EXTREME POINTS OF LATTICE INTERVALS IN THE MINKOWSKI–RÅDSTRÖM–HÖRMANDER LATTICE

JERZY GRZYBOWSKI AND RYSZARD URBAŃSKI

Abstract. In this paper we characterize extreme points of any symmetric interval in the Minkowski–Rådström–Hörmander lattice $\mathbb{X}$ over any Hausdorff topological vector space $X$ (Theorem 1). Then we prove that the unit ball in the Minkowski–Rådström–Hörmander lattice $\mathbb{X}$ over any normed space $X$, $\dim X \geq 2$, has exactly two extreme points (Theorem 2).

Let $(G, +, \leq)$ be a commutative lattice-ordered group $\mathbb{I}$, where $x \lor y = \sup(x, y)$, $x \land y = \inf(x, y)$, $x_+ = x \lor 0$, $x_- = (-x) \lor 0$, and $|x| = x \lor (-x)$ for $x, y \in G$. In a commutative lattice-ordered group $G$ the following properties hold true: $x \leq y$ implies $z + x \leq z + y$, $|x| = x_+ + x_-$, $x = x_+ - x_-$, $|x| \geq 0$, $x_+ + x_- = (x + x)_+$, and $x + (y \lor z) = (x + y) \lor (x + z)$ for any $x, y, z \in G$.

The interval $[a, b]$ is the set $\{x \in G \mid a \leq x \leq b\}$ where $a, b \in G$, $a \leq b$. In particular, $[-a, a] = \{x \in G \mid |x| \leq a\}$ for $a \in G$, $a \geq 0$.

We say that $x \in G$ is an extreme point of a subset $A$ of $G$ if for any $y, z \in A$ the equality $x + x = y + z$ implies that $x = y = z$. By ext $A$ we denote the set of all extreme points of $A$. If $G$ is a vector space, this definition of the extreme point of a subset $A$ coincides with the usual one.

The following proposition characterizes extreme points of symmetric intervals in $G$.

Proposition 1. Let $G$ be a commutative lattice-ordered group, $a \in G$, $a \geq 0$. Then $x \in \text{ext } [-a, a]$ if and only if $|x| = a$.

Proof. Let $x \in \text{ext } [-a, a]$. Since $x \leq a$, $x_+ \leq a$ and $x_+ + x_- - a \leq a$; and since $x_+ \geq 0$, $x_+ + x_- - a \geq -a$. Denote $y = x_+ + x_- - a$. We have $|y| \leq a$. Also $-x \leq a$. Hence $-x_+ \geq -a$ and $-x_- \leq a$. Then $a - x_+ - x_- \geq -a$. Since $-x_- \leq 0$, $a - x_+ - x_- \leq a$. Denote $z = a - x_+ - x_-$. We have $|z| \leq a$. Since $y + z = x + z$, $x = y = x_+ + x_- - a$. Hence $x + |x| = x + (x \lor (-x)) = (x + x) \lor 0 = (x + x)_+ = x_+ + x_- = x + a$. Then $|x| = a$.
Assume that \(|x| = a, y, z \in [-a, a], x + x = y + z\). Denote \(w = y - x\). Then \(y = x + w, z = x - w\). We have
\[
|x| = a \geq |y| \lor |z| = (x + w) \lor (-x + x) \lor (-x + w) = \left[ (x + w) \lor (x - w) \right] \lor (-x + w) = \left[ (x + w) \lor (x - w) \right] \lor (x + w) = |x + w| \lor (x - w) = |x + w| = |x| + |w|.
\]
Hence \(|w| = 0\) and \(w = 0\). Therefore, \(x = y = z\) and \(x \in \text{ext}[-a, a]\).

**Remark.** In the case of convex subsets in a real vector space the classical definition of the extreme points coincides with our definition. The intervals in a vector lattice are convex sets. Therefore, the following corollary holds true.

**Corollary 1.** Let \(Y\) be a vector lattice, \(a \in Y, a \geq 0\). Then we have \(\text{ext}[-a, a] = \{x \in Y \mid |x| = a\}\).

A subset \(E\) of a vector lattice \(Y\) is solid if for all \(x \in E\) the interval \([-|x|, |x|]\) is contained in \(E\). A topological vector lattice \(Y\) is locally solid if there exists a neighborhood basis of the origin consisting of solid sets [3].

We say that a point \(x\) of a topological vector space \(Y\) is a strongly extreme point of \(E \subset Y\) if for all generalized sequences \((y_\alpha)_{\alpha \in \Lambda}, (z_\alpha)_{\alpha \in \Lambda} \subset E\) such that \(\lim_{\alpha \in \Lambda} \frac{y_\alpha + z_\alpha}{2} = x\), we have \(\lim_{\alpha \in \Lambda} y_\alpha = \lim_{\alpha \in \Lambda} z_\alpha = x\).

The following proposition identifies extreme points and strongly extreme points of symmetric intervals in topological vector lattices.

**Proposition 2.** Let \(Y\) be a locally solid topological vector lattice. Then for any \(a \in Y, a \geq 0\), every extreme point of \([-a, a]\) is strongly extreme.

**Proof.** Let \(x \in \text{ext}[-a, a]\). By Proposition 1, \(|x| = a\). Let \(y_\alpha, z_\alpha \in [-a, a], \alpha \in \Lambda\), \(\lim_{\alpha \in \Lambda} \frac{y_\alpha + z_\alpha}{2} = x\). We have
\[
|x| = a \geq |y_\alpha| \lor |z_\alpha| = \frac{|y_\alpha + z_\alpha|}{2} + \frac{|y_\alpha - z_\alpha|}{2}.
\]

Since \(Y\) is locally solid, \(|x|\) is continuous and \(\lim_{\alpha \in \Lambda} \frac{|y_\alpha + z_\alpha|}{2} = a\). Since \(\frac{|y_\alpha - z_\alpha|}{2} \leq a - \frac{|y_\alpha + z_\alpha|}{2}\) and \(\lim_{\alpha \in \Lambda} (a - \frac{|y_\alpha + z_\alpha|}{2}) = 0\), by the corollary after Proposition 1 in Section 7.1.3 of [3], \(\lim_{\alpha \in \Lambda} \frac{y_\alpha + z_\alpha}{2} = 0\). Therefore,
\[
\lim_{\alpha \in \Lambda} y_\alpha = \lim_{\alpha \in \Lambda} \frac{y_\alpha + z_\alpha}{2} + \lim_{\alpha \in \Lambda} \frac{y_\alpha - z_\alpha}{2} = x.
\]

In a similar way \(\lim_{\alpha \in \Lambda} z_\alpha = x\). Hence the point \(x\) is a strongly extreme point of the interval \([-a, a]\).

Let \(X = (X, \tau)\) be a Hausdorff topological vector space over the field \(\mathbb{R}\) and \(\mathcal{B}(X)\) be the family of all nonempty closed bounded convex subsets of \(X\). For \(A, B, C, D \in \mathcal{B}(X)\) we have \(A + B = \{a + b \mid a \in A, b \in B\}\), \(A + B = A + B\), \(A \lor B = \text{conv}(A \cup B)\), \(A \lor B = \{x \in X \mid x + B \subset A\}\), \((A, B) \sim (C, D)\) if and only if \(A + D = B + C\). The relation “\(\sim\)” is a relation of equivalence and \([A, B] = [(A, B)]\) denotes a class of equivalence. Also \([A, B] \leq [C, D]\) if and only if \(A + D \subset B + C, [A, B] + [C, D] = [A + C, B + D]\), \([A, B] \lor [C, D] = \sup\{[A, B], [C, D]\} = ([A + D) \lor (B + C), B + D]\).

The quotient space \(\tilde{X} = \mathcal{B}^2(X)/\sim\) is called the Minkowski–Råström–Hörmander (M–R–H) space over \(X\). Moreover, \((\tilde{X}, +, \leq)\) is a topological vector lattice. For the definition of an M–R–H lattice we refer to [14] and to paragraph 3.4 in [10].
Moreover, and in calculating an Aumann integral \cite{2}. The M–R–H lattices were also studied in bounded-valued correspondences \cite{4}, \cite{6}; in quasidifferential calculus \cite{5}, \cite{10}, \cite{13};

$$\bigcup$$

By Corollary 4.5.5 in \cite{10} we have

$$\{ x \in \tilde{X} | x = [A, B], A \cup B = K, A \cap B = L \}.$$  

Moreover, all extreme points of the interval \([-\tilde{a}, \tilde{a}]\) are strongly extreme.

**Proof.** Let \(A, B \in \mathcal{B}(X)\), \(A \cup B = K, A \cap B = L\), \(\tilde{x} = [A, B]\). Then

$$|\tilde{x}| = \tilde{x} \vee (-\tilde{x}) = [A, B] \vee [B, A] = [2(A \cup B), A + B].$$

By Corollary 4.5.5 in \cite{10} \(A \cup B + A \cap B = A + B\). Hence \(|\tilde{x}| = [A \cup B, A \cap B] = [K, L] = \tilde{a}\). By Corollary 1, \(\tilde{x} \in \text{ext} \{ a, \tilde{a} \}\).

Assume that \(\tilde{x} = [A_1, B_1] \in \text{ext} \{ a, \tilde{a} \}\). Let \(v \in A_1 + B_1\). Denote \(A_2 = A_1 + A_1 \vee B_1 + K - v, B_2 = B_1 + A \vee B_1 + K - v\). Then \(\tilde{x} = [A_2, B_2]\) and \(K \subset A_2, L \subset K \subset B_2\).

By Corollary 1 we have \(|\tilde{x}| = \tilde{a}\). Hence \(|[A_2, B_2]| = [A_2 \cup B_2, A_2 \cap B_2] = [K, L]|\) by Theorem 6.3.3, p. 136 in \cite{10}, there exist \(A, B \in \mathcal{B}(X)\) such that \(\tilde{x} = [A, B], A \cup B = K, A \cap B = L\). The last assertion of the theorem follows from Proposition 2 and local solidity of the M–R–H lattice \(\tilde{X}\).

**Remark.** Let \(a = [K, L], K, L \in \mathcal{B}(X), L \subset K\). Every extreme point \(\tilde{x} = [A, B], A, B \in \mathcal{B}(X), A \cup B = K, A \cap B = L\) of the interval \([-\tilde{a}, \tilde{a}]\) is determined by the family of all connected components of \(K \setminus L\) that are contained in \(A\) and disjoint with \(B\). Therefore, the number of all extreme points of the interval \([-\tilde{a}, \tilde{a}]\) is equal to the number of all subsets of the family of all connected components of \(K \setminus L\).

**Example.** If \(K\) is the closed disc circumscribed around the square \(L\) in the plane, then the set \(K \setminus L\) has four connected components: \(C_1, C_2, C_3, C_4\). The interval \([-\tilde{a}, \tilde{a}]\), \(\tilde{a} = [K, L]\) has \(2^4 = 16\) extreme points. Four extreme points, \(\tilde{a}, \tilde{x}_1 = [L \cup C_3 \cup C_4, L \cup C_1 \cup C_2], \tilde{x}_2 = [C_2 \cup C_4, C_1 \cup C_3]\) and \(\tilde{x}_3 = [C_1 \cup C_3 \cup C_4, L \cup C_2]\), are presented in Figure 1.

The following proposition characterizes the subset \(U\) of \(\tilde{X}\), where \(U\) is a closed convex neighborhood of the origin in a topological vector space \(X\) that is not normable.

**Proposition 3.** If a Hausdorff topological vector space \(X\) is not normable and \(U\) is a closed convex neighborhood of the origin, then the set \(U\) in \(\tilde{X}\) has no extreme points.

**Proof.** Let \(\tilde{x} \in U\). Then \(|\tilde{x}| \leq \tilde{a}\), for some \(\tilde{a} = [K, \{0\}], 0 \in K \in \mathcal{B}(X), K \subset U\). By the theorem of Kolmogorov the set \(U\) is not bounded. Then there exists \(u \in U \setminus K\).

The convex hull of the union of two bounded convex sets in \(X\) is a bounded set.
Therefore, the set $K \cup u$ is bounded and $K_1 = K \cup u \in \mathcal{B}(X)$. Also $K_1 \subset U$ and $K$ is a proper subset of $K_1$. Then $|\bar{x}| \leq \bar{b} = [K_1, \{0\}], |\bar{x}| \neq \bar{b}$ and $\bar{x} \notin \text{ext} [-\bar{b}, \bar{b}]$, where $[-\bar{b}, \bar{b}] \subset U$.

Let $(X, \| \cdot \|)$ be a normed space and $\mathbb{B}$ be the closed unit ball in $X$. If $x^* \in X^*$, then $\|x^*\| = \text{sup} \{x^*(x) \mid x \in X, \|x\| \leq 1\}$. The pair $(X^*, \| \cdot \|)$ is a Banach space. Let $\bar{x} = [A, B]$. The function $\| \cdot \| : X \rightarrow \mathbb{R}_+$ defined by $\|\bar{x}\|_H = d_H(A, B) = \inf \{\alpha > 0 \mid A \subset B + \alpha \mathbb{B}, B \subset A + \alpha \mathbb{B}\}$ is a norm in $X$. Let us notice that $d_H$ is a Hausdorff metric in $\mathcal{B}(X)$ and the topology in $\bar{X}$ generated by $\| \cdot \|_H$ is identical with the topology introduced before Theorem 1. The normed $\text{M–R–H}$ lattice $(\bar{X}, \| \cdot \|_H)$ is not a Banach space with the exception of $\dim X = 1$. The unit ball $\mathbb{B}$ in the $\text{M–R–H}$ lattice $\bar{X}$ is equal to the interval $[-\bar{a}, \bar{a}]$ where $\bar{a} = [\mathbb{B}, \{0\}]$ (\cite{12}, \cite{14}). The following theorem is the major consequence of Theorem 1.

**Theorem 2.** Let $X$ be a normed space. Unless $\dim X = 1$ the two endpoints $\bar{a} = [\mathbb{B}, \{0\}]$ and $-\bar{a}$ are the only extreme points of the unit ball $\mathbb{B}$ in the Minkowski–Rådström–Hörmander lattice $\bar{X}$. In $\mathbb{R}$ we have $\text{ext} \mathbb{B} = \{\bar{a}, -\bar{a}, \bar{b}, -\bar{b}\}$, where $\bar{b} = \{1\}, \{0\}$. Moreover, all extreme points of the ball $\mathbb{B}$ are strongly extreme.

**Proof.** Let $\bar{x} = [A, B]$ be an extreme point of the ball $\mathbb{B}$. We may assume that $A \cup B$ is convex. By Theorem 1 we have $|\bar{x}| = \bar{a}$. Then $[A \cup B, A \cap B] = [\mathbb{B}, \{0\}]$. Hence $A \cup B = A \cap B + \mathbb{B}$. By Theorem 3 of \cite{8}, $A \subset B$ or $B \subset A$ or $\mathbb{B}$ is an interval. If $A \subset B$, then $\bar{x} = -\bar{a}$. If $B \subset A$, then $\bar{x} = \bar{a}$. If $\mathbb{B}$ is an interval, then $\dim X = 1$. For $X = \mathbb{R}$ the $\text{M–R–H}$ lattice $\bar{X}$ is two-dimensional and the ball $\mathbb{B}$ is a square. □

**Remarks.** 1. The $\text{M–R–H}$ lattice $\bar{X}$ is not strictly convex nor uniformly convex. Moreover, the ball $\mathbb{B}$ is not the convex hull of the set $\text{ext} \mathbb{B}$, with the exception of $\dim X = 1$.

2. If $\bar{a} = [K, L], L \subset \text{int} K$, then the interval $[-\bar{a}, \bar{a}]$ in the completion of $\bar{X}$ has exactly the same extreme points. If $L \not\subset \text{int} K$ we do not know whether it is true.

3. The unit ball in the completion of $\bar{X}$ has exactly the same extreme points. Hence for $\dim X \geq 2$ the completion of $\bar{X}$ is not a reflexive space due to the Krein–Milman theorem.

4. Let $\mathcal{K}(X)$ be the family of all nonempty compact convex subsets of the Hausdorff topological vector space $X$. The quotient space $\mathcal{K}^2(X)/\sim$ found an application in the quasidifferential calculus \cite{3}. The space $\mathcal{K}^2(X)/\sim$ is a sublattice of the $\text{M–R–H}$ lattice $\bar{X}$. If the normed space $X$ is infinitely dimensional, then the unit ball in $\mathcal{K}^2(X)/\sim$ has no extreme points by the reasoning analogous to Proposition 3.
Finally, we show that in the case of the dual space $X^*$ being separable, all the extreme points of any interval in $\overline{X}$ are exposed.

**Theorem 3.** Let $(X, \| \cdot \|)$ be a normed space such that the dual space $X^*$ is separable. If $\overline{a} \in \overline{X}$, $\overline{a} \geq 0$, then $\text{ext} [-\overline{a}, \overline{a}] = \exp [-\overline{a}, \overline{a}]$.

**Proof.** Let $\{f_n\}_{n \in \mathbb{N}}$ be a dense subset of the unit sphere in the dual space $\{f \in X^* \mid \|f\| = 1\}$. For any mapping $\epsilon : \mathbb{N} \rightarrow [-1, 1]$ define the function $b_\epsilon : \mathcal{B}(X) \rightarrow \mathbb{R}$ by

$$b_\epsilon(A) = \sum_{n=1}^{\infty} 2^{-n} \epsilon(n) \sup f_n(A).$$

The function $b_\epsilon$ is additive, positively homogeneous and continuous, because $|b_\epsilon(A) - b_\epsilon(B)| \leq d_H(A, B)$.

Let $\overline{b}_\epsilon(\overline{x}) = b_\epsilon(A) - b_\epsilon(B)$, where $\overline{x} = [A, B]$. The function $\overline{b}_\epsilon$ is well-defined on $\overline{X}$, and linear and continuous with $|\overline{b}_\epsilon(\overline{x})| \leq \|\overline{x}\|$ for $\overline{x} \in \overline{X}$. Let $\overline{a} = [K, L]$, $L \subset K$ and $\overline{x} \in \text{ext} [-\overline{a}, \overline{a}]$. By Theorem 1, $\overline{x} = [A, B]$, $A \cup B = K$, $A \cap B = L$. Let

$$\epsilon(n) = \begin{cases} 1 & \text{if } \sup f_n(A) \leq \sup f_n(B), \\ -1 & \text{if } \sup f_n(B) > \sup f_n(A). \end{cases}$$

Let $\epsilon^+(n) = \max(\epsilon(n), 0)$, $\epsilon^-(n) = \max(-\epsilon(n), 0)$. For any $\overline{y} \in [-\overline{a}, \overline{a}]$, $\overline{y} = [C, D]$ we have $L + D \subset K + C$ and $C + L \subset D + K$. Notice that $b_\epsilon+(A) = b_\epsilon+(A \cup B)$ and $b_\epsilon-(B) = b_\epsilon-(A \cap B)$. Then

$$b_\epsilon+(C) + b_\epsilon+(B) = b_\epsilon+(C) + b_\epsilon+(L) \leq b_\epsilon+(D + K) = b_\epsilon+(D) + b_\epsilon+(K) = b_\epsilon+(D) + b_\epsilon+(A).$$

Hence $\overline{b}_\epsilon+(\overline{y}) \leq \overline{b}_\epsilon+(\overline{x})$. In an analogous way, $b_\epsilon-(D) + b_\epsilon-(A) \leq b_\epsilon-(C) + b_\epsilon-(B)$. Hence $\overline{b}_\epsilon-(\overline{y}) \geq \overline{b}_\epsilon-(\overline{x})$. Therefore,

$$\overline{b}_\epsilon+(\overline{y}) = \overline{b}_\epsilon+(\overline{y}) - \overline{b}_\epsilon-(\overline{y}) \leq \overline{b}_\epsilon+(\overline{x}) - \overline{b}_\epsilon-(\overline{x}) = \overline{b}_\epsilon(\overline{x}).$$

Now, assume that $\overline{b}_\epsilon(\overline{y}) = \overline{b}_\epsilon(\overline{x})$. Then $\overline{b}_\epsilon(\overline{y}) = \overline{b}_\epsilon(\overline{x})$ and $\overline{b}_\epsilon-(\overline{y}) = \overline{b}_\epsilon-(\overline{x})$. By $(*)$ for any $n \in \mathbb{N}$, such that $\epsilon(n) = 1$,

$$\sup f_n(C + B) = \sup f_n(C + L) = \sup f_n(D + K) = \sup f_n(A + D).$$

In a similar way for $\epsilon(n) = -1$,

$$\sup f_n(D + A) = \sup f_n(L + D) = \sup f_n(K + C) = \sup f_n(C + B).$$

Then by the fact that $\{f_n \mid n \in \mathbb{N}\}$ is dense in the unit sphere in $X^*$ we have for all $f \in X^*$,

$$\sup f(D + A) = \sup f(B + C).$$

Hence $A + D = B + C$ and $\overline{y} = \overline{x}$. Therefore, $\overline{x}$ is an exposed point of the interval $[-\overline{a}, \overline{a}]$. \(\Box\)

**Corollary 2.** Let $(X, \| \cdot \|)$ be a normed space such that the dual space $X^*$ is separable. Then every extreme point of the unit ball $\overline{B}$ in the Minkowski–Rådström–Hörmander lattice $\overline{X}$ is exposed.
References


Faculty of Mathematics and Computer Science, Adam Mickiewicz University, Umultowska 87, 61-614 Poznań, POLAND
E-mail address: jgrz@amu.edu.pl

Faculty of Mathematics and Computer Science, Adam Mickiewicz University, Umultowska 87, 61-614 Poznań, Poland
E-mail address: rich@amu.edu.pl