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A ROUGH DIFFERENTIABLE FUNCTION

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ABSTRACT. A real-valued continuously differentiable function f on the unit interval is constructed such that

$$\sum_{k=1}^{\infty} \beta_f(x, 2^{-k}) = \infty$$

holds for every $x \in [0,1]$. Here $\beta_f(x, 2^{-k})$ measures the distance of f to the best approximating linear function at scale 2^{-k} around x.

1. INTRODUCTION

A fundamental theorem of H. Lebesgue states that a Lipschitz function $f : [0,1] \to \mathbb{R}$ is differentiable at almost every point. Approximation by linear functions at the scale r is measured by

$$(1) \qquad \beta_f(x,r) = \inf_{a,b\in\mathbb{R}} \sup\{\frac{1}{r}|f(y) - (ay+b)| : y \in (x-r, x+r) \cap \operatorname{dom}(f)\}.$$

Thus Lebesgue's theorem implies that

$$\lim_{r\to 0} \beta_f(x,r) = 0,$$

for almost every $x \in [0,1]$. This conclusion was profoundly strengthened by C. Bishop and P. Jones, who proved in [J, B-J] that

$$\sum_{k=1}^{\infty} \beta_f^2(x, 2^{-k}) < \infty,$$

for almost every $x \in [0,1]$. They also demonstrated that this result is optimal within the class of estimates that hold almost everywhere. This however does not rule out the possibility that a better estimate holds on a small subset of $[0,1]$.

In particular, the question remains open whether for each Lipschitz function $f : [0,1] \to \mathbb{R}$ the estimate

$$(2) \qquad \sum_{k=1}^{\infty} \beta_f^p(x, 2^{-k}) < \infty, \qquad 1 \le p < 2,$$

holds in *at least one point* $x \in [0,1]$. Clearly, the smaller the value of p the harder it is to find a point $x \in [0,1]$ for which (2) holds, or equivalently the easier it is to find a counterexample.

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In this paper we focus on $p = 1$ for which the problem is linked to the ongoing efforts to provide some geometric understanding for J. Bourgain's results ([B-1], [B-2]), asserting that there exist points $x \in [0,1]$, at which a given bounded harmonic function has finite radial variation. That is, when u is bounded and harmonic in the unit disk, then there has to exist a point $x \in [0,1]$ such that

$$\int_0^1 |\nabla u(re^{2\pi i x})| dr < \infty.$$

The link between (2) (with $p = 1$) and J. Bourgain's results is P.W. Jones' estimate [J1] that

$$\int_0^1 |\nabla u(re^{2\pi i x})| dr \leq C \sum_{k=1}^{\infty} \beta_f(x, 2^{-k}),$$

where f is the Lipschitz function obtained by integrating the boundary values of u; in other words

$$f(x) = \int_0^x u(e^{2\pi i y}) dy.$$

In the paper presented here we exhibit a Lipschitz function, in fact even a continuously differentiable $f : [0,1] \to \mathbb{R}$ for which $\sum_{k=1}^{\infty} \beta_f(x, 2^{-k}) = \infty$ at *every* $x \in [0,1]$. The feedback to the result on radial variation is the clarification that Bourgain's proof does not find points where the function f is particularly flat, but rather it establishes for a general *Lipschitz* function f the existence of points around which f is remarkably symmetric. Indeed, by our example points where f is flat might not exist; by Bourgain's results (see [B-1], [B-2]) points of symmetry do exist, and they even form a set of Hausdorff dimension one.

The estimate (2) with $p = 1$ is also closely related to the original result about differentiability. In fact, it is the only condition in terms of the decay of $\beta_f(x, \cdot)$'s that implies differentiability at the point x. This can be seen from the following:

Remark. i) Given an arbitrary function f and a point x from the interior of its domain, then the condition (2) implies that f is differentiable at x. This is a consequence of the estimate

$$\left| \frac{f(y) - f(z)}{y - z} - \frac{f(y') - f(z')}{y' - z'} \right| \leq 16\, \beta_f(x, R)$$

if $0 < \frac{R}{4} < (y - z), (y' - z')$ and $y, y', z, z' \in [x - R, x + R]$, which is in turn implied by

$$\left| \frac{f(y) - f(z)}{y - z} - a_R \right| \leq 8\, \beta_f(x, R)$$

where a_R is an optimal slope in definition (1) of $\beta_f(x, R)$. (Indeed, an easy argument shows that the infinimum in (1) is attained and the absolute value of the corresponding a does not exceed the Lipschitz constant of f.)

ii) If $\beta_{k+1} \in (0, 2\,\beta_k)$, $|s_k - s_{k+1}| \leq \beta_k$ and $\sup_k |s_k| < \infty$, then the Lipschitz function f which satisfies $f(x) = s_k x$ for $|x| = 2^{-k}$ and is affine on any of the intervals $\pm[2^{-k-1}, 2^{-k}]$ also fulfills $\beta_f(0, 2^{-k}) \leq \beta_k$ for all k.

2. The construction of a rough C^1-function

The main result of the paper is this:

Theorem. *We consider the Banach space $C^1([0,1])$ of all continuously differentiable functions $f : [0,1] \to \mathbb{R}$ with the norm $\|f\|_{C^1} = \|f\|_\infty + \|f'\|_\infty$. Then for a residual set Y in this space we have that*

$$\sum_{k=1}^{\infty} \beta_f(x, 2^{-k}) = \infty \text{ for all } x \in [0,1] \text{ and all } f \in Y.$$

We deduce this theorem from a slightly weaker but more uniform existence result for an auxiliary Lipschitz function. It is stated in Proposition 3 below and has the crucial feature that only a finite number of scales is involved.

Before entering the proofs leading to the theorem, we would like to point out that around 1980 M. Talagrand showed how to construct an interesting collection $\mathcal{E}$ of pairwise disjoint intervals in $(0,1)$. These cover a set E of arbitrarily small measure and for every $x \in (0,1) \setminus E$ there exists a sequence of intervals $I_n \in \mathcal{E}$ satisfying

$$\sum_n \frac{|I_n|}{|I_n| + \operatorname{dist}(x, I_n)} \geq K \gg 1.$$

Our proof below relies on Talagrand's method of construction. It seems that the only published account of it is in [J-M-T]. In this paper we employ a version of Talagrand's example, kindly suggested to us by the referee, that simplifies the presentation given in [J-M-T].

In the sequel, the term interval refers to a nontrivial and bounded subinterval of the real line. We write $|M|$ for the Lebesgue measure of a set M and, given a collection of intervals $\mathcal{A}$, we use $\mathcal{A}^*$ to denote the set $\bigcup_{I \in \mathcal{A}} I$.

Lemma 1. *Let I be an interval. Then for every $K > 1$ there is a finite collection $\mathcal{E}$ of pairwise disjoint intervals in I such that $E = \mathcal{E}^*$ has measure*

$$|E| < |I|/K$$

and for each $x \in I \setminus E$ there is a sequence $\{I_n^x\}_{n=1}^{N_x} \subseteq \mathcal{E}$ which satisfies $|I_{n+1}^x| < 2^{-n-4}|I_n^x|$ for $n < N_x$ and

$$\sum_{n=1}^{N_x} \frac{|I_n^x|}{|I_n^x| + \operatorname{dist}(x, I_n^x)} \geq K.$$

Proof. We choose a decreasing sequence $\varepsilon_p \in (0,1)$ such that

$$\sum_{p=1}^{\infty} \varepsilon_p \leq \frac{|I|}{8K}$$

but

$$\sum_{p=1}^{\infty} \varepsilon_p |\log \varepsilon_p| = \infty.$$

We start defining our collection of intervals by setting $\mathcal{D}_0 = \emptyset$ and $H_0 = I$. Suppose after p steps of the construction we have obtained collections

$$\mathcal{D}_0, \mathcal{D}_1, \cdots, \mathcal{D}_p$$

where each $\mathcal{D}_i$ is a family of pairwise disjoint compact subintervals of I of equal length $l_i > 0$ and also the covered closed sets $\mathcal{D}_i^*$ are pairwise disjoint subsets of I satisfying $|\bigcup_{i=1}^p \mathcal{D}_i^*| \leq \sum_{i=1}^p \varepsilon_p$. Together with these families we consider the functions

$$b_i(x) = \frac{l_i}{\operatorname{dist}(x, \mathcal{D}_i^*) + l_i}, \text{ where } b_i \equiv 0 \text{ if } \mathcal{D}_i = \emptyset.$$

Moreover, we also have a sequence of relatively open subsets of I

$$H_0 \supseteq H_1 \supseteq \cdots \supseteq H_{p-1} \text{ such that } (\mathcal{D}_1^* \cup \cdots \cup \mathcal{D}_j^*) \cap H_j = \emptyset \text{ for } j = 1, \ldots, p-1.$$

Now we define the new set

$$H_p = \{x \in I : \sum_{i=1}^p b_i(x) < K + 1 \text{ and } x \notin (\mathcal{D}_1^* \cup \cdots \cup \mathcal{D}_p^*)\} \subset H_{p-1}.$$

We note that $\sum_1^p b_i$ is piecewise monotone (because its derivative is piecewise rational) and conclude that H_p is the disjoint union of finitely many intervals.

If $|H_p| \leq |I|/2K$, we arrange these intervals into a collection $\widehat{\mathcal{D}}_{p+1}$. Since

$$|\bigcup_{i=1}^p \mathcal{D}_i^*| < |I|/2K$$

by our choice of the ε_p's, we see that

$$\mathcal{E} = \bigcup_{i=1}^p \mathcal{D}_i \cup \widehat{\mathcal{D}}_{p+1}$$

does what is required in the conclusion of the lemma.

Otherwise, if $|H_p| > |I|/2K$ we pick a collection $\mathcal{H}_p$ of disjoint open intervals of equal length $L_{p+1} \in (0, 2^{-p-4} l_p)$ such that $|\mathcal{H}_p^*| > |H_p|/2$ and that $\mathcal{H}_p^* \subseteq \operatorname{int}(H_p)$. For each $J \in \mathcal{H}_p$ we define $\mathcal{D}_{p+1}(J)$ to contain the single compact intervals of length $l_{p+1} = L_{p+1}\varepsilon_{p+1}/|\mathcal{H}_p^*| < L_{p+1}/2$ concentric with J. We put

$$\mathcal{D}_{p+1} = \bigcup_{J \in \mathcal{H}_p} \mathcal{D}_{p+1}(J)$$

and

$$b_{p+1}(x) = \frac{l_{p+1}}{\operatorname{dist}(x, \mathcal{D}_{p+1}^*) + l_{p+1}}.$$

For b_{p+1} we have the crucial estimate:

$$\int_{H_p} b_{p+1}(x) \geq \varepsilon_{p+1} \log \frac{|\mathcal{H}_p^*|}{\varepsilon_{p+1}}. \tag{3}$$

Indeed, for each $J \in \mathcal{H}_p$ we obtain, essentially integrating $\frac{1}{x}$, that

$$\int_J b_{p+1}(x) \geq l_{p+1} \left(1 + 2 \log \frac{L_{p+1} + l_{p+1}}{2 l_{p+1}}\right) \geq |\mathcal{D}_{p+1}(J)^*| \log \frac{|\mathcal{H}_p^*|}{\varepsilon_{p+1}},$$

since $(t+1)^2/4 > t/e$ for all t. Then we use the fact that

$$\sum_{J \in \mathcal{H}_p} |\mathcal{D}_{p+1}(J)^*| = \varepsilon_{p+1}$$

to arrive at (3) and $|\bigcup_{i=1}^{p+1} \mathcal{D}_i^*| \leq \sum_{i=1}^{p+1} \varepsilon_i$.

We keep iterating our construction and want to show that after a finite time the first case $|H_p| < |I|/2K$ occurs and finishes the proof. Indeed, otherwise $h = \inf_p |\mathcal{H}_p^*| > 0$ necessarily. Since there exists a p_0 with $\varepsilon_{p_0} < h$ we use (3) and the fact that the sequence $\{H_p\}_p$ is decreasing to obtain the following chain of estimates:

$$\begin{aligned}\sum_{p=p_0}^{\infty} \varepsilon_{p+1}(|\log(\varepsilon_{p+1})| + \log(h)) &\leq \sum_{p=p_0}^{\infty} \varepsilon_{p+1} \log \frac{|\mathcal{H}_p^*|}{\varepsilon_{p+1}} \leq \sum_{p=1}^{\infty} \int_{H_p} b_{p+1}(x) \\ &= \sum_{p=1}^{\infty} \int_{H_p \setminus H_{p+1}} \sum_{j=2}^{p+1} b_j(x) \leq K + 2.\end{aligned}$$

This clearly contradicts our assumptions on the ε_p's. □

We also need to understand how the functions b_p defined in the previous proof provide a lower bound on β_f. We have the following simple statement.

Lemma 2. *Let $x, a, a + 2b \in [0, 1]$ and $b > 0$. Suppose the measurable set $M \subset [a, a + 2b]$ satisfies $|M \Delta [a + b, a + 2b]| < b/49$ and that the Lipschitz function $f : [0, 1] \to \mathbb{R}$ fulfills $f' = \chi_M$ a.e. on $[a, a + 2b]$. Then*

$$\beta_f(x, 2^{-k}) \geq \frac{2}{49} \frac{2b}{2b + \operatorname{dist}(x, [a, a + 2b])},$$

if $\max(|x - a|, |x - a - 2b|) \in (2^{-k-1}, 2^{-k}]$.

For the proof it is sufficient to notice the following. If s is the best approximating slope as occurring in (1), then $s \in [0, 1]$. Moreover, at one of the subintervals $[a, a + b]$, $[a + b, a + 2b]$ the gradient of f differs, up to a subset of measure $b/49$, with a fixed sign and in modulus at least $1/2$ from s.

This shows that f minus its best affine approximation oscillates over the interval $[a, a + 2b] \subset [x - 2^{-k}, x + 2^{-k}]$ at least by an amount of $(1/2)(48b/49) - (b/49) = 23b/49$ and so the difference cannot be smaller than $11b/49$. Since $2^{-k-1} < 2b + \operatorname{dist}(x, [a, a + 2b])$, the lemma follows.

Now we are ready to show the existence of 1-Lipschitz functions with a large "sum of the betas" over a finite scale of radii.

Proposition 3. *For every $K > 1$ there is an integer N and a 1-Lipschitz function $f : [0, 1] \to \mathbb{R}$ such that*

$$\sum_{k=1}^{N} \beta_f(x, 2^{-k}) \geq K \textit{ for } \text{every } x \in [0, 1].$$

Proof. Without loss of generality, $K > 49$. We put $\mathcal{E}_0 = \{[0, 1]\}$. By an iterated use of Lemma 1 we obtain a "nested" sequence of finite collections $\mathcal{E}_k$, $k = 0, \ldots, K$, consisting of disjoint intervals such that

a) $\mathcal{E}_{k+1}^* \subseteq \mathcal{E}_k^*$, and
b) for every $I \in \mathcal{E}_k$, $k < K$, $\mathcal{E} = \mathcal{E}_{k+1}(I) = \{J \in \mathcal{E}_{k+1} : J \subseteq I\}$ does satisfy the conclusion of Lemma 1.

We put $\mathcal{E}_{K+1} = \emptyset$ and define the set

$$M_0 = \bigcup_{k=1}^{K} \bigcup \{ (\frac{a + b}{2}, b) \setminus \mathcal{E}_{k+1}^* : J \in \mathcal{E}_k, a = \inf J, b = \sup J \},$$

and define the 1-Lipschitz function by $f(x) = |[0,x] \cap M_0|$.

Clearly, for every $I \in \mathcal{E}_k$ and $M = M_0 \cap I$ we can apply Lemma 2 to conclude that

$$\beta_f(x, 2^{-n}) \geq \frac{2}{49} \frac{|I|}{|I| + \operatorname{dist}(x, I)} \tag{4}$$

if $n = n(x, I)$ satisfies $\bar{I} \subseteq [x - 2^{-n}, x + 2^{-n}]$ but $\bar{I} \nsubseteq [x - 2^{-n-1}, x + 2^{-n-1}]$. We fix N such that $2^{-N+2} < \inf\{|I| : I \in \mathcal{E}_K\}$.

First, we consider a point $x \in E = \bigcap_{k=1}^{K} \mathcal{E}_k^*$. Then there exists a sequence $I_k \in \mathcal{E}_k$ with $x \in I_K \subset I_{K-1} \subset \ldots \subset I_1$ and clearly $|I_{k+1}| < |I_k|/K < |I_k|/49$ for $k < K$. This immediately gives that for the $n(x, I_k)$ defined right after (4) the inequality $n(x, I_k) < n(x, I_{k+1}) < N$ holds. Therefore, we infer

$$\sum_{k=1}^{N} \beta_f(x, 2^{-k}) \geq K \frac{2}{49}.$$

Now, let $x \in [0,1] \setminus E$; then there is some $k_0 < K$ and $I \in \mathcal{E}_{k_0}$ with $x \in I \setminus (\mathcal{E}_{k_0+1}(I))^*$. Therefore, we find a sequence $\{I_l^x\}_{l=1}^{L_x} \subseteq \mathcal{E}_{k_0+1}(I)$ such that $|I_{l+1}^x| < 2^{-l-4}|I_l^x|$ for $l < L_x$ and that

$$\sum_{l=1}^{L_x} \frac{|I_l^x|}{|I_l^x| + \operatorname{dist}(x, I_l^x)} \geq K.$$

Let $S_x = \{l \leq L_x : |I_l^x| \geq 2^{-l} \operatorname{dist}(x, I_l^x)\}$; then clearly

$$\sum_{l \notin S_x}^{L_x} \frac{|I_l^x|}{|I_l^x| + \operatorname{dist}(x, I_l^x)} \leq \sum_{1}^{\infty} 2^{-l} \leq 1.$$

Thus

$$\sum_{l \in S_x} \frac{|I_l^x|}{|I_l^x| + \operatorname{dist}(x, I_l^x)} \geq K - 1.$$

Moreover, if $l \in S_x$ and if $m = \min\{s \in S_x : s > l\}$, then

$$|I_m^x| + \operatorname{dist}(x, I_m^x) \leq (2^m + 1)|I_m^x| \leq 2^{-2}|I_{m-1}^x| \leq |I_l^x|/4 \leq 2^{-n(x, I_l^x)-1}.$$

Thus $n(x, I_m^x) \geq n(x, I_l^x) + 1$ and clearly $n(x, I_m^x) < N$. We conclude that for all $x \in [0,1] \setminus E$

$$\sum_{k=1}^{N} \beta_f(x, 2^{-k}) \geq \frac{2}{49} \sum_{l \in S_x} \frac{|I_l^x|}{|I_l^x| + \operatorname{dist}(x, I_l^x)} \geq \frac{2K - 2}{49},$$

which proves the proposition. $\square$

Hence, all is prepared to give the

Proof of the theorem. We have for all x and $r > 0$ that $\beta_{f+g}(x, r) \leq \beta_f(x, r) + \beta_g(x, r)$ and $\beta_h(x, r) = \beta_{-h}(x, r)$. This clearly implies a triangle type inequality

$$|\beta_f(x, r) - \beta_g(x, r)| \leq \beta_{(f-g)}(x, r) \leq \frac{1}{r}\|f - g\|_\infty.$$

From this it is obvious that every set

$$\mathcal{U}_K = \{f \in C^1([0,1]) ; \text{ there is } N \text{ such that } \inf_{x \in [0,1]} \sum_{n=1}^{N} \beta_f(x, 2^{-n}) > K\}$$

is open with respect to the $\|\cdot\|_\infty$-norm and hence open in C^1. To finish, we only need to show each $\mathcal{U}_K$ is dense in $C^1([0,1])$. So fix any $g \in C^1([0,1])$ and $\varepsilon > 0$. We find $h \in C^2([0,1])$ with $\|g-h\|_{C^1} < \varepsilon/2$ and note that $\beta_h(x,r) \le \|h''\|_\infty (r/2)$, giving

$$\sum_{n=1}^{\infty} \beta_h(x, 2^{-n}) < \|h''\|_\infty \text{ for all } x.$$

Using Proposition 3 we can pick a 1-Lipschitz $F : [0,1] \to \mathbb{R}$ and N with $F(0) = 0$ and

$$\sum_{n=1}^{N} \beta_F(x, 2^{-n}) > 4\frac{K + \|h''\|_\infty}{\varepsilon} + 2 \text{ for all } x.$$

Extending F constantly outside $[0,1]$, we can consider mollifications f of F. Clearly, $\|f\|_{C^1} \le 2$ and if $\|f - F\|_\infty$ is sufficiently small, then

$$\sum_{n=1}^{N} \beta_f(x, 2^{-n}) > 4\frac{K + \|h''\|_\infty}{\varepsilon} + 1 \text{ for all } x \in [0,1].$$

By all we said before, $G = h + (\varepsilon/4)f$ satisfies $\|g - G\|_{C^1} < \varepsilon$ and $G \in \mathcal{U}_K$. $\square$

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