{j=m+1}^{m+n}r_{i_j}\chi_{\Delta\cap D}\Big\|_X\ge\alpha\Big\|\sum_{j=m+1}^{m+n}r_j\chi_{[0,2^{-m}]}\Big\|_X,$$

where $\alpha>0$ depends only on the constant C and on the space X. Therefore, applying (2) to the dyadic interval $[0,2^{-m}]$, we have that, for large enough n,

$$\Big\|\sum_{j=m+1}^{m+n}r_{i_j}\chi_{\Delta\cap D}\Big\|_X\ge\alpha C_2\Big\|\sum_{j=m+1}^{m+n}r_j\Big\|_X=\alpha C_2\Big\|\sum_{j=m+1}^{m+n}r_{i_j}\Big\|_X.$$

It follows that

$$\|\chi_D\|_{\Lambda(\mathcal{R},X)}\ge\|\chi_{D\cap D}\|_{\Lambda(\mathcal{R},X)}\ge\alpha C_2.$$

Since $\Lambda(\mathcal{R},X)$ is a Banach function space, the above inequality implies that$\Lambda(\mathcal{R},X)$ $\subset L^\infty$. We always have the opposite embedding. Hence, $\Lambda(\mathcal{R},X)=L^\infty$. □

Corollary 3. *If X and Y are r.i. spaces on $[0,1]$ with $X\subset Y$, then $\Lambda(\mathcal{R},X)\subset\Lambda(\mathcal{R},Y)$.*

Proof. If $\log_2^{1/2}(2/t)\notin X_0$, then by Theorem 1 we have $\Lambda(\mathcal{R},X)=L^\infty$, and so $\Lambda(\mathcal{R},X)\subset\Lambda(\mathcal{R},Y)$. Suppose that $\log_2^{1/2}(2/t)\in X_0$. Then, $\log_2^{1/2}(2/t)\in X\subset Y$, and so, by [12], we have $\mathcal{R}(X)\approx\ell^2$ and $\mathcal{R}(Y)\approx\ell^2$. Hence,

$$\begin{aligned}\|x\|_{\Lambda(\mathcal{R},Y)}&\asymp\sup\Big\{\Big\|x\sum a_nr_n\Big\|_Y:\|(a_n)\|_2\le1\Big\}\\ &\le C\cdot\sup\Big\{\Big\|x\sum a_nr_n\Big\|_X:\|(a_n)\|_2\le1\Big\}\\ &\asymp\|x\|_{\Lambda(\mathcal{R},X)}.\end{aligned}$$

□

4. Concluding remarks

Remark 4. A somewhat surprising feature of Theorem 1 is that, together with Theorem 3.2 in [4], it implies that the symmetric kernel $\mathrm{Sym}\,(\mathcal{R},X)$ of $\Lambda(\mathcal{R},X)$ reduces to L^∞ if and only if the multiplicator space $\Lambda(\mathcal{R},X)$ does. That is, the condition $\log_2^{1/2}(2/t)\notin X_0$ guarantees the equivalence

$$\begin{aligned}&\sup\Big\{\Big\|\chi_D\sum a_nr_n\Big\|_X:\Big\|\sum a_nr_n\Big\|_X\le 1\Big\}\\ &\asymp\sup\Big\{\Big\|\chi_{[0,m(D)]}\Big(\sum a_nr_n\Big)^*\Big\|_X:\Big\|\sum a_nr_n\Big\|_X\le 1\Big\}\\ &\asymp 1,\end{aligned}$$

with constants independent of a set $D\subset[0,1]$ with $m(D)>0$. Moreover, the same example as in Remark 3.5 in [4] shows that L_N is not the largest r.i. space satisfying $\Lambda(\mathcal{R},X)=L^\infty$.

Remark 5. The proof of Theorem 1 shows that when $\Lambda(\mathcal{R},X)=L^\infty$ the norm in $\Lambda(\mathcal{R},X)$ of the characteristic function of any dyadic interval of $[0,1]$ is attained (up to equivalence) at an appropriate Rademacher tail sum. More precisely, there exists a constant $M_1>0$, depending only on X, such that, for all $n\ge 1$,

$$\sup\Big\{\Big\|\chi_\Delta\sum_{i=n+1}^\infty a_ir_i\Big\|_X:\Big\|\sum_{i=n+1}^\infty a_ir_i\Big\|_X\le 1\Big\}\ge M_1, \tag{12}$$

where Δ is any dyadic interval of rank n. Moreover, if X is any interpolation space for the couple (L^∞,L_N), the proof of Theorem 1 in [2] indicates that a relation analogous to (12) also holds for Rademacher partial sums; that is, there exists a constant $M_2>0$, depending only on X, such that, for all $n\ge 1$ and any dyadic interval Δ of rank n, we have

$$\sup\Big\{\Big\|\chi_\Delta\sum_{i=1}^n a_ir_i\Big\|_X:\Big\|\sum_{i=1}^n a_ir_i\Big\|_X\le 1\Big\}\ge M_2. \tag{13}$$

Example 6. The situation noted in Remark 5 concerning the behaviour of Rademacher partial sums, (13), does not hold for all spaces X with $\Lambda(\mathcal{R},X)=L^\infty$. Indeed, we present an appropriate space X satisfying $\log_2^{1/2}(2/t)\notin(X)_0$ for which (13) does not hold. Recall the following construction from [1]. Given an r.i. space X on $[0,1]$, consider the sequence space $E=E(X)$ given by the norm

$$\|(a_k)\|_E:=\Big\|\sum a_kr_k\Big\|_X.$$

We always have $E\in\mathcal{I}(\ell^1,\ell^2)$, and since interpolation spaces for the couple (ℓ^1,ℓ^2) are described by the real method, there exists a Banach lattice F of two-sided sequences satisfying $(\min\{1,2^k\})_{-\infty}^\infty\in F$ such that

$$E=(\ell^1,\ell^2)_F^K:=\big\{x:(K(2^k,x;\ell^1,\ell^2))_{-\infty}^\infty\in F\big\}; \tag{14}$$

see [6, Theorems 4.4.5 and 4.4.38]. Consider the r.i. space $Y:=(L^\infty,L_N)_F^K$. Then, $Y\subset X$ and

$$\|\sum a_kr_k\|_Y\asymp\|\sum a_kr_k\|_X,\quad (a_k)\in\ell^2. \tag{15}$$

Moreover, since $Y\in\mathcal{I}(L^\infty,L_N)$, then $Y\ne X$ whenever $X\notin\mathcal{I}(L^\infty,L_N)$.

Let X be the Lorentz space $\Lambda(\varphi_0)$ generated by $\varphi_0(t) := \log_2^{-1/2}(2/t)$ for $0 < t \leq 1$. The fundamental function of Y is

$$\varphi_Y(t) = \|(K(2^k, \chi_{[0,t]}; L^\infty, L_N))_{-\infty}^{\infty}\|_F.$$

Using the formula

$$K(t, x; L^\infty, L_N) \asymp t \sup_{0<s<2^{1-t^2}} x^*(s) \log_2^{-1/2}(2/s), \quad t > 0$$

(see [2]), it can be easily checked that

$$K(2^k, \chi_{[0,t]}; L^\infty, L_N) \asymp \begin{cases} 2^k \log_2^{-1/2}(2/t) & \text{if } k \leq \frac{1}{2}\log_2\log_2(2/t), \\ 1 & \text{if } k > \frac{1}{2}\log_2\log_2(2/t). \end{cases}$$

In order to identify the Banach lattice F in (14) corresponding to $X = \Lambda(\varphi_0)$, using the results from [11] mentioned in the proof of Theorem 1, we have

$$\begin{aligned} \Big\|\sum_{k=1}^{\infty} a_k r_k\Big\|_{\Lambda(\varphi_0)} &= \int_0^1 \Big(\sum_{k=1}^{\infty} a_k r_k\Big)^*(t)\, d\varphi_0(t) \\ &\asymp \int_0^1 K(\log_2^{1/2}(2/t), (a_k); \ell^1, \ell^2)\, d(\log_2^{-1/2}(2/t)) \\ &\asymp \sum_{i=0}^{\infty} K(2^i, (a_k); \ell^1, \ell^2) \cdot 2^{-i}. \end{aligned}$$

Thus, the fundamental function of Y, for $0 < t \leq 1$, is given by

$$\begin{aligned} \varphi_Y(t) &\asymp \sum_{i=0}^{\infty} K(2^i, \chi_{[0,t]}; L^\infty, L_N) \cdot 2^{-i} \\ &\asymp \log_2^{-1/2}(2/t) \sum_{i \leq \frac{1}{2}\log_2\log_2(2/t)} 1 + \sum_{i > \frac{1}{2}\log_2\log_2(2/t)} 2^{-i} \\ &\asymp \log_2^{-1/2}(2/t)\log_2\log_2(2/t). \end{aligned}$$

Since $X = \Lambda(\varphi_0)$ satisfies the condition of Theorem 1, we have $\Lambda(\mathcal{R}, X) = L^\infty$. However, (13) does not hold in this case. Indeed, by (15), for any $n \geq 2$ and any $a_1, a_2, \ldots, a_n$, we have

$$\begin{aligned} \Big\|\chi_{[0,2^{-n}]} \sum_1^n a_i r_i\Big\|_{\Lambda(\varphi_0)} &= \Big|\sum_1^n a_i\Big| \varphi_{\Lambda(\varphi_0)}(2^{-n}) \\ &= \frac{\varphi_0(2^{-n})}{\varphi_Y(2^{-n})} \Big\|\chi_{[0,2^{-n}]} \sum_1^n a_i r_i\Big\|_Y \\ &\leq \frac{C}{\log n} \Big\|\sum_1^n a_i r_i\Big\|_Y \\ &= \frac{C}{\log n} \Big\|\sum_1^n a_i r_i\Big\|_{\Lambda(\varphi_0)}, \end{aligned}$$

whence, for $n \geq 2$,

$$\sup\Big\{\Big\|\chi_{[0,2^{-n}]} \sum_{i=1}^n a_i r_i\Big\|_{\Lambda(\varphi_0)} : \Big\|\sum_{i=1}^n a_i r_i\Big\|_{\Lambda(\varphi_0)} \leq 1\Big\} \leq \frac{C}{\log n}.$$

References

1. S. V. Astashkin, About interpolation of subspaces of rearrangement invariant spaces generated by Rademacher system, *Int. J. Math. Math. Sci.* **25** (2001) 451–465. MR1823608 (2002e:46029)
2. S. V. Astashkin, On the multiplier space generated by the Rademacher system, *Math. Notes* **75** (2004) 158–165. MR2054550 (2004k:46035)
3. S. V. Astashkin, Systems of random variables equivalent in distribution to the Rademacher system and $\mathcal{K}$-closed representability of Banach pairs, *Sb. Math.* **191** (2000) 779–807. MR1777567 (2001g:60034)
4. S. V. Astashkin and G. P. Curbera, Symmetric kernel of Rademacher multiplicator spaces, *J. Funct. Anal.* **226** (2005) 173–192. MR2158179 (2006b:46032)
5. C. Bennett and R. Sharpley, *Interpolation of Operators* (Academic Press, Boston, 1988). MR928802 (89e:46001)
6. Yu. A. Brudnyĭ and N. Ya. Krugljak, *Interpolation Functors and Interpolation Spaces* (North–Holland, Amsterdam, 1991). MR1107298 (93b:46141)
7. G. P. Curbera, A note on function spaces generated by Rademacher series, *Proc. Edinburgh Math. Soc.* **40** (1997) 119–126. MR1437816 (98e:46036)
8. G. P. Curbera and V. A. Rodin, Multiplication operators on the space of Rademacher series in rearrangement invariant spaces, *Math. Proc. Cambridge Phil. Soc.* **134** (2003) 153–162. MR1937800 (2003h:46039)
9. M. A. Krasnosel'skii and Ya. B. Rutickii, *Convex Functions and Orlicz Spaces* (Noordhoff, Gröningen, 1961). MR0126722 (23:A4016)
10. S. G. Kreĭn, Ju. Ī. Petunīn and E. M. Semënov, *Interpolation of Linear Operators* (Amer. Math. Soc., Providence, RI, 1982). MR649411 (84j:46103)
11. S. J. Montgomery–Smith, The distribution of Rademacher sums, *Proc. Amer. Math. Soc.* **109** (1990) 517–522. MR1013975 (91a:60034)
12. V. A. Rodin and E. M. Semenov, Rademacher series in symmetric spaces, *Anal. Math.* **1** (1975) 207–222. MR0388068 (52:8905)
13. A. Zygmund, *Trigomometric series.* Vols. I, II (Cambridge University Press, Cambridge, 1977). MR0617944 (58:29731)

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