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A HOPF ALGEBRA HAVING A SEPARABLE GALOIS EXTENSION IS FINITE DIMENSIONAL

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To José Luis Gómez Pardo on the occasion of his 60th birthday

Abstract. It is shown that a Hopf algebra H over a field admitting a Galois extension A separable over its subalgebra of coinvariants B is of finite dimension. This answers in the affirmative a question posed by Beattie et al. in [*Proc. Amer. Math. Soc.* 128, No. 11 (2000), 3201-3203]. It is also proven that this result holds true if H has bijective antipode and the extension A/B is Frobenius.

Introduction

The notion of Hopf-Galois extension, as known nowadays, is due to Kreimer and Takeuchi [7] and it is a mainstay of Hopf algebra theory. It originated in the work of Chase and Sweedler about actions of Hopf algebras on rings [3]. When the Hopf algebra is the coordinate algebra of an affine group scheme that acts on an affine algebra, the above notion may be interpreted in geometric terms and it is linked with the concept of torsor or principal homogeneous space [10, page 168]. Faithfully flat Hopf-Galois extensions are currently widely accepted as a noncommutative counterpart of this geometric concept.

For $H = k[G]$, the group algebra of a group G over a field k, [8, Theorem 8.1.7] shows that an H-Galois extension is precisely a strongly graded algebra, that is, a k-algebra A admitting a decomposition $A = \bigoplus_{\sigma\in G} A_\sigma$ as k-vector space and satisfying $A_\sigma A_\tau = A_{\sigma\tau}$ for all $\sigma, \tau \in G$. The subalgebra of coinvariants of A is A_e (e the identity element of G). Năstăsescu et al. characterized in [9, Proposition 2.1] when the extension A/A_e is separable. In particular, they found that if A is separable over A_e, then G is finite. It was investigated in [1] if an analogous result could hold for general Hopf algebras. To be more precise, suppose that H is a Hopf algebra over k having a Hopf-Galois extension separable over its subalgebra of coinvariants. Is H necessarily finite dimensional? A positive answer was given under the additional assumption that H is co-Frobenius.

In this short note we answer this question in the affirmative. Our proof relies on a combination of the properties of the separability idempotent, the Galois maps

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and an old result of Sweedler. As often happens in Hopf algebra theory, the new proof seems more natural and simpler than the original proof for $H = k[G]$. Combining our result with one of Cohen and Fischman we provide a characterization of separable Hopf-Galois extensions that generalizes to Hopf algebras the above-mentioned one of Năstăsescu et al. for strongly graded rings. We finally show that if a Hopf algebra H with bijective antipode admits a Hopf-Galois extension A with subalgebra of coinvariants B such that A/B is Frobenius, then H is also finite dimensional.

We fix some notation and recall the definition of Hopf-Galois extension. We expect that the reader is familiar with the rudiments of Hopf algebra theory. Our conventions and notation are those of [8]. Throughout H stands for a Hopf algebra over a field k. All vector spaces considered in the sequel are over k, map means linear map, and $\otimes$ denotes the tensor product over k.

For a right H-comodule algebra A with structure map $\rho : A \to A \otimes H$, its subalgebra of coinvariants $A^{co(H)} = \{a \in A : \rho(a) = a \otimes 1_H\}$ is denoted by B. The canonical Galois maps are given by

$$can : A \otimes_B A \to A \otimes H, \qquad a \otimes_B a' \mapsto \sum_{(a')} aa'_{(0)} \otimes a'_{(1)},$$

$$can' : A \otimes_B A \to A \otimes H, \qquad a \otimes_B a' \mapsto \sum_{(a)} a_{(0)}a' \otimes a_{(1)}.$$

Recall from [8, Definition 8.1.1] that A/B is said to be an *H-Galois extension* if *can* is bijective. If H has bijective antipode, then *can* is surjective or bijective if and only if *can′* is [8, page 124].

1. The main theorem

Theorem. *Let H be a Hopf algebra and A a right H-comodule algebra with subalgebra of coinvariants B. Suppose that $can : A \otimes_B A \to A \otimes H$ is surjective and there is a nonzero element $e = \sum_{i=1}^n e_i \otimes_B e'_i \in A \otimes_B A$ such that $ae = ea$ for all $a \in A$ and $can'(e) \neq 0$. Then H is finite dimensional.*

Proof. We will prove that H has a nonzero finite dimensional left ideal. In virtue of Sweedler's result [11, Corollary 2.7] (see alternatively [5, Lemma 5.3.1(i)]), this will imply that H is finite dimensional.

We pick nonzero elements $a_j \in A, h_j \in H$ for $j = 1, ..., m$ such that

$$\sum_{j=1}^m a_j \otimes h_j = can'(e) = \sum_{i=1}^n \sum_{(e_i)} e_{i(0)}e'_i \otimes e_{i(1)} \tag{1}$$

and the a_j's are linearly independent. Take $h \in H$ arbitrary. Since *can* is surjective we may find $c_l, d_l \in A$ for $l = 1, ..., r$ satisfying

$$1 \otimes h = can(\sum_{l=1}^r c_l \otimes_B d_l) = \sum_{l=1}^r \sum_{(d_l)} c_l d_{l(0)} \otimes d_{l(1)}. \tag{2}$$

For $l = 1, ..., r$ we have $d_l e = ed_l$. Applying *can′* to this equality we get

$$\sum_{i=1}^n \sum_{(e_i)} \sum_{(d_l)} d_{l(0)}e_{i(0)}e'_i \otimes d_{l(1)}e_{i(1)} = \sum_{i=1}^n \sum_{(e_i)} e_{i(0)}e'_i d_l \otimes e_{i(1)}. \tag{3}$$

Then,

$$\begin{aligned}\sum_{j=1}^{m} a_j \otimes hh_j &\overset{(1),(2)}{=} \sum_{l=1}^{r}\sum_{(d_l)}\sum_{i=1}^{n}\sum_{(e_i)} c_l d_{l(0)} e_{i(0)} e_i' \otimes d_{l(1)} e_{i(1)} \\ &\overset{(3)}{=} \sum_{l=1}^{r}\sum_{i=1}^{n}\sum_{(e_i)} c_l e_{i(0)} e_i' d_l \otimes e_{i(1)} \\ &\overset{(1)}{=} \sum_{l=1}^{r}\sum_{j=1}^{m} c_l a_j d_l \otimes h_j.\end{aligned}$$

Let $\varphi_t \in A^*$ be such that $\varphi_t(a_j) = \delta_{tj}$, the Kronecker symbol, for $t = 1, ..., m$. Evaluating $\varphi_t \otimes id_H$ on the preceding set of equalities we obtain

$$hh_t = \sum_{l=1}^{r}\sum_{j=1}^{m} \varphi_t(c_l a_j d_l) h_j.$$

This yields that the subspace spanned by the h_j's is a finite dimensional nonzero left ideal of H, as required. □

Assume that A/B is separable and let $e = \sum_{i=1}^{n} e_i \otimes_B e_i' \in A \otimes_B A$ be the separability idempotent. Then $\sum_{i=1}^{n} e_i e_i' = 1_A$ and $ae = ea$ for all $a \in A$. Notice that $can'(e)$ is nonzero since

$$1_A = \sum_{i=1}^{n} e_i e_i' = \sum_{i=1}^{n}\sum_{(e_i)} e_{i(0)} \varepsilon(e_{i(1)}) e_i' = (id_A \otimes \varepsilon) can'(e),$$

where ε denotes as usual the counit of H. Applying the above theorem we get the announced result.

Corollary 1. *Let H be a Hopf algebra and A a right H-Galois extension separable over its subalgebra of coinvariants B. Then H is finite dimensional.*

Cohen and Fischman provided in [4, Theorem 1.8] several characterizations of separable Hopf-Galois extensions for a finite dimensional Hopf algebra. These characterizations together with our result allow us to characterize such extensions for an arbitrary Hopf algebra.

Corollary 2. *Let H be a Hopf algebra and A a right H-Galois extension. Then, $A/A^{co(H)}$ is separable if and only if H is finite dimensional and one of the equivalent conditions (2)-(6) in* [4, Theorem 1.8] *holds.*

This corollary may be viewed as a generalization to Hopf algebras of [9, Proposition 2.1] characterizing strongly graded rings separable over the degree one component. Under the assumption that H has bijective antipode, Corollary 1 holds true for Frobenius extensions. We would like to thank the referee for pointing out this fact.

Corollary 3. *Let H be a Hopf algebra having bijective antipode and A a right H-Galois extension with subalgebra of coinvariants B. If A/B is Frobenius, then H is finite dimensional.*

Proof. The hypothesis on A/B assures the existence of a nonzero element $e \in A \otimes_B A$ such that $ae = ea$ for all $a \in A$, [6, Remarks 1.2(a) and 1.4(c)]. Since H has bijective antipode, can' is bijective. Our theorem now applies. □

When B is a field and A a B-algebra, A/B separable implies A/B is finite dimensional and semisimple, which in turn implies A/B is Frobenius [2, Section 1.3].

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